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# NOTETAKER CHECKLIST FORM

**(Complete one for each talk.)**

Name: Elizabeth Gross Email/Phone: egross7@uic.edu

Speaker's Name: Claudia Polini

Talk Title: Hilbert coefficients, generalized Hilbert functions, & associated graded rings.

Date: 12 / 3 / 12 Time: 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: Hilbert coefficients, generalized Hilbert function, j-multiplicity, depth of graded module, normalization

Please summarize the lecture in 5 or fewer sentances: Discusses relationship between Hilbert coefficients, the depth of associated graded rings, & the normalization of ideals.

## CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
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  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

MSRI 12/3/12

$(R, m)$ Noetherian local ring $d = \dim R > 0$ $|R/m| = \infty$

$I \subset m$ R-ideal

$$I \rightsquigarrow G = gr_I(R) = \oplus I^n/I^{n+1}$$

- $\sqrt{I} = m$ $\quad n \mapsto \lambda\left(I^n/I^{n+1}\right)$

- any $I$ $\quad n \mapsto \lambda\left(\left[H^0_m(G)\right]_n\right) = \lambda\left(H^0_m\left(I^n/I^{n+1}\right)\right)$

$m^t H^0_m(G) = 0$ for some $t$ $\Rightarrow$ $H^0_m(G)$ finite graded module over $G \otimes R/m^t$ of dimension $\leq d$

$$\Rightarrow \quad HF_I(n) \underset{n \gg 0}{=} \sum_0^{d-1} (-1)^i j_i(I) \binom{n+d-i-1}{d-i-1}$$

$j_i(I) :=$ generalized Hilbert coefficients of $I$

- $\sqrt{I} = m$ : $j_i(I) = e_i(I)$

$j_0(I) = j(I)$ $\quad$ j-multiplicity of $I$ (Achilles - Manaresi)

Applications
- criteria for integral dependence
- depth of associated graded rings
- effective bounds for normalization

Properties
- bounds for $j_i$
- how to compute them?
- behavior under hyperplane sections

## §1 Multiplicity and depth of G

$k = \bar{k}$ $\quad R = k[x_1, \ldots, x_n]/\mathfrak{a}$ $\quad \mathfrak{a} \subset (x_1, \ldots, x_n)^2$ hom. prime $\quad ht\,\mathfrak{a} = h = n - d$

$\Rightarrow \deg \mathrm{Proj}(R) = e(R) = e_0(m_R) \geq h+1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \quad \Rightarrow R$ CM

• local $R$ CM $\quad G = gr_m(R)$

$h = \mathrm{ecodim}(R) = \mu(m) - d$

(Abhyankar) $\quad e(R) \geq h+1$

(Sally) $\qquad\qquad = \quad \Rightarrow G$ CM $\Rightarrow HS(z) = \dfrac{1+hz}{(1-z)^d}$

<u>Thm</u> (Sally, Rossi-Valla, Wang)

$$\left.\begin{array}{l} e \leq h+2 \\ e = h+3, \ \mathrm{type}(R) < h \end{array}\right\} \Rightarrow \mathrm{depth}\, G \geq d-1 \Rightarrow HS_{\checkmark}$$

<u>OPEN</u> $\quad e \leq h+3 \Rightarrow \mathrm{depth}\, G \geq d-2$

Similar results hold for $gr_I(R)$ where $\sqrt{I} = m$
(Rossi-Valla, Corso-P-Vaz Pinto, Elias, –)

• $\sqrt{I} = m$, $a_1, \ldots, a_d \subset I$ general : $e_0(I) = \lambda\left(\dfrac{R}{(a_1, \ldots, a_d)}\right) \Leftrightarrow R$ CM

• any $I$ : $\quad \bar{R} = \dfrac{R}{(a_1, \ldots, a_{d-1}) : I^\infty}$ $\quad \begin{cases} 0 \\ \text{1-dim CM and } a_d \text{ NZD on } \bar{R} \end{cases}$

$\Rightarrow j(I) = \lambda\left(\dfrac{\bar{R}}{a_d \bar{R}}\right)$ $\qquad$ [AM, Nishida-Ulrich, Xie]

$\qquad = \lambda\left(\bar{I}/\bar{I}^2\right) + \lambda\left(\bar{I}^2/a_d\bar{I}\right)$

$j(I)$ minimal if $j(I) = \lambda(\overline{I}/\overline{I^2}) \iff \overline{I^2} = a_d \overline{I}$

Thm [P-Xie]

- $I\ G_d$ : $\mu(I_p) \le \dim R_p \quad \forall p \in V(I) \setminus \{m\}$

(*) - $I$ weakly $(d-2)$-residually CM

- depth $R/I \geq \min\{1, \dim R/I\}$

(a) $j(I)$ minimal $\Rightarrow$ $G$ CM

(b) $j(I)$ almost minimal $\Rightarrow$ depth $G \geq d-1$

(*) holds if : $\cdot \dim R/I \le 1$

$\cdot$ $I$ licci

Ex: $R = k[[x,y,z,v,w]]$ $\quad I = I_2 \begin{pmatrix} x & y & z & v \\ y & z & w & w \end{pmatrix}$

$HS_I(z) = \dfrac{z+z^2+z^3+z^4}{(1-z)^5}$ $\quad$ $HS_I(z) = \dfrac{1+3z}{(1-z)^5}$

Mantero - Xie

## §2 Properties of generalized Hilbert polynomials

$$P_I(n) = \sum_0^{d-1} (-1)^i j_i(I) \binom{n+d-i-1}{d-i-1}$$

Thm [PX, Colome-P-Ulrich-Xie]

$\left.\begin{array}{l} a_1,..,a_s \text{ sequentially general in } I \\ (a_1,..,a_s) \subset K \subset (a_1,..,a_s) : I^\infty \end{array}\right\} \Rightarrow \quad P_{IR/K} = \Delta^s P_I$

- $\sqrt{I} = m \quad a$ general in $I$

$HF_{I/aR} = \Delta HF_I \iff \text{grade } G_+ > 0$

**Thm** [CPUX] grade $G_+ > 0$ : $HF_{I/aR} = \Delta HF_I \iff \operatorname{depth} \overset{G'}{\overset{\shortparallel}{G/H^0_m(G)}} \geq 2$

**Pf:** $a^* = a + I^2 \in G_+$

$$0 \to G(-1) \xrightarrow{a^*} G \to G(I/aR) \to 0$$

$$0 \to H^0_m(G)(-1) \xrightarrow{a^*} H^0_m(G) \to H^0_m(G(I/aR)) \to H^1_m(G)(-1) \xrightarrow{a^*} H^1_m$$

$\therefore$ $HF_{I/aR} = \Delta HF_I \iff a^*$ NZD on $H^1_m(G) \iff H^0_{G+}(H^1_m(G)) = 0$

$\iff H^0_{G+}(H^1_m(G')) = 0 \iff H^1_M(G') = 0 \iff \operatorname{depth} G' \geq 2$

Let $M = mG + G_+ \Rightarrow H^0_M = H^0_{G+} \circ H^0_m$

$$H^1_{G+}(\cancel{H^0_m(G')}) \longrightarrow H^1_M(G') \xrightarrow{\sim} H^0_{G+}(H^1_m(G')) \to H^2_{G+}(\cancel{H^0_m(G')})$$

#

• $\sqrt{I} = m$ : $e_0 > 0$

$j(I) > 0 \iff \dim H^0_m(G) = d \iff \underbrace{\dim G/mG}_{\ell(I) \text{ analytic spread of } I} = d$

• $\sqrt{I} = m$ $R$ CM : $e_1 \geq 0$

$= 0 \iff I$ c.i.

**COR** [CPUX] $I$ $G_d$, weakly $(d-2)$-residually $S_2$

$\ell = d$, excellent $\quad j_1(I) \geq 0$

$= 0 \iff \sqrt{I} = m$, $I$ c.i.

Ex $R = k[[x,y,z]]/(x^3 - x^2 y)$ $I = (xy^t, z)$ $t \geq 0$

$I$ ~~$G_2$~~ and $j_1(I) = 2 - t$

Recall: $e_0 > 0$, $e_1 \geq 0$, $e_2 \geq 0$ $e_3$ can be negative

## §3 Normalized Hilbert coefficients

$\hat{R}$ reduced, CM

$\overline{I} = \{x \in R \,/\, x^k + a_1 x^{k-1} + \dots + a_k = 0 \quad a_i \in I^i\}$

$\mathbb{F} = \{\overline{I^n}\}_{n \geq 0}$ $G_{\mathbb{F}} = gr_{\mathbb{F}} = \oplus \overline{I^n}/\overline{I^{n+1}}$ finite over $G$

$\cdot \sqrt{I} = m$ $\lambda\left(\overline{I^n}/\overline{I^{n+1}}\right) \underset{n \gg 0}{=} \sum_0^{d-1} (-1)^i \overline{e}_i(I) \binom{n+d-i-1}{d-i-1}$

Thm (Itoh) (WLG $I$ c.i) $d \geq 3$

$\overline{e}_3 \geq 0$

$\overline{I} = m$, $R$ GOR: $= 0 \iff \overline{I^{n+2}} = I^n \overline{I^2} \quad \forall n$ $(\Rightarrow G_{\mathbb{F}}$ CM$)$

OPEN $\overline{e}_i \geq 0$

$\overline{e}_3 = 0 \iff \overline{I^{n+2}} = I^n \overline{I^2}$

Thm [Corso - P. Rossi]

type$(R) \leq 2$, $\overline{I} = m$: $\overline{e}_3 = 0 \iff \overline{I^{n+2}} = I^n \overline{I^2}$ $(\Rightarrow G_{\mathbb{F}}$ CM, $G$ CM for all $I \subseteq \overline{I})$

## §4 Bounds for normalization

$R$ regular

$$R[It] = \oplus I^n \subset \overline{R[It]} = \oplus \overline{I^n}$$

$A$ domain, essentially of finite type over $k$, $ch(k)=0$

compute $\overline{A}$

normal $\equiv S_2 + R_1$

$$A \subset A_0 = \underbrace{End(\omega_A)}_{S_2} \subset A_1 \subset \dots \subset A_t = \overline{A} \quad [\text{Vasconcelos}]$$

$A_{i+1} = End(J(A_i)^{-1})$ $S_2$

$A_i = A_{i+1} \Rightarrow J(A_i)$ is principal in codim 1 $\overset{\text{Lipman}}{\Rightarrow}$ $A_i$ is $R_1$

[CPUX] **Thm** $I_p$ normal $\forall p \in V(I) \setminus \{m\}$

for instance if $\overline{R}_I$ has isolated sing?

$\Rightarrow$ length of any chain of graded $S_2$-algebras between $R[It]$ and $\overline{R[It]}$ is

$$\leq \overline{j}_1(I) - j_1(I)$$

**Thm** [P-Ulrich-Vasconcelos]

$$\overline{e}_1 \leq (d-1) \min \{ \frac{e_0}{2}, e_0 - \lambda(R/I) \}$$

**COR** $\overline{R[It]} = End_{R[It]}(\omega_{R[It]}) \Leftrightarrow j_1(I_p) = \overline{j}_1(I_p)$ $\forall p \in V(I)$

Ciupercă