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Lee Mosher "Quasi-Actions on Trees"
Joint with M. Sageev and K. Whyte

<u>Theorem</u> (Stallings) Let G = torsion free group, f.g. If G has coh. dim 1, then G is free.

$\boxed{\text{A. Cayley 1878}}$
"quasi-geometrical repres."

X, Y are metric spaces, $f: X \to Y$ is a <u>quasi-isometry</u> if $\exists K \geq 1, C \geq 0$ s.t.

- $\frac{1}{K} d(x,y) - C \leq d(f(x), f(y)) \leq K(d(x,y)) + C \quad \forall x, y \in X$
- $\forall y \; \exists x$ s.t. $d(f(x), y) \leq C$

<u>Examples</u>: G, H f.g. groups with word metric, then they are Q.I. if
① G has finite index in H
② $G/\underset{\text{finite}}{N} \simeq H$
③ G, H act prop. disc., cocompactly, isometrically on the same proper geodesic metric space X.

<u>Theorem</u> (Gersten; Block-Weinberger)
Coh dim. 1 is a quasi-isometry invariant (idea - not precise)

<u>Theorem</u> If G is a f.g. free group, and H is a f.g. torsion free group qi to G, then H is free.

<u>Fact</u>: (Karrass-Pietowski-Solitar)
Every virtually free group is the fundamental group of a graph of finit groups

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<u>Graphs of groups</u>

- a finite graph Γ
- vertex groups Γ_v, $v \in Vert(\Gamma)$
- Edge groups Γ_e, $e \in Edge(\Gamma)$
- $e \overset{\eta}{\longrightarrow} v$ then $\gamma_\eta : \Gamma_e \to \Gamma_v$ injective

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- Take spaces with these fundamental groups, glued in pattern of Γ
  "Graph of Spaces" = B
- $\pi_1(\Gamma) \underset{\text{def}}{=} \pi_1(\text{Graph of Spaces})$

• Let $X = \widetilde{B}$, there is a $\pi_1(\Gamma)$-equivariant map
$\underbrace{X \to T}_{\text{tree of spaces}}$ – Bass Serre tree

<u>Theorem</u>: Let $G = \pi_1$ (graph of finite groups)
Let H be f.g. QI to G. Then, H is π_1(graph of finite groups)

One way to generalize:
- Keep finite edge groups
- Allow infinite vertex groups

<u>Theorem</u>: (Stallings, ...)
Let G be π_1(graph of groups with finite edge groups)
Let H = f.g. group QI to G.
Then, $H = \pi_1$(graph of gps w/ finite edge groups)

<u>Def</u>: Γ = graph of groups
Γ is <u>geometrically homogeneous</u> if all edge–vertex injections have finite index image.

<u>Examples</u>: Graphs of $\mathbb{Z}$'s
Graphs of $\mathbb{Z}^n$'s
Fix n, graphs of $PD(n)$ (Poincaré Duality groups)

Reason for terminology:
1. All vertex and edge groups are in the same QI class.
2. In $X \to T$, any vertex or edge space has ~~bounded~~ finite Hausdorff distance from any other.
("fibrations in the coarse category" in a sense)

Theorem (M, Sageev, Whyte)
If $G = \pi_1$(geom. hom graph of with ~~free~~ bushy Bass-Serre (important) coarse $PD(n)$'s)
and if H f.g. group QI to G, then
$H = \pi_1$(" ")
Moreover, vertex and edge groups of H are QI to those of G.

Bass-Serre Trichotomey (T Bass-Serre tree)
- T bounded
- T "line-like" (QI to line)
- T is "bushy" (∞ many ends)

Fix $\mathcal{C}$ = QI class of coarse $PD(n)$ groups
(e.g. [redacted] 2-ended gps, gps QI to $\mathbb{H}^{400}$, etc)

$\Gamma(\mathcal{C}) = \{\pi_1$(graphs of groups in $\mathcal{C}$ w/ bushy Bass-Serre tree)$\}$
Then, $\Gamma(\mathcal{C})$ is closed under quasi-isometry.

$n=0$ graphs of finite gps closed under QI
and $\Gamma(\mathcal{C})$ forms a single QI class

Theorem Every group in $\Gamma(\mathbb{Z})$ is QI to exactly one of the following:
1. $BS(1,n) = \langle a,t \mid tat^{-1} = a^n \rangle$, n is power free (amenable) (Farb-M)
2. $F_2 \times \mathbb{Z}$
3. $BS(2,3) = \langle a,t \mid ta^2t^{-1} = a^3 \rangle$

(2), (3): non-amenable, Whyte

(most fall into 3)

Theorem (Farb-M)
- Amenable graphs of $\mathbb{Z}^n$'s are classified
- Word-hyperbolic [higher genus surface $\rtimes$ free (rank ≥ 2) groups are classified]

<u>Proof</u> $G = \pi_1(\Gamma)$, Γ graph of course PD(n)'s w/ bushy BS tree
$X \to T$ Bass Serre tree of spaces
T is <u>bushy</u> and has bounded valence
$\Rightarrow$ Ends(T) is a Cantorset

Step 1: Let H be QI to G,
H QI to X

$$\begin{tikzcd} H \arrow[r, "f"] & X \arrow[l, bend left, "\text{coarse inverse}"] \end{tikzcd}$$

Conjugating left action of H on itself get
① <u>Quasi-action</u>

$$H \times X \to X$$
$$(h, x) \mapsto h \cdot x$$

- $\exists K, C$ s.t. $\forall h \in H$, the map
  $x \mapsto h \cdot x$ is a K, C QI.

- $\exists C$ s.t. $\forall h_1, h_2, x$ $d(h_1 \cdot (h_2 \cdot x), (h_1 \cdot h_2) \cdot x) \leq C$

② which is <u>Cobounded</u>
$\exists C$ s.t. $\forall x, y$ $\exists h$ s.t.
$d(hx, y) \leq C$

③ and <u>Proper</u> $\forall R$, $\exists M$ s.t $\forall x, y$
$\{h \mid h \cdot B(x, R) \cap B(y, R) \neq \emptyset\} \leq M$

<u>Step 2</u> : <u>Theorem (Farb - M)</u>
The Quasi-action $H \times X \to X$ coarsely respects vertex and edge spaces
$\Rightarrow$ descends to a quasi-action $H \times T \to T$

<u>Theorem</u> If H is a f.g. group, T a bushy tree of bounded valence, $H \times T \to T$ a cobounded quasiaction, then this is quasiconjugate to an action $H \times T' \to T'$