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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Stephen Hermes **Email/Phone:** SHHERMES@BRANDEIS.EDU

Speaker's Name: O. Iyama

Talk Title: Tau-Tilting Theory II

Date: 11 / 1 / 12 **Time:** 3 : 30 am / pm (circle one)

List 6-12 key words for the talk: Tau-tilting, tilting theory, quivers, representation theory, Hasse diagrams, cluster tilting

Please summarize the lecture in 5 or fewer sentances: The speaker recalled the notions of tau-tilting and support tau-tilting modules. He showed that support tau tilting modules are in bijection w/ torsion theories of a certain type and thus are partially ordered. He showed that the exchange graph of support tau-tilting modules is the Hasse diagram of this partial ordering.

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Tau-Tilting Theory II

O. Iyama November 1, 2012

Joint with T. Adachi, I. Reiten.

1 Tilting Theory

Let Λ be a finite dimensional algebra over a algebraically closed field k. If $T = T_1 \oplus \cdots \oplus T_n$ is a tilting module, sometimes we can replace a summand T_i with a new one to get a new tilting module. We can't always do this in general. We can 'fix' this problem by extending in a few different ways:

1. Cluster tilting objects in the cluster category, but this only works for special algebras Λ.
2. Silting theory (roughly replace tilting modules with complexes) but this makes the category too big.
3. τ-tilting theory (c.f. Reiten's lecture 'Tau-Tilting Theory I').

There is the Nakayama functor $\nu : \mathsf{proj}\Lambda \to \mathsf{inj}\Lambda$ (categories of finite dimensional projective and injective modules respectively) which is an equivalence of categories. We can use ν to construct the Auslander-Reiten translate. Let M be a Λ-module. Take a (minimal) projective presentation

$$P_1 \xrightarrow{f} P_0 \to M \to 0$$

(the minimality is to make this unique up to isomorphism). This induces $\nu(f) : \nu P_1 \to \nu P_0$. Define the *Auslander-Reiten translate* to be $\tau M = \ker \nu(f)$. This gives a bijection between the indecomposable non-projective Λ-modules and the indecomposable non-injective Λ-modules.

Definition 1.1. A module M is *τ-rigid* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$. M is *τ-tilting* if it is τ-rigid and $|M| = |\Lambda|$ (the number of indecomposable summands equal).

Definition 1.2. M is *support τ-tilting* if there is an idempotent $e \in \Lambda$ such that M is a τ-tau $\Lambda/\Lambda e\Lambda$-module

Example 1.3. Let Q be the quiver

$$1 \underset{a}{\overset{a}{\rightleftarrows}} 2$$

and $\Lambda = kQ/(a^2)$. Then $\binom{1}{2} \oplus \binom{2}{1}$, $2 \oplus \binom{2}{1}$, and $\binom{1}{2} \oplus 1$ are τ-tilting. These and additionally, $2, 1$ and the 0 module are support τ-tilting.

2 Results

Theorem 2.1 (AIR, cf. Smalø)**.** Let $s\tau{-}\mathsf{tilt}\Lambda$ be the set of isoclasses of basic support τ-tilting Λ-modules (basic means multiplicity of each indecomposable summand is 1). Then $s\tau{-}\mathsf{tilt}\Lambda$ is in bijection with the set of *torsion classes* $\mathcal{T} \subset \mathsf{mod}\Lambda$ (i.e. $\mathcal{T}$ is a full subcategory, closed under factor modules and extensions) which are functorially finite.

One direction is given by sending a support τ-tiling module M to its factor category $\mathsf{Fac}M$, and the reverse direction is given by sending $\mathcal{T}$ to the direct sum of all relative projectives in $\mathcal{T}$.

In this way, we can regard $s\tau{-}\mathsf{tilt}\Lambda$ as a poset. Specifically, $M \geq N$ iff $\mathsf{Fac}(N) \subset \mathsf{Fac}(M)$.

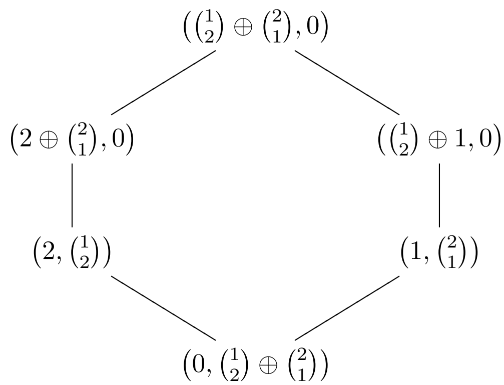
Theorem 2.2 (AIR, cf. Bongartz)**.** Any τ-rigid tilting module is a summand of some τ-tilting module.

The construction is given as follows: A τ-rigid module N is a summand of the projective cogenerator of the kernel of the functor $\mathrm{Hom}_\Lambda(-, \tau N)$.

Theorem 2.3 (AIR, cf. Reidtmann-Schofield)**.** For every basic τ-rigid $\Lambda/\Lambda e \Lambda$-module N such that $|N| = |\Lambda/\Lambda e\Lambda| - 1$ (i.e. M is almost support τ-tilting), there are precisely 2 support τ-tilting modules (M_i, P_i) $i = 1, 2$ (c.f. Reiten for this notation) such that N is a summand of M_i $(i = 1, 2)$ and Λe is a summand of P_i $(i = 1, 2)$.

Definition 2.4. Call (M_1, P_1) and (M_2, P_2) *mutations* of each other. From this we can construct an *exchange graph* in the usual way.

Example 2.5. For the algebra in the first example, $\left(2, \binom{1}{2}\right)$ and $\left(2 \oplus \binom{2}{1}, 0\right)$ are mutations of each other (here we use the (M, P) notation). The full exchange graph is the hexagon:



Theorem 2.6 (AIR, cf. Happel-Unger, Aihara-I)**.** The exchange graph is the same as the Hasse graph with respect to the partial ordering above.

Corollary 2.7. If there are only finitely many support τ-tilting modules, the exchange graph is connected. The proof proceeds by taking the maximal module and mutating to the minimal module.

Example 2.8 (Adachi)**.** Take Q to be a cycle with n vertices, and an arbitrary $r > 0$. Let

$$\Lambda = kQ/(a^r : a \in Q^1).$$

Then the number of support τ-tilting modules is $\binom{2n}{n}$ if $r \geq n$ and $\frac{3n-1}{n}\binom{2n-2}{n-1}$ for $r = n - 1$.

Example 2.9 (Mizuno)**.** Take Λ a preprojective algebra of Dynkin type (i.e. add reverse arrows and mod out by mesh relations). Then elements of the Weyl group W are in bijection with $s\tau\mathrm{-tilt}\Lambda$. The bijection is given as follows: Take a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$. Let $I_i = \Lambda(1 - e_i)\Lambda$. Then $I_{i_1} \cdots I_{i_\ell}$ is the corresponding module.

The partial ordering corresponds to the weak Bruhat ordering.

Remark 2.10. There is a natural one-to-one correspondence between $s\tau\mathrm{-tilt}\Lambda$ and $s\tau\mathrm{-tilt}\Lambda^{\mathrm{op}}$. One way to interpret this is via silting theory.

Theorem 2.11. There are bijections between the following:

1. $s\tau\mathrm{-tilt}\Lambda$
2. basic two-term silting objects (a bounded projective complex T in the homotopy category is *silting* if $\mathrm{Hom}(T, T[i]) = 0$ for every $i > 0$ and it generates $\mathcal{K}^b(\mathrm{proj}\Lambda)$).

If $\Lambda = \text{End}_{\mathcal{C}}(T)$ is a cluster-tilted algebra (for some 2-CY category $\mathcal{C}$) then we also get a bijection with basic cluster-tilting objects.

Remark 2.12.

1. Derksen-Fei give some of [AIR] results in terms of two-term tilting complexes.
2. Using the bijection above, we can show many results in cluster-tilting theory.

Definition 2.13. The *g-vector* of (M, P) is $g^{(M,P)} = [P_0^M] - [P_1^M] - [P] \in K_0(\mathsf{proj}\Lambda)$ where

$$P_1^M \to P_0^M \to M \to 0$$

of M is a minimal projective presentation.

Proposition 2.14.

1. τ-rigid pairs are determined by their g-vectors.
2. g-vectors of indecomposable summands of (M, P) (support τ-tilting) form a basis of $K_0(proj\Lambda)$.