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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Elizabeth Gross Email/Phone: egross7@uic.edu

Speaker's Name: June Huh

Talk Title: Positivity of algebraic cycles & complexity of combinatorial geometries

Date: 12 / 3 / 12 Time: 2 : 00 am / pm (circle one)

List 6-12 key words for the talk: representable matroids, characteristic polynomial of a matroid, ~~representa~~ representable algebraic cycles, Milnor number, Rota's conjectura, combinatorial geometry

Please summarize the lecture in 5 or fewer sentances: Gives a positive answer to Rota's conjecture for representable matroids. Discusses the connection between Rota's conjecture & representable algebraic cycles. Introduces the permutohedral variety & relation to matroid theory.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - <u>**Computer Presentations**</u>: Obtain a copy of their presentation
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Positivity of algebraic cycles and convexity of combinatorial geometries

June Huh

University of Michigan at Ann Arbor

MSRI, December 3, 2012

This slide can be found at:

http://www-personal.umich.edu/∼junehuh/

1. Combinatorial geometries

Hassler Whitney (1935).

A *matroid* on a finite set E is a collection of subsets of E, called independent sets, with the following properties:

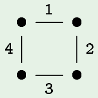
- Every subset of an independent set is independent.
- If an independent set A has more elements than independent set B, then there is an element in A, when added to B, gives a larger independent set.

Let G be a finite graph, and E the set of edges.

Call a subset of E independent if it does not contain a circuit.

This defines a *graphic matroid* M .

Example



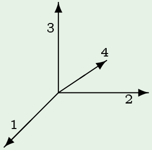
Independent sets of M: All subsets of $E = \{1, 2, 3, 4\}$ except $\{1, 2, 3, 4\}$.

Let V be a vector space over a field k, and E a finite set of vectors.

Call a subset of E independent if it is linearly independent.

This defines a matroid M *representable* over k.

Example



Independent sets of M: All subsets of $E = \{1, 2, 3, 4\}$ except $\{1, 2, 3, 4\}$.

Graphic matroids are representable over every field.

Without loss of generality, we may assume that every set

with ≤ 2 elements are independent.

These matroids are called *combinatorial geometry*.

In the representable case, a combinatorial geometry is defined from

- a finite set of points in $\mathbb{P}(V)$, or equivalently,
- a finite set of hyperplanes in $\mathbb{P}(V^\vee)$.

2. Concavity

- A sequnce $a_0, \ldots, a_r$ is *log-concave* if for all i

$$a_{i-1}\, a_{i+1} \leq a_i^2.$$

- If there are *no internal zeroes*, log-concavity implies *unimodality*:

$$a_0 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \cdots \geq a_r \quad \text{for some } i.$$

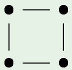
George Birkhoff (1912).

- The chromatic polynomial of G is the function

$$\chi_G(q) = (\text{number of proper colorings of } G \text{ with } q \text{ colors}).$$

- $\chi_G(q)$ is a polynomial in q with integer coefficients.
- $\chi_G(4) > 0$ for planar G?

Example



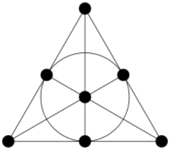
$$\chi_G(q) \quad = \quad 1q^4 - 4q^3 + 6q^2 - 3q$$

Conjecture (Read and Hoggar 1968)

The coefficients of $\chi_G(q)$ form a log-concave sequence for any graph G.

- Let E be a finite set of points in $\mathbb{P}(V)$.
- Define the f-vector by

$$f_i = (\text{number of independent subsets of } E \text{ with size } i).$$



Example (Fano plane)

For $E = \mathbb{F}_2^3 \setminus \{\mathbf{0}\}$, we have

$$f_0 = 1, \quad f_1 = 7, \quad f_2 = 21, \quad f_3 = 28.$$

Conjecture (Welsh and Mason 1969)

The sequence $f_0, f_1, \ldots, f_r$ is log-concave for any finite subset of $\mathbb{P}(V)$.

- The analogous conjectures on the f-vector of simplicial polytopes, another representative class of shellable simplicial complexes, were disproved by Björner (1981).

- The conjecture on $\chi_G(q)$ was computer verified for all graphs with ≤ 13 vertices by Lundow and Markström (2006).

Björner's example

Examples 8.40. The unimodality conjecture fails for a simplicial polytope of dimension $d = 20$ with the following f-vector, for which $f_{11} > f_{12} < f_{13}$.

$$
\begin{array}{lcrl}
f_{-1} & = & 1 & \\
f_0 & = & 4203045807626 & \\
f_1 & = & 84060916163336 & \\
f_2 & = & 798578704207074 & \\
f_3 & = & 4791472253296106 & \\
f_4 & = & 20363758019368323 & \\
f_5 & = & 65164051780016980 & \\
f_6 & = & 162910744316489788 & \\
f_7 & = & 325834059588060117 & \\
f_8 & = & 529707205213463823 & \\
f_9 & = & 709935971390166248 & \\
f_{10} & = & 805494832051588614 & \\
f_{11} & = & 821976324224631043 & / \\
f_{12} & = & 821976324224611712 & \longleftarrow \\
f_{13} & = & 822000129478641948 & \backslash \\
f_{14} & = & 747383755288236256 & \\
f_{15} & = & 546761228419958342 & \\
f_{16} & = & 293715859557026466 & \\
f_{17} & = & 106920718330384544 & \\
f_{18} & = & 23458617733909980 & \\
f_{19} & = & 2345861773390998 &
\end{array}
$$

Results

Theorem (-)

The conjecture on the chromatic polynomial is true for all graphs.

Theorem (Lenz, H-Katz)

The conjecture on the f-vector is true for any set of vectors.

Rota (1964).

Rota noted the significance of the *characteristic polynomial* $\chi_M(q)$

of a combinatorial geometry M.

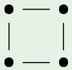
"*One of the most useful principles of enumeration in discrete probability and combinatorial theory is the celebrated principle of inclusion-exclusion.*"

Rota, The first line of *On the Foundations of Combinatorial Theory I*.

Conjecture (Rota 1970)

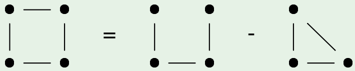
The coefficients of $\chi_M(q)$ form a log-concave sequence for any graph G.

Example



$$\chi_G(q) = 1q^4 - 4q^3 + 6q^2 - 3q$$

We may write



and compute

$$\begin{aligned} \chi_{G'}(q) &= q(q-1)^3 \\ \chi_{G''}(q) &= q(q-1)(q-2). \end{aligned}$$

Therefore

$$\chi_G(q) = \chi_{G'}(q) - \chi_{G''}(q) = 1q^4 - 4q^3 + 6q^2 - 3q.$$

- If M', and M'' are deletion and contraction of M respectively, then

$$\chi_M(q) = \chi_{M'}(q) - \chi_{M''}(q).$$

- In fact, the characteristic polynomial is the finest ‘additive’ invariant of combinatorial geometries.

- $\chi_M(q)$ is really the χ of M.

- $\chi_G(q) = \chi_M(q)$ if M is the matroid of G, and
- $f_M(q) = \chi_{M^\dagger}(q)$ if $M^\dagger$ is the coextension M (Brylawski '77, Lenz '12).
- There are many other interesting specializations of $\chi_M(q)$.

Theorem (-)

If M is representable over a field of characteristic zero, then the coefficients of $\chi_M(q)$ form a log-concave sequence.

- Our proof is algebro-geometric in nature and cannot be generalized to non-representable combinatorial geometries. However, all except one argument ‘tropicalize’, and this allows us to drop the characteristic zero assumption (H-Katz).

3. Milnor numbers of projective hypersurfaces

Milnor (1968).

If $f : (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}, \mathbf{0})$ is a germ of an isolated singularity at $\mathbf{0}$,

then the Milnor fiber of f is homotopic to the bouquet of spheres.

The number of spheres in the Milnor fiber is the degree of the gradient map

of f restricted to a small sphere.

- If h is a homogeneous polynomial, consider the complement

$$D(h) = \{h \neq 0\} \subset \mathbb{P}^r.$$

  The complement $D(h)$ is d-fold covered by the Milnor fiber $\{h = 1\}$ of

$$h : \mathbb{C}^{r+1} \to \mathbb{C}.$$

- In this projective setting, the theory of Milnor can be carried out without any assumption on h.

Let Γ be the graph of the gradient map $\mathbb{P}^r \dashrightarrow \mathbb{P}^r$

$$\begin{tikzcd} & \Gamma \subset \mathbb{P}^r \times \mathbb{P}^r \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^r \arrow[rr, dashed, "\left(\frac{\partial h}{\partial z_0} : \frac{\partial h}{\partial z_1} : \cdots : \frac{\partial h}{\partial z_r}\right)"'] & & \mathbb{P}^r \end{tikzcd}$$

Define $\mu^i(h)$ by

$$[\Gamma] = \sum_{i=0}^{r} \mu^i(h) \left[\mathbb{P}^{r-i} \times \mathbb{P}^i\right] \in H_{2r}\left(\mathbb{P}^r \times \mathbb{P}^r; \mathbb{Z}\right).$$

Theorem (-)

For any homogeneous polynomial $h \in \mathbb{C}[z_0, \ldots, z_n]$,

1. $\mu^i(h)$ *is the number of i-cells in a CW-model of $D(h)$.*

   More precisely, $D(h) \cap \mathbb{P}^i$ is obtained from $D(h) \cap \mathbb{P}^{i-1}$ by attaching $\mu^i(h)$ cells of dimension i.

2. *If h is a product of linear forms, then the attaching maps are trivial:*

$$\mu^i(h) = b_i\big(D(h)\big)$$

Writing $\mathcal{A}$ for the hyperplane arrangement defined by h,

$$\chi_{\mathcal{A}}(q) = \sum_{i=0}^{r} (-1)^i \mu^i(h) q^{r-i}.$$

In other words, $\mu = \mu$.

This provides new intuition on the coefficients of $\chi_M(q)$, because homology classes of subvarieties are very special.

4. Representable algebraic cycles

Let X be a complex algebraic variety. It is interesting to ask

"Which even dimensional homology classes of X come from a subvariety?"

(Hartshorne 1974)

Let's say that such homology classes are *representable* by a subvariety.

This is an extremely difficult problem, even for very simple X...

Example ($\mathbb{P}^n$)

Let
$$\xi = d[\mathbb{P}^k] \in H_{2k}(\mathbb{P}^n; \mathbb{Z}).$$

1. If k is 0 or n, then ξ is representable iff $d = 1$.
2. If otherwise, then ξ is representable iff $d \geq 1$.

Example ($\mathbb{P}^1 \times \mathbb{P}^1$)

Let

$$\xi = d_0[\mathbb{P}^1 \times \mathbb{P}^0] + d_1[\mathbb{P}^0 \times \mathbb{P}^1] \in H_2(\mathbb{P}^1 \times \mathbb{P}^1; \mathbb{Z}).$$

Then ξ is representable iff

1. d_0, d_1 are nonnegative, and
2. if one of the d_i is 0, the other d_i is 1.

Theorem (in preparation)

Let

$$\xi = d_0[\mathbb{P}^2 \times \mathbb{P}^0] + d_1[\mathbb{P}^1 \times \mathbb{P}^1] + d_2[\mathbb{P}^0 \times \mathbb{P}^2] \in H_4(\mathbb{P}^2 \times \mathbb{P}^2; \mathbb{Z}).$$

1. *If $d_1 = 0$, then ξ is representable iff (d_0, d_1, d_2) is $(1,0,0)$ or $(0,0,1)$.*
2. *If otherwise, then ξ is representable iff $d_0, d_1, d_2 \geq 0$ and $d_1^2 \geq d_0 d_2$.*

This is really the question on the shape of bigraded Hilbert polynomials

of standard bigraded domains.

I don't think this statement has a very short proof.

The current argument is

(delicate argument on representing a collection of integers as sums of squares)

$$+$$

(delicate argument on free divisors on a blowup of $\mathbb{P}^2$ at many points).

Theorem (-)

Let

$$\xi = \sum_i d_i[\mathbb{P}^{k-i} \times \mathbb{P}^i] \in H_{2k}(\mathbb{P}^m \times \mathbb{P}^n; \mathbb{Z}).$$

1. *If ξ is of the form*

$$d[\mathbb{P}^0 \times \mathbb{P}^0], d[\mathbb{P}^m \times \mathbb{P}^0], d[\mathbb{P}^0 \times \mathbb{P}^n], d[\mathbb{P}^m \times \mathbb{P}^n],$$

   then ξ is representable iff $d = 1$.

2. *If otherwise, some positive multiple of ξ is representable iff $\{d_i\}$ form a log-concave sequence of nonnegative integers with no internal zeros.*

In particular, the sequence $\mu^i(h)$ is log-concave for any h.

Surprisingly, the homology class in $H_{10}(\mathbb{P}^5 \times \mathbb{P}^5; \mathbb{Z})$

$$\xi = 1[\mathbb{P}^5 \times \mathbb{P}^0] + 2[\mathbb{P}^4 \times \mathbb{P}^1] + 3[\mathbb{P}^3 \times \mathbb{P}^2] + 4[\mathbb{P}^2 \times \mathbb{P}^3] + 2[\mathbb{P}^1 \times \mathbb{P}^4] + 1[\mathbb{P}^0 \times \mathbb{P}^5]$$

is not representable.

This can be deduced from Pirio and Russo's recent result (2012),

which asserts that there is a bijective correspondence between

rank 3 Jordan algebras and quadro-quadric Cremona transformations.

Algebra	Jordan adjoint $(x,y,z,t,u,v)^{\#}$	Type	Multidegree	$\dim(R)$
$\mathcal{J}_{s}$	$(yz-v^2, xz-u^2, xy-t^2, uv-zt, tv-uy, tu-xv)$	IV	$(2,4,4,2)$	0
$\mathcal{J}_{ss}$	$(y^2+z^2+t^2+u^2+v^2, xy, -xz, -xt, -xu, -xv)$	I	$(2,2,2,2)$	0
$\mathbb{C}\times\mathcal{J}^5_{q,4}$	$(y^2+z^2+t^2+u^2, xy, -xz, -xt, -xu, -xv)$	I	$(2,2,2,2)$	1
$\mathbb{C}\times\mathcal{J}^5_{q,3}$	$(y^2+z^2+t^2, xy, -xz, -xt, -xu, -xv)$	I	$(2,2,2,2)$	2
$\mathcal{J}_{92}$	$(yz-t^2, xz, xy, -xt, tv-uz, tu-yv)$	IV	$(2,4,4,2)$	2
$\mathcal{J}_{100}$	$(yz, xz, xy, -xt, -xu, -xv)$	I	$(2,2,2,2)$	3
$\mathcal{J}_{101}$	$(yz, xz, xy, -zt, -zu, -yv)$	II	$(2,3,3,2)$	3
$\mathcal{J}^0_{102}$	$(yz, xz, xy, -zt, -yu, -xv)$	IV	$(2,4,4,2)$	3
$\mathcal{J}^1_{102}$	$(yz, xz, xy, -zt+2uv, -yu, -xv)$	IV	$(2,4,4,2)$	3
$\mathcal{J}^{000}_{113}$	$(xy, x^2, -yz, -ty, -yu, -xv)$	II	$(2,3,3,2)$	4
$\mathcal{J}^{100}_{100}$	$(xy, x^2, -yz+v^2, -yt, -yu, -xv)$	III	$(2,4,4,2)$	4
$\mathcal{J}^a_{122}$	$(xy, x^2, -yz, -ty, -xu, -xv)$	II	$(2,3,3,2)$	4
$\mathcal{J}^b_{122}$	$(xy, x^2, -yz+u^2, -yt, -xu, -xv)$	IV	$(2,4,4,2)$	4
$\mathcal{J}^c_{122}$	$(xy, x^2, -yz+2uv, -yt, -xu, -xv)$	IV	$(2,4,4,2)$	4
$\mathcal{J}^d_{122}$	$(xy, x^2, -yz+u^2, -yt+v^2, -xu, -xv)$	IV	$(2,4,4,2)$	4
$\mathcal{J}^e_{122}$	$(xy, x^2, -yz+u^2, -yt+2uv, -xu, -xv)$	IV	$(2,4,4,2)$	4
$\mathcal{J}^0_{124}$	$(xy, x^2, -yz, -yt, -xu, -xv+2tu)$	IV	$(2,4,4,2)$	4
$\mathcal{J}^1_{124}$	$(xy, x^2, -yz+u^2, -yt, -xu, -xv+2tu)$	IV	$(2,4,4,2)$	4
$\mathcal{J}^0_{126}$	$(xy, x^2, -yz, -yt, -xu, -xv+2zu+2tu)$	IV	$(2,4,4,2)$	4
$\mathcal{J}^1_{126}$	$(xy, x^2, -yz+u^2, -yt-u^2, -xu, -xv+2zu+2tu)$	IV	$(2,4,4,2)$	4
$\mathcal{J}^{(0)}_{140}$	$(xy, x^2, -yz, -xt, -xu, -xv)$	I	$(2,2,2,2)$	4
$\mathcal{J}^{(1)}_{140}$	$(xy, x^2, -yz+t^2, -xt, -xu, -xv)$	I	$(2,2,2,2)$	4
$\mathcal{J}^{(2)}_{140}$	$(xy, x^2, -yz+t^2+u^2, -xt, -xu, -xv)$	I	$(2,2,2,2)$	4
$\mathcal{J}^{(3)}_{140}$	$(xy, x^2, -yz++t^2+u^2+v^2, -xt, -xu, -xv)$	I	$(2,2,2,2)$	4
$\mathcal{J}_{143}$	$(xy, x^2, -yz, -xt, -xu, -xv+2zt)$	II	$(2,3,3,2)$	4
$\mathcal{J}^a_{ES}$	$(x^2, -xy, -xz, -xt+2yu-zv, -xu+z^2, -xv+2yz)$	IV	$(2,4,4,2)$	5
$\mathcal{J}^b_{ES}$	$(x^2, -xy, -xz, -xt+2yz, -xu+y^2, -xv+z^2)$	IV	$(2,4,4,2)$	5
$\mathcal{J}^{c,(-1)}_{ES}$	$(x^2, -xy, -xz, -xt, -xu-2yz+2zt, -xv-y^2+t^2)$	II	$(2,3,3,2)$	5
$\mathcal{J}^{c,(0)}_{ES}$	$(x^2, -xy, -xz, -xt, -xu+2zt, -xv+t^2)$	II	$(2,3,3,2)$	5
$\mathcal{J}^{c,(d)}_{ES}$	$(x^2, -xy, -xz, -xt, -xu+2zt, -xv+z^2+t^2)$	IV	$(2,4,4,2)$	5
$\mathcal{J}^{c,(1)}_{ES}$	$(x^2, -xy, -xz, -xt, -xu+y^2+2zt, -xv+t^2)$	IV	$(2,4,4,2)$	5
$\mathcal{J}^{c,(2)}_{ES}$	$(x^2, -xy, -xz, -xt, -xu+2zt, -xv+y^2+t^2)$	IV	$(2,4,4,2)$	5
$\mathcal{J}^{c,(3)}_{ES}$	$(x^2, -xy, -xz, -xt, -xu+2yz+2zt, -xv+t^2)$	IV	$(2,4,4,2)$	5
$\mathcal{J}^{c,(4)}_{ES}$	$(x^2, -xy, -xz, -xt, -xu+2zt, -xv+2yz+t^2)$	IV	$(2,4,4,2)$	5
$\mathcal{J}^{c,(5)}_{ES}$	$(x^2, -xy, -xz, -xt, -xu+y^2+2zt, -xv+z^2+t^2)$	IV	$(2,4,4,2)$	5
$\mathcal{J}^{d,000}_{ES}$	$(x^2, -xy, -xz, -xt, -xu, -xv+y^2)$	I	$(2,2,2,2)$	5
$\mathcal{J}^{d,100}_{ES}$	$(x^2, -xy, -xz, -xt, -xu, -xv+y^2+z^2)$	I	$(2,2,2,2)$	5
$\mathcal{J}^{d,110}_{ES}$	$(x^2, -xy, -xz, -xt, -xu, -xv+y^2+z^2+t^2)$	I	$(2,2,2,2)$	5
$\mathcal{J}^{d,111}_{ES}$	$(x^2, -xy, -xz, -xt, -xu, -xv+y^2+z^2+t^2+u^2)$	I	$(2,2,2,2)$	5

TABLE 10. Involutive normal forms for quadro-quadric Cremona transformations of $\mathbb{P}^5$ ($\dim R$ stands for the dimension of the radical of the corresponding Jordan algebra, this one being labelled with the notation used in [30]).

REFERENCES

[1] Allison B. N., A class of nonassociative algebras with involution containing the class of Jordan algebras, Math. Ann. 237 (1978).

Perhaps this is part of a general phenomenon:

Representability questions tend to have ‘answers’ if we allow multiples.

We have similar experiences when we studied very ample divisors,

graded free resolutions over a polynomial ring, stable homotopy theory,

Hodge conjecture, etc.

- Tension between axioms of field theory and axioms of combinatorial geometry goes back at least to Whitney (1935) and even Hilbert (1899).

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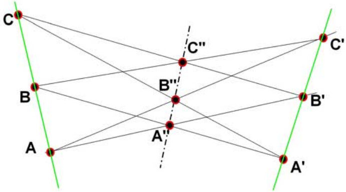
- Vamos (1978),

  The missing axiom of matroid theory is lost forever.
- Mayhew, Newman, and Whittle (2012),

  Is the missing axiom of matroid theory lost forever?

5. Is the missing axiom of matroid theory lost forever??



(A nonrepresentable combinatorial geometry violating the Pappus theorem)

- The gradient map is the blowup of the singular locus of

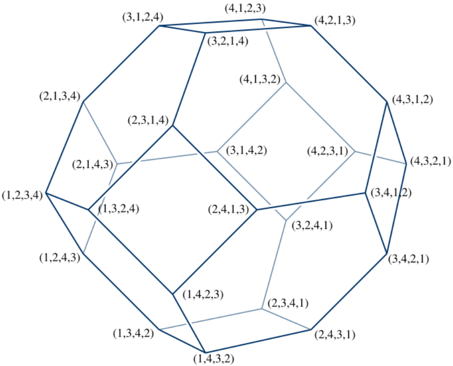
$$\{L_0 L_1 \cdots L_n = 0\} \subseteq \mathbb{P}^r.$$

- Over a field of any characteristic, we consider the diagram

$$\begin{tikzcd}
 & \Gamma \arrow[r] \arrow[dl] & \widetilde{\mathbb{P}}^n \arrow[r] \arrow[dl] \arrow[drr] & \mathbb{P}^n \times \mathbb{P}^n & \\
\mathbb{P}^r \arrow[r, "L"'] & \mathbb{P}^n \arrow[rrr, dashed, "(\frac{1}{x_0} : \frac{1}{x_1} : \cdots : \frac{1}{x_n})"] & & & \mathbb{P}^n
\end{tikzcd}$$

where $L = [L_0 : \cdots : L_n]$ and $\widetilde{\mathbb{P}}^n$ is the standard resolution of the Cremona transformation.

$\widetilde{\mathbb{P}}^n$ is the toric variety of the permutohedron:



$L : \mathbb{P}^r \to \mathbb{P}^n$ defines a subvariety of $\widetilde{\mathbb{P}}^n$ (the maximal wonderful model of the hyperplane arrangement), and hence a homology class.

- In general, a homology class of a toric variety can be described as a combinatorial object called *tropical variety*.
- Any combinatorial geometry M on $n+1$ elements of rank $r+1$ can be tropicalized to define
$$\Delta_M \in A_r\left(\widetilde{\mathbb{P}}^n\right).$$
- Δ_M is representable as a homology class of $\widetilde{\mathbb{P}}^n$ iff

  M is representable over the field of definition of $\widetilde{\mathbb{P}}^n$.

Therefore, Hartshorne's problem is impossibly difficult even for

relatively simple toric varieties, because it is at least as difficult as

Problem (Whitney's missing axiom)

Find an axiom of matroid theory which characterizes representability over an infinite field (say $\mathbb{C}$).

Theorem (H-Katz)

For any combinatorial geometry M,

$$\widetilde{\mathbb{P}}^n \longrightarrow \mathbb{P}^n \times \mathbb{P}^n, \qquad \Delta_M \mapsto \chi_M(q).$$

From our point of view, Rota's conjecture precisely says that some multiple of $\chi_M(q)$ is representable by a subvariety.

If the conjecture is true, I believe it is because the same is true for Δ_M in $\widetilde{\mathbb{P}}^n$.

Wild speculation

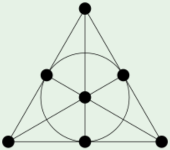
Every combinatorial geometry is representable over every field (if we allow multiples and small perturbations).

Whitney's original axioms guarantee representability, and hence there is no missing axiom.

This statement is valid if several 'expected properties' hold for the effective cone and the nef cone of algebraic cycles in $\widetilde{\mathbb{P}}^n$.

In general, the effective cone of a toric variety is a polyhedral cone spanned by closures of torus orbits. In particular, the effective cone depends only on the fan of the toric variety.

Example



There is no subvariety of $\widetilde{\mathbb{P}}^7_{\mathbb{C}}$ representing Δ_M.

However, Δ_M is in the effective cone of $\widetilde{\mathbb{P}}^7_{\mathbb{C}}$.

- Various cones of positive cycles are thoroughly explored in dimension and codimension 1: Ample cone, nef cone, big cone, pseudoeffective cone, cone of effective curves, movable curves, etc.

- Grothendieck asked several questions concerning higher codimension cones. Debarre, Ein, Lazarsfeld, Voisin (2012) showed that the answers are 'no, no, and no'.

- If that is too difficult, I propose to study the positive cones only for toric varieties.
- There are computational evidences (Maclagan and Smith) which suggest 'no, no'.
- If that is too difficult, I propose to study the positive cones of this one toric variety $\widetilde{\mathbb{P}}^n$, that of the permutohedron.
- There are computational evidences which says 'perhaps yes'.

Conclusion

1. *Rota's conjecture leads us to the study of positive cycles in $\widetilde{\mathbb{P}}^n$.*
2. *The permutohedral variety $\widetilde{\mathbb{P}}^n$ is a nice playground for everyone:*
   - *matroid theorists,*
   - *commutative algebraists (who like matroids and multigraded Hilbert polynomials),*
   - *toric experts,*
   - *G/B experts,*
   - *birational geometers interested in various positive cones,*
   - *geometers studying general questions concerning algebraic cycles.*
3. *$\widetilde{\mathbb{P}}^n$ is a simple variety with many mysteries.*