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NOTETAKER CHECKLIST FORM

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Name: Elizabeth Gross **Email/Phone:** egross@uic.edu

Speaker's Name: Jürgen Herzog

Talk Title: On the stable set of associated prime ideals of a monomial ideal.

Date: 12 / 3 / 2012 **Time:** 9 : 30 (am) / pm (circle one)

List 6-12 key words for the talk: associated prime ideals, index of stability, monomial ideals, polymatroidal ideals, primary decomposition, analytic spread

Please summarize the lecture in 5 or fewer sentances:

Introduces & defines the stable set of associated prime ideals, $\operatorname{Ass}^\infty(I)$, & the index of stability. Shows that for any set of non-zero monomial prime ideals there exists a monomial ideal I such that $\mathcal{P} = \operatorname{Ass}^\infty(I)$. Describes $\operatorname{Ass}^\infty(I)$ for polymatroidal ideals & shows the index of stability for these ideals is bounded above by the analytic spread.

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On the stable set of associated prime ideals of a monomial ideal

Jürgen Herzog
Universität Duisburg-Essen

Combinatorial Commutative Algebra and Applications
MSRI, Dec 3 – 7, 2012

Outline

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Algebraic background and history

The story begins with a question of Ratliff who in the 70'th asked:

What happens to $\mathrm{Ass}(R/I^n)$ as n gets large?

It is a general phenomenon that algebraic and homological properties of I^n stabilize for large n.

(Ratliff, 76,84) Let $\overline{J}$ denote the integral closure of an ideal J. Then

- $\mathrm{Ass}(\overline{I^n}) \subset \mathrm{Ass}(\overline{I^{n+1}})$ for all n. (Persistence)
- $\mathrm{Ass}(\overline{I^n})$ stabilizes, i.e., there exists an integer k_0 such that

$$\mathrm{Ass}(\overline{I^{k_0+\ell}}) = \mathrm{Ass}(\overline{I^{k_0}}) \text{ for all } \ell \geq 0.$$

We set $$\overline{\mathrm{Ass}^{\infty}}(I) = \mathrm{Ass}(\overline{I^{k_0}}).$$

(Brodmann, 79) $\mathrm{Ass}(I^n)$ stabilizes.

The smallest integer for which $\mathrm{Ass}(I^n)$ stabilizes is called the index of stability of I. We denote this number by

$$\mathrm{astab}(I),$$

and set

$$\mathrm{Ass}^\infty(I) = \mathrm{Ass}(I^n) \quad \text{where } n \geq \mathrm{astab}(I).$$

(McAdam, 83)

$$\overline{\mathrm{Ass}^\infty}(I) = \{P |\ I \subset P,\ \mathrm{height}\, P = \ell(I_P)\} \subset \mathrm{Ass}^\infty(I).$$

Here, if $J \subset (R, \mathfrak{m})$, then $\ell(J)$ denotes the analytic spread of J, i.e., $\dim \mathcal{R}(J)/\mathfrak{m}\mathcal{R}(J)$.

What can be said about $\mathrm{Ass}^\infty(I)$ and $\mathrm{astab}(I)$ when I is a monomial ideal?

The following simple remarks are useful:

(1) Associated prime ideals of a monomial ideal are monomial prime ideals, i.e., generated by variables.

(2) Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal, P a monomial prime ideal.
We set $S(P) = K[\{x_j \mid x_j \in P\}]$ and let $I(P) \subset S(P)$ be the monomial ideal which is obtained from I by the substitution

$$x_j \mapsto 1 \text{ for } x_j \notin P.$$

$I(P)$ is called the monomial localization of I with respect to P.

One has $I^k(P) = I(P)^k$ for all k, and
$P \in \mathrm{Ass}(I) \iff \mathfrak{m}_P \in \mathrm{Ass}(I(P))$, where $\mathfrak{m}_P$ is the graded maximal ideal of $S(P)$.

In this lecture we want to address the following questions:

(1) Given any set $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$ of non-zero monomial prime ideals. Does there exist a monomial ideal such that

$$\operatorname{Ass}^{\infty}(I) = \mathcal{P}?$$

(2) Does there exists a global upper bound for $\operatorname{astab}(I)$, not depending on I but only on S?

Problem (1) is completely open for squarefree monomial ideals. In that case the minimal prime ideals in $\mathcal{P}$ determine already the monomial ideal.

For example, let $\mathcal{P} = \{(x_1), (x_2), (x_1, x_2)\}$. Suppose there exists a squarefree monomial ideal with $\operatorname{Ass}^\infty(I) = \mathcal{P}$. Then $I = (x_1) \cap (x_2) = (x_1x_2)$, and so $(x_1, x_2) \notin \operatorname{Ass}(I^k)$ for all k. Thus $\mathcal{P} = \{(x_1), (x_2), (x_1, x_2)\}$ is not $\operatorname{Ass}^\infty(I)$ for a squarefree monomial ideal.

(–, Bandari, Rinaldo, 2011) For any $\mathcal{P}$ there exists a monomial ideal with $\operatorname{Ass}^\infty(I) = \mathcal{P}$.

Construction: Let $\mathcal{P} = \{P_1, \ldots, P_r\}$, $|G(P_i)| \leq |G(P_j)|$ for $i < j$.

For $s = 1, \ldots, r$ we choose an integer k_s which is bigger than the minimal degree of

$$J_{s-1} := P_1^{k_1} \cap P_2^{k_2} \cap \cdots \cap P_{s-1}^{k_{s-1}}.$$

Then for any integer $t \geq 1$, tk_s is bigger than the minimal degree of $P_1^{tk_1} \cap P_2^{tk_2} \cap \cdots \cap P_{s-1}^{tk_{s-1}}$, since $J_{s-1}^t \subset P_1^{tk_1} \cap P_2^{tk_2} \cap \cdots \cap P_{s-1}^{tk_{s-1}}$.

It follows that

$$\mathrm{Ass}(P_1^{tk_1} \cap P_2^{tk_2} \cap \cdots \cap P_r^{tk_r}) = \mathcal{P}$$

for all t.

There exists (!) an integer d such that

$$(P_1^{dk_1} \cap P_2^{dk_2} \cap \cdots \cap P_r^{dk_r})^c = P_1^{cdk_1} \cap P_2^{cdk_2} \cap \cdots \cap P_r^{cdk_r}$$

for all $c \geq 1$. This is a consequence of the fact, that the symbolic power algebra of a monomial ideal id finitely generated, as shown by Lyubeznik.

Therefore,

$$\mathrm{Ass}^\infty(P_1^{dk_1} \cap P_2^{dk_2} \cap \cdots \cap P_r^{tdk_r}) = \mathcal{P}.$$

The following yields an algorithm to compute $\mathrm{Ass}^\infty(I)$ for a monomial ideal: There are only finitely many monomial prime ideals. We test all. Let $P = (x_{i_1}, \ldots, x_{i_k})$. Then

$$P \in \mathrm{Ass}^\infty(I) \iff \text{Krulldim } H_{k-1}(x_{i_1}, \ldots, x_{i_k}; \mathcal{R}(I(P))) > 0.$$

An implementation can be found under

http://ww2.unime.it/algebra/rinaldo/stableset/

Examples (–, Vladoiu, Rauf, 2011)

(1) Let $I = P_{F_1} P_{F_2} \cdots P_{F_r}$ where $P_{F_i} = (x_j : j \in F_i)$.

The presentation of I as a product of monomial prime ideals is unique.

(–, Conca, 2002) $I = \bigcap_A P_A^{|A|}$, where $P_A = \sum_{i \in A} P_{F_i}$.

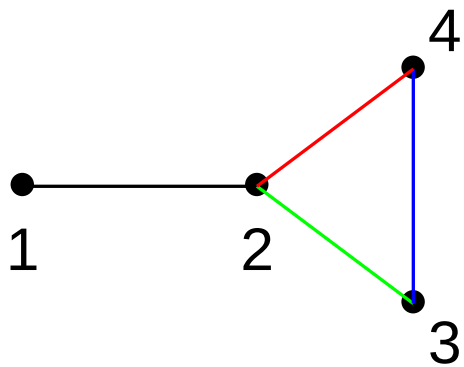
We define the intersection graph G_I on $[r]$ for which $\{i, j\}$ is an edge of G_I if and only if $F_i \cap F_j \neq \emptyset$. Then

$$\mathrm{Ass}^\infty(I) = \mathrm{Ass}(I^k) \quad \text{for all } k$$

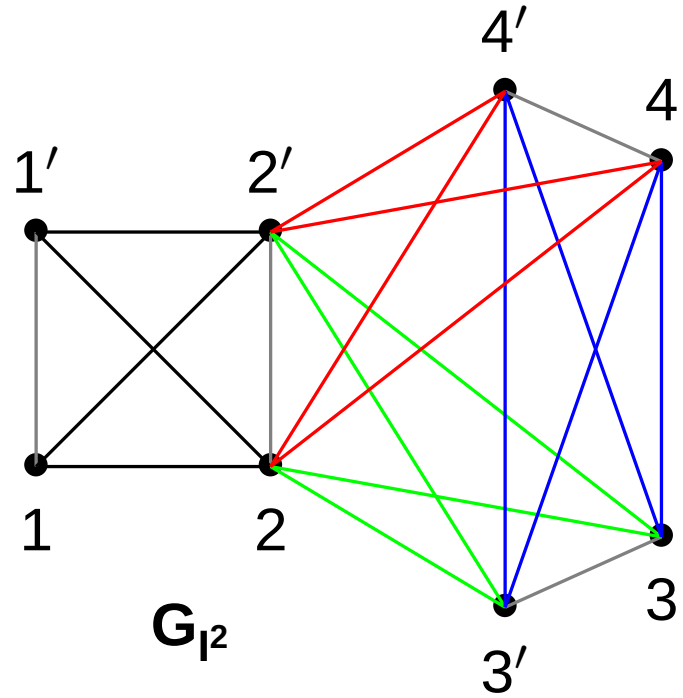
and

$$\mathrm{Ass}(I) = \{P_A \mid A = V(T), \text{ where } T \subset G_I \text{ is a tree}\}.$$

For example, consider $I = (x_1, x_2)(x_1, x_2, x_3, x_4)(x_3, x_5)(x_4, x_5)$. Notice that for any ideal I of this type the graph G_{I^k} is just the k-th expansion of G_I.



G_I



G_{I^2}

The trees of G_I have one, two, three, or four vertices. The one–vertex trees, that is, the vertices, correspond to the associated primes $P_{F_1}, \ldots, P_{F_4}$.

The two–vertex trees correspond to the associated primes

$$P_{F_1} + P_{F_2}, P_{F_2} + P_{F_3}, P_{F_2} + P_{F_4}, P_{F_3} + P_{F_4}.$$

All trees with three and four vertices generate the maximal ideal. Consequently we obtain that

$$\operatorname{Ass}(I) = \{(x_1, x_2), (x_1, x_2, x_3, x_4), (x_3, x_5), (x_4, x_5), (x_3, x_4, x_5), (x_1, x_2, x_3, x_4, x_5)\}.$$

(2) Let $I_{d;a_1,\ldots,a_n}$ be the ideal generated by all monomials

$$x_1^{c_1}x_2^{c_2}\cdots x_n^{c_n}$$

of degree d with $c_i \leq a_i$. Then

$$\mathrm{Ass}^{\infty}(I_{d;a_1,\ldots,a_n}) = \{P \ : \ I_{d;a_1,\ldots,a_n} \subset P\}.$$

In particular, if $I = I_{d;1,\ldots,1}$, then

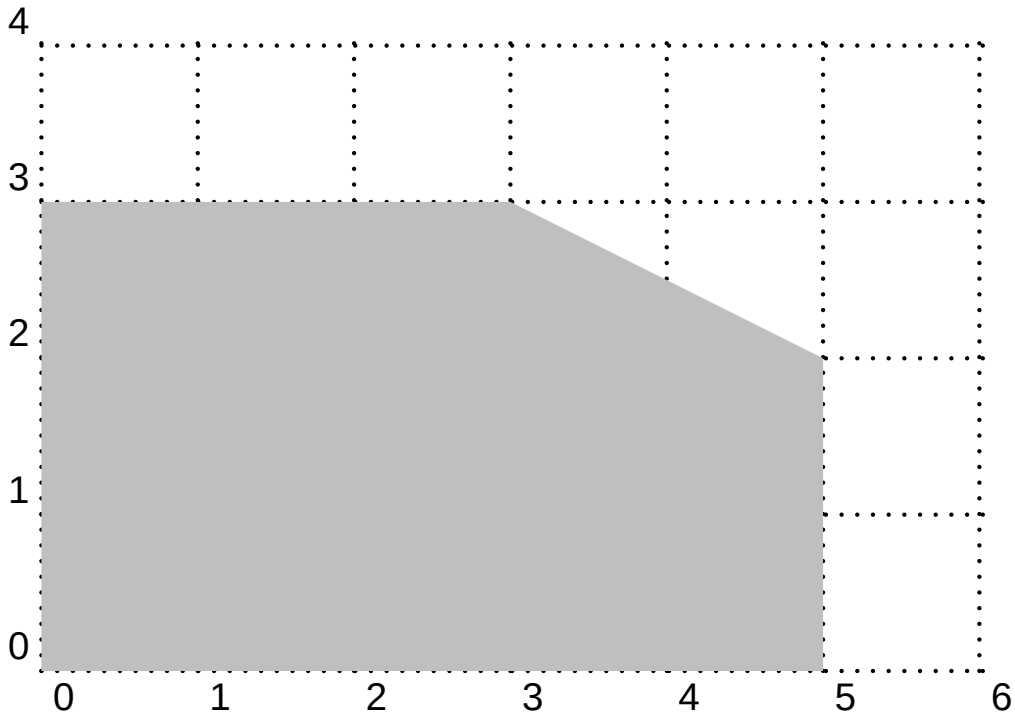
$$\mathrm{Ass}(I) = \{P_F : \ F \subset [n], \ |F| = n-d+1\},$$

while

$$\mathrm{Ass}^{\infty}(I) = \{P_F : \ F \subset [n], \ |F| \geq n-d+1\}.$$

Both examples before are examples of polymatroidal ideals.

A polymatroid is a (special) convex polytope $\mathcal{P}$ in $\mathbb{R}^n$. It is called discrete, if all vertices of $\mathcal{P}$ are integer vectors.



A polymatroidal ideal is a monomial ideal whose exponent vectors correspond to the bases of a polymatroid.

Nice properties of polymatroidal ideals:

(1) A monomial ideal I is polymatroidal, if it satisfies the following exchange property: for any

$$u, v \in G(I) \text{ with } \nu_i(u) > \nu_i(v),$$

there exists j such that

$$\nu_j(u) < \nu_i(v) \text{ and } x_j(u/x_i) \in G(I).$$

(2) Polymatroidal ideals have linear resolutions.

(3) If I and J are polymatroidal, then IJ is polymatroidal. In particular, I^k is polymatroidal for all k.

(4) Monomial localizations of polymatroidal ideals, are polymatroidal.

(5) $\mathcal{R}(I)$ is normal for any polymatroidal ideal.

What can we say about astab(I)?

McAdam in his Lecture Notes "Asymptotic prime divisors" quotes an example of Sathaye:

$$I \subset K[x, z_1, \ldots, z_{2n}]/(xz_{2i-1}^{2i-1} - z_{2i}^{2i}, z_j^j z_i),$$

$$I = (z_1, \ldots, z_{2n}) \subset P = (x, z_1, \ldots, z_{2n}).$$

Then for $1 \leq k \leq 2n$,

$$P \in \mathrm{Ass}(I^k) \quad \Longleftrightarrow \quad k \text{ is even}$$

Is such a behaviour also possible in a regular ring?

(Bandari,–, Hibi, 2012) Let $n \geq 0$ be an integer and

$$I \subset S = K[a, b, c, d, x_1, y_1, \ldots, x_n, y_n]$$

be the monomial ideal in the polynomial ring S with generators

$$a^6, a^5b, ab^5, b^6, a^4b^4c, a^4b^4d, a^4x_1y_1^2, b^4x_1^2y_1, \ldots, a^4x_ny_n^2, b^4x_n^2y_n.$$

Then

$$\operatorname{depth}(S/I^k) = \begin{cases} 0, & \text{if } k \text{ is odd and } k \leq 2n+1; \\ 1, & \text{if } k \text{ is even and } k \leq 2n; \\ 2, & \text{if } k > 2n+1. \end{cases}$$

In particular, the depth function of this ideal has precisely n strict local maxima.

In both examples one needs sufficiently many variables to produce such examples.

This is not the case for the regularity of powers of ideals. Conca constructed a family of monomial ideal in 4 variables whose regularities stabilize for arbitrarily high powers.

On the other hand, in all known cases $\operatorname{astab}(I) < n$ for all monomial ideals $I \subset K[x_1, \ldots, x_n]$.

(Martinez-Bernal, Morey, Villarreal, 2005) If G is a finite graph, and $I(G)$ its edge ideal. Then $\operatorname{astab}(I(G)) = 1$, if G is bipartite, and

$$\operatorname{astab}(I(G)) \leq n - k - s \quad \text{if } G \text{ is not bipartite,}$$

where the smallest odd cycle of G has length $2k + 1$ and where s is the number of leaves of G.

Actually we expect the following inequality:

$$\operatorname{astab}(I) < \ell(I) \ (\leq n).$$

(–, Qureshi, 2012) The inequality is true for any polymatroidal ideal.

Let I be a monomial ideal with $G(I) = \{u_1, \ldots, u_m\}$.

The linear relation graph $\Gamma = \Gamma(I)$ of I is the graph with egdes (i,j) for which there exist $u_k, u_\ell \in G(I)$ such that $x_i u_k = x_j u_\ell$.

(–, Qureshi, 2012) If Γ has r vertices and s connected components, then

(a) $\operatorname{depth} S/I^t \leq n - t - 1$ for $t = 1, \ldots, r - s$.

(b) $\ell(I) \geq r - s + 1$. Equality holds if I is polymatroidal.

Proof of the fact that $\operatorname{astab}(I) < \ell(I)$ for polymatroidal ideals:

(a) and (b) imply that

$$\operatorname{depth} S(P)/I(P)^{\ell(I(P))-1} \leq \dim S(P) - \ell(I(P))$$

for any polymatroidal ideal I and any monomial prime ideal P with $I \subset P$.

(Eisenbud-Huneke, 83) If $\mathcal{R}(J)$ is Cohen–Macaulay, then

$$\min_t \{depth S/J^t\} = n - \ell(J).$$

Since $\mathcal{R}(I(P))$ is a normal toric ring it follows by a theorem of Hochster that $\mathcal{R}(I(P))$ is Cohen–Macaulay. Hence

$$\operatorname{depth} S(P)/I(P)^{\ell(I(P))-1} = \dim S(P) - \ell(I(P)).$$

Now let $P \in \operatorname{Ass}^{\infty}(I)$, then $\ell(I(P)) = \dim S(P)$ by McAdam.

Therefore,

$$\operatorname{depth} S(P)/I(P)^{\ell(I(P))-1} = 0,$$

and hence

$$P \in \operatorname{Ass}(I^{\ell(I(P))-1}).$$

By Ratliff,

$$P \in \operatorname{Ass}(I^k) \quad \text{for all} \quad k \geq \ell(I) - 1.$$

This shows that

$$\operatorname{astab}(I) < \ell(I).$$

- S. Bandari, J. Herzog, T. Hibi, Monomial ideals whose depth function has any given number of strict local maxima, Preprint 2011, arXiv:1205.1348v1 [math.AC], to appear in Arkiv för matematik.
- M. Brodmann, Asymptotic stability of $\mathrm{Ass}(M/I^nM)$, Proc. AMS **74**, 16–18 (1979)
- D. Eisenbud, C. Huneke, Cohen–Macaulay Rees algebras and their specialization, J. Algebra **81**, 202–224 (1983)
- J. Herzog, T. Hibi, Discrete polymatroids, J. Algebraic Combinatorics **16**, 239–268 (2002)
- J. Herzog, A. Rauf, M. Vladoiu, The stable set of associated prime ideals of a polymatroidal ideal, J. Algebraic Combinatorics, DOI 10.1007/s10801-012-0367-z.

- J. Herzog, A. Qureshi, Persistence and stability properties of powers of ideals. Preprint 2012, arXiv:1208.4684v2 [math.AC]
- C. Huneke, On the associated graded ring of an ideal, Ill. J. Math. **26**, 121–137, (1982)
- G. Lyubeznik, On the Arithmetical Rank of Monomial ideals. J. Algebra **112**, 86–89 (1988)
- J. Martinez-Bernal, S. Morey, R. H. Villarreal, Associated primes of powers of edge ideals, Preprint 2011, arXiv:1103.0992v3 [math.AC], to appear in Collectanea Mathematics.

- M. Hochster, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes, Ann. of Math. **96**, 228–235 (1972)
- S. McAdam, *Asymptotic Prime Divisors*, Lecture Notes in Mathematics 1023, SpringerVerlag, New York, 1983.
- S. McAdam, Asymptotic prime divisors and analytic spreads, Proc Amer. Math. Soc, **80**, 555–559,(1980)
- L. J. Ratliff Jr. On prime divisors of I^n, n large, Michigan Math. J., **23**, 337–352, (1976)
- L. J. Ratliff Jr. On asymptotic prime divisors, Pacific J. Math., **111**, 396–413, (1984)