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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Stephen Hermes **Email/Phone:** SRHERMES@BRANDEIS.EDU

Speaker's Name: M. Shapiro

Talk Title: Generalized Cluster Algebras and Teichmüller Spaces of Riemannian Surfaces w/ Orbifold Points of Arbitrary Order

Date: 1 / 2 / 12 **Time:** 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: Orbifolds, Generalized cluster Algebra, Teichmüller space, hyperbolic geometry, cluster algebra, mutation

Please summarize the lecture in 5 or fewer sentances: The speaker introduced a class of algebras generalising the class of cluster algebras. This generalization allows one to extend surface cluster algebras to surfaces w/ orbifold points of arbitrary order. These algebras satisfy a modified mutation relation.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
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Generalized Cluster Algebras and Orbifolds

M. Shaprio November 2, 2012

Joint with L. Chekhov.

Plan:

1. Cluster algebras and compatible 2-forms.
2. Generalized mutation rule compatible with 2-form.
3. Generalized cluster algebra.
4. Properties.
5. Main example: Teichmüller space of surfaces with orbifold points (of arbitrary order).
6. Open questions.

1 Compatible 2-Forms

Compatible Poisson structures (c.f. Gekhtman) don't always exist. So we consider more generally compatible 2-forms.

Definition 1.1. A skew 2-form ω is *compatible* with functions $\{f_1, \ldots, f_n\}$ if

$$\omega = \sum_{i,j} \frac{df_i}{f_i} \wedge \frac{df_j}{f_j}$$

for some $\omega_{ij} \in \mathbb{Z}$.

A skew 2-form ω is *compatible* with a cluster algebra $\mathcal{A}(B)$ if it is compatible with any cluster $\mathbf{x}$ of $\mathcal{A}(B)$. That is, for $\mathbf{x} = \{x_1, \ldots, x_n\}$

$$\omega = \sum_{i,j} \frac{dx_i}{x_i} \wedge \frac{dx_j}{x_j}$$

(where ω_{ij} depends on the cluster $\mathbf{x}$).

Surfaces with orbifold points of order 2 correspond bijectively with cluster algebras of finite mutation type (cf. Felikson). What if we consider more generally orbifold points of higher order?

1. Order 2,3 considered by L. Chekhov and M. Mazzocco ('09).
2. Arbitrary order by S-Chekhov. On the cluster algebra side we need to consider "generalized" cluster algebras.

Question ([GSV09])**:** Assume that you have a manifold with 2-form and log-canonical coordinates. Can we define an analogue of cluster transformations which preserve the type of this 2-form and satisfy some natural axioms?

Look for an infinite bi-indexed sequence of transformations $T_{n,k} : (B, x_1, \ldots, x_n) \to (B', x'_1, \ldots, x'_n)$ for $1 \le k \le n \le \infty$ (B skew-symmetric $n \times n$ integer matrix, x_i rational functions) satisfying:

1. Locality: $x'_j = x_j$ for $j \neq k$ and x'_k is a polynomial in $x_1, \ldots, \hat{x}_k, \ldots, x_n$.

2. $T^2_{n,k} = id$

3. $\omega = \sum b_{ij} \frac{dx_i}{x_i} \wedge \frac{dx_j}{x_j} = \sum b'_{ij} \frac{dx'_i}{x'_i} \wedge \frac{dx'_j}{x'_j}$

4. S_n-Equivariance:
$$T_{n,\sigma(k)}(\sigma B, x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (\sigma B', x'_{\sigma(1)}, \ldots, x'_{\sigma(n)})$$
for $\sigma \in S_n$

5. Universality:
$$T_{n,k}(B^j, x_1, \ldots, \hat{x_j}, \ldots x_{n+1}) = T_{n+1,k'}(B, x_1, \ldots, x_{n+1})\big|_{x_j=1}$$
where $k' = k$ if $k < j$ and $k' = k+1$ if $k > j$, and B^j denotes B with the j-th cow and column deleted.

Theorem 1.2. Then $T_{n,k}$ as above are given by formulas

$$x'_k = \frac{v(x, b_k)^d P(y(x, b_k))}{x_k}$$

where $v(x, b_k) = \prod_{b_{ik}<0} x_i^{-b_{ki}}$ and $y(x, b_k) = \prod_i x_i^{b_{ki}}$, and P is a reciprocal polynomial of degree d (i.e. $t^d P(1/t) = P(t)$).

2 Generalized Cluster Algebras

Assume that B is an integer skew-symmetriseable $n \times n$ matrix. Assume that $d_1, \ldots, d_n \in \mathbb{N}$ such that b_{ki} is divisible by d_k, so that $b_{ki} = d_k \beta_{ki}$ for some $\beta_{ki} \in \mathbb{Z}$.

For $\mathbf{p} = \{p_0, \ldots, p_d\}$ define $\Theta_{\mathbf{p}}(u, v) = \sum p_j u^j v^{d-j}$ and $\rho_{\mathbf{p}} = \sum p_j t^j$. A *seed* is a triple

$$(\mathbf{x} = \{x_1, \ldots, x_n\}, \mathbf{p} = \{p_{10}, \ldots, p_{1,d_1}; p_{2,0}, \ldots, p_{2,d_2}; \ldots, p_{n,d_n}\}, B).$$

Mutation $(\mathbf{x}, \mathbf{p}, B) \leftrightarrow (\mathbf{x}', \mathbf{p}', B')$ in the direction k:

1. $B \leftrightarrow B'$ is standard (transpose to standard).

2. $x'_i = \begin{cases} x_i & \text{if } i \neq j \\ \frac{\Theta_{\mathbf{p}_k}(u_k^+, u_k^-)}{x_k} & i = k \end{cases}$

3. $p'_{k,l} = p_{k,d_k - l}$

4. $\frac{p'_{ij}}{p'_{i,r}} = \begin{cases} (p_{k,0})^{\beta_{ki}} \frac{p_{ir}}{p_{ij}} & if b_{ki} > 0 \\ (p_{k,0})^{-\beta_{ki}} \frac{p_{ir}}{p_{ij}} & if b_{ki} < 0 \end{cases}$

where $\mathbf{p}_k$ os the k-th row of all the coefficients, $u_k^+ = \prod_{\beta_{ki}>0} x_i^{\beta_{ki}}$, $u_k^- = \prod_{\beta_{ki}<0} x_i^{-\beta_{ki}}$.

Remark 2.1. If $p'_{kl} = p_{kl}$ then ρ_k is a palindromic polynomial.

Properties:

1. If the 2-form is compatible with the cluster algebra $\mathcal{A}(B)$, it is compatible with the generalized cluster algebra $\mathcal{A}_\Theta(B)$.

2. From caterpillar lemma (FZ) follows Laurent phenomenon.

3. Finite type classification: Generalized cluster algebras of finite type satisfy Cartain-Killing classification.

3 Main Example

An orbifold point on a surface has a neighbourhood isomorphic to D/Z_p ($D \subset \mathbb{R}^2$the unit disk, $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$). This gives a cone point with angle $2\pi/p$.

Let $\Sigma_{g,s,r}$ be a Riemann surface of genus g with s holes and r orbifold points of order p_i and assume $\partial\Sigma \neq \varnothing$. For each orbifold point, consider a path around the orbifold point that tends towards the boundary tangentially. Remove the "petals" i.e. the areas enclosed by these paths to get a regular surface, which we can triangulate and introduce λ-coordinates on as usual. (It will give coordinates on decorated Teichmüller space with orbifold points.)

A *petal surface* is a surface with such chosen arcs around the orbifold points. The pedals can be mutated, and we have for a triangle with side lengths a, b, c, the triangle obtained by mutation at c satisfies the exchange relation

$$cc' = a^2 + 2\cos(\pi/p)ab + b^2.$$