MSRI
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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Elizabeth Gross **Email/Phone:** egross7@uic.edu

Speaker's Name: Satoshi Murai

Talk Title: On generalized lower bound conjecture for simplicial polytopes

Date: 12 / 4 / 12 **Time:** 11 : 30 am / pm (circle one)

List 6-12 key words for the talk: stacked triangulations, simplicial polytopes, h-vector, face numbers, Stanley-Reisner rings, monomial ideals

Please summarize the lecture in 5 or fewer sentances:

Shows that if $h_{r-1} = h_r$ for some $r \le d/2$ then the polytope P has an $(r-1)$-stacked triangulation by examing Stanley-Reisner rings.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
    - **Computer Presentations**: Obtain a copy of their presentation
    - **Overhead**: Obtain a copy or use the originals and scan them
    - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
    - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Stacked Triangulations & face numbers

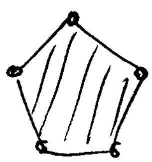
joint work with Evan Nevo

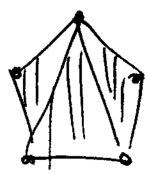
§ Stacked triangulations

<u>Def</u> P : d-polytope in $\mathbb{R}^N$

A triangulation of P is a geometric simplicial complex in $\mathbb{R}^N$ whose underlying space is P.

<u>Ex</u> ($d=2$)

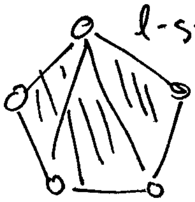




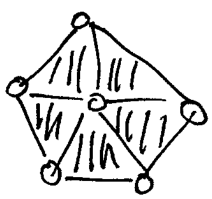
triangulation

<u>Def</u> A triangulation T of a d-polytope P is r-stacked if all its interior faces have $\dim \geq d-r$

<u>Ex</u>



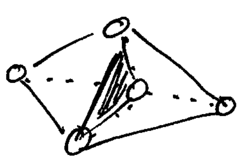
1-stacked



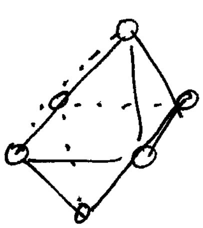
Not 1-stacked

<u>Def</u> A simplicial d-polytope P is r-stacked if it has an r-stacked triangulation T such that $\partial T = \partial P$.

<u>Ex</u>



1 stacked



not 1 stacked

Remember

① every d-polytope is $(d-1)$-stacked

② 1-stacked polytope = stacked polytope

Q Fix $r \geq 1$

Which simplicial polytopes are r-stacked?

§ 2 stackedness & face numbers

P is simplicial d-polytope

$f_i(P) = \#\ i$-dim face of P

$h(P) = (h_0(P), \ldots, h_d(P))$ h-vector

$$h_i = \sum_{j=0}^{n} \binom{d-j}{n-j} (-1)^{n-j} f_{j-1}(P)$$

Rem $h(P)$ is symmetric ($h_i = h_{d-n}$)

Conj ~~[illegible]~~ (McMullen-Walkup, 1971)

If P is a simplicial d-polytope

(1) $h_{r-1}(P) \leq h_r(P)$ for $r \leq d/2$

(2) if $h_{r-1}(P) = h_r(P)$ for some $r \leq d/2$ then P is $(r-1)$-stacked

$$h_0 \leq h_1 \leq \cdots \leq h_{d/2} \geq \cdots \geq h_d$$

Rem ① Conj (1) was proved by Stanley (1980)

② McMullen & Walkup proved if P is $(r-1)$-stacked for some $r \leq d/2$ then $h_{r-1}(P) = h_r(P)$

③ Conj (2) for $r=2$ was proved by Barnette (1971) & Billeratee (1980

<u>Result</u> Conj (2) holds

§ shape of stacked triangulations

An abstract simplicial complex is a combinatorial triangulation of a simplicial d-polytope P if

(1) $|\Delta|$ is homeomorphic to d ball

and

(2) $\partial\Delta$ is combinatorially isomorphic to ∂P

<u>Thm A</u> (McMullen 2004)

Let $r \leq \frac{d+1}{2}$. An $(r-1)$-stacked combinatorial triangulation of P is geometric & unique.

Δ is simplicial complex on V.

$S = k[x_v \mid v \in V]$ (k field of char 0)

$I_\Delta = (x_{v_1} \cdots x_{v_k} \mid \langle v_1, \ldots, v_k \rangle \notin \Delta)$

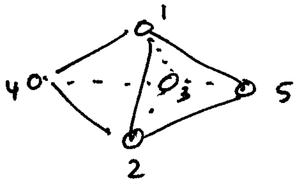
S/I_Δ Stanley Reisner ring of Δ

<u>Thm B</u> (Bagchi, Datta 2011)

In thm A, the $(r-1)$-stacked triangulation Σ is defined by

(#) $I_\Sigma = (m \in I_{\partial P} \mid \deg m \leq r)$

Ex ($d=3$, $r=2$)



$I_{\partial P} = (x_1x_2x_3,\ x_4x_5)$

$I_{\Sigma} = (x_4x_5)$

Thm (M-Nevo)

Let $r \le d/2$. If a simplicial d-polytope P satisfies $h_{r-1}(P) = h_r(P)$ the simplicial complex Σ. Then (#) is an $(r-1)$-stacked triangulation of P.

Conj (McMullen 2004)

The triangulation Σ of P is regular.

Outline of Pf

Key lem 1 If Σ is d-dim, pure and $H_{d-1}(\Sigma, k) = 0$ then Σ is a triangulation of P.

Key lem 2 The Stanley-Reisner ring of Σ has Krull dim $d+1$ & Cohen-Macaulay.

Idea of pf By the Hard Lefschetz thm, $\exists\ \theta = \theta_1, \ldots, \theta_d$ linear system of parameter of $S/I_{\partial P}$ & a linear form w st

$$\times w \mid (S/I_{\partial P}+(\theta))_{k-1} \to (S/I_{\partial P}+(\theta))_k$$

is injective for $k \le d/2$ & surjective for $k \geq d/2$