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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Elizabeth Gross **Email/Phone:** egross7@uic.edu

Speaker's Name: Uwe Nagel

Talk Title: Enumerations deciding the Weak Lefschetz Property

Date: 12 / 4 / 2012 **Time:** 11 : 30 am / pm (circle one)

List 6-12 key words for the talk: weak lefschetz property, monomial ideals, lozenge tilings, lattice paths, Mahonian determinants, syzygy bundles, laplace equations, Togliatti system

Please summarize the lecture in 5 or fewer sentances: Discusses an approach for studying monomial ideals in three variables using lozenge tilings. Gives a combinatorial interpretation of the weak Lefschetz property. Explores Laplace equations as an application

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Enumerations of the Weak Lefschetz Property

joint work with

David Cook II

(University of Notre Dame)

Uwe Nagel

(University of Kentucky)

MSRI, December 4, 2012

Outline

- Lefschetz Properties
- Lozenge tilings, perfect matchings, and lattice paths
- Mahonian Determinants
- Type 2 algebras
- Existence of Laplace equations

$R = K[x_1, \ldots, x_n]$, K an infinite field
$I \subset R$ homogeneous, artinian ideal ($\dim_K R/I < \infty$)

Lefschetz Properties

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$I \subset R$ homogeneous, artinian ideal ($\dim_K R/I < \infty$)

Definition

$A = R/I$ has the Weak Lefschetz Property (WLP) if there is a linear form $\ell \in R$ such that the multiplication

$$\times \ell : [A]_i \to [A]_{i+1}$$

has maximal rank for all i (i.e. is injective or surjective).
A has the Strong Lefschetz Property (SLP) if

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Remark: (i) ℓ general.
(ii) WLP implies restrictions on Hilbert function
 (g-Theorem (Stanley)).
(iii) WLP and SLP are related to Fröberg's conjecture.

Known results

Theorem

- *(Harima, Migliore, N., Watanabe, 2003) If $n \leq 2$ and* char $K = 0$*, then A has the SLP.*
- *(Migliore, Zanello, 2007) If $n \leq 2$, then A always has the WLP.*

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Theorem (Stanley, 1980; ...)

If $\operatorname{char} K = 0$*, then each monomial c.i., $I = (x_1^{a_1}, \ldots, x_n^{a_n})$, has the SLP.*

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Theorem (Harima, Migliore, N., Watanabe, 2003)

If $n = 3$, char $K = 0$*, then* every *c.i. $I = (f_1, f_2, f_3)$ has the WLP.*

Monomial ideals in three variables

$I \subset R = K[x, y, z]$ artinian monomial ideal.

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Theorem (Boij, Migliore, Miró-Roig, N., Zanello, 2012)

If $n = 3$, char $K = 0$, and R/I is level of type 2, then R/I has the WLP.

Counterexamples if R/I is not level or if char $K > 0$.

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Example

- If $I = (x^7, y^7, z^7, x^2y^2z^2)$, then R/I has the WLP if and only if the characteristic of K is not 2 or 7.

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- If $I = (x^{20}, y^{20}, z^{20}, x^3y^8z^{13})$, then R/I has the WLP if and only if the characteristic of K is not 2, 3, 5, 7, 11, 17, 19, 23, or

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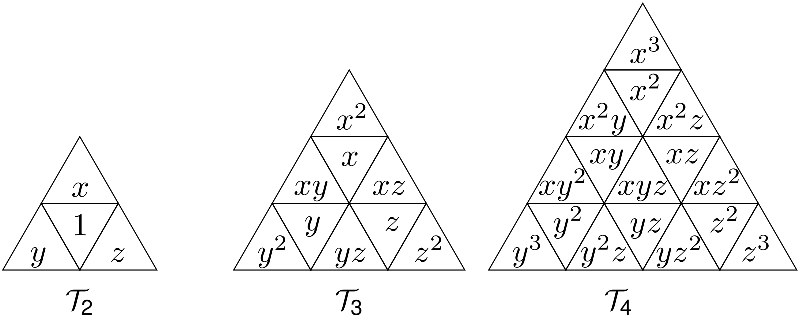
Triangular regions

Triangular region $\mathcal{T}_d$: equilateral triangle of side length d, subdivided into equilateral unit triangles:

- $\binom{d}{2}$ downward-pointing ($\bigtriangledown$) - labeled by monom. in $[R]_{d-2}$,

  and

- $\binom{d+1}{2}$ upward-pointing ($\bigtriangleup$) - labeled by monom. in $[R]_{d-1}$.

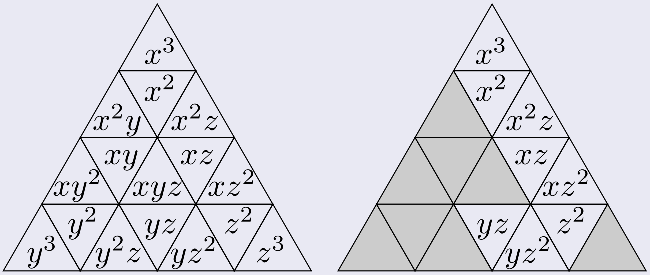


Triangular regions

$I \subset R$ any monomial ideal
$d \geq 1$ any integer
triangular region $T_d(I)$: obtained from $\mathcal{T}_d$ by removing triangles with labels in I.

Example 1

$I = (xy, y^2, z^3)$, $d = 4$.

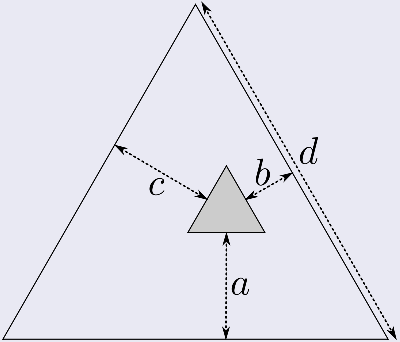


$\mathcal{T}_4$ $\qquad$ $T_4(xy, y^2, z^3)$

Triangular regions

Example 2

$I = (x^a y^b z^c)$.



$T_d(x^a y^b z^c)$

Lozenge tilings

$T \subset \mathcal{T}_d$ any subregion

Lozenge (diamond, callisson, rhombus):
glue an $\bigtriangledown$- and an $\triangle$-triangle along the common edge

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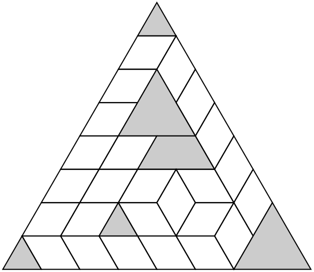
Tile T by lozenges if possible



A tiling of $T_8(x^7, y^7, z^6, xy^4z^2, x^3yz^2, x^4yz)$

Necessary tileability condition: balanced ($\#\bigtriangledown = \#\bigtriangleup$)

Bipartite graphs and perfect matchings

$T \subset \mathcal{T}_d$ any subregion

$G(T)$ bipartite graph:
B = set of centers of $\triangledown$-triangles, ordered revlex by labels,
W = set of centers of $\triangle$-triangles, ordered revlex by labels

- Vertices: $B \cup W$
- Edges: (B_i, W_j) if the corresponding triangles share an edge

Bi-adjacency matrix $Z(T)$: zero-one matrix of size $\#B \times \#W$:

$$Z(T)_{(i,j)} = \begin{cases} 1 & \text{if } (B_i, W_j) \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

Bipartite graphs and perfect matchings

Assume T is balanced ($\#B = \#W$):

Perfect matching of $G(T)$: a set of pairwise non-adjacent edges of $G(T)$ such that each vertex is matched

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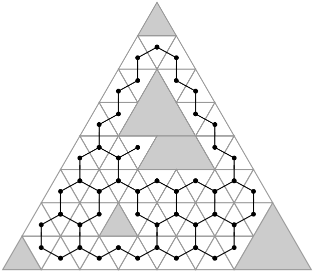
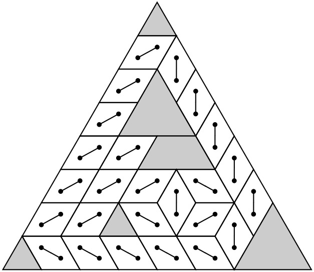
$$\Big\updownarrow 1-1$$

lozenge tiling of T



The graph $G(T)$



A perfect matching

Bipartite graphs and perfect matchings

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If T is balanced, then

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Definition

A lozenge tiling τ of T induces a bijection $B \to W,\ B_i \mapsto W_{\sigma(i)}$, where $\sigma \in \mathfrak{S}_{\#B}$. The perfect matching sign of τ is

$$\operatorname{msgn} \tau := \operatorname{sgn} \sigma.$$

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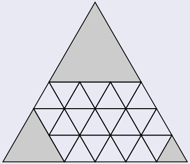
Corollary

$$\sum_{\tau \text{ tiling of } T} \operatorname{msgn} \tau := \det Z(T).$$

Bipartite graphs and perfect matchings

Example

Consider $T = T_6(x^3, y^4, z^5)$.



$$Z(T) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

$\operatorname{perm} Z(T) = \det Z(T) = 10$.

Lattice paths

$T \subset \mathcal{T}_d$ any subregion

$L(T)$: set of midpoints of NE edges of triangles in T

- Label the vertices of $L(T)$ that are only on $\triangle$-triangles by $A_1, \ldots, A_m$ according to the revlex order of the monomials, beginning with the smallest.
- Label the vertices of $L(T)$ are only on $\bigtriangledown$-triangles by $E_1, \ldots, E_n$ according to the revlex order of the monomials, beginning with the smallest.

A lattice path from A_i to E_j is a path in $L(T)$ where each single move is to the East ($\rightarrow$) or to the South-East ($\searrow$).

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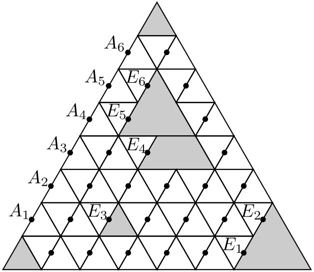

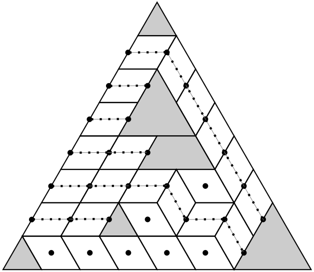

Lattice paths

Lattice path matrix $N(T)$: size $m \times n$

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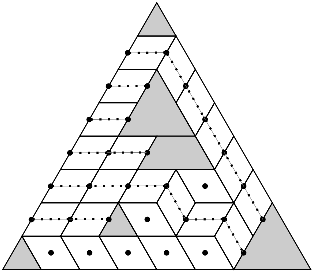

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Definition

The lattice path sign of a lozenge tiling τ of T is

$$\operatorname{lpsgn} \tau := \operatorname{sgn} \sigma,$$

where $\sigma \in \mathfrak{S}_m$ is the permutation such that, for all i, the path starting in A_i ends in $E_{\sigma(i)}$.

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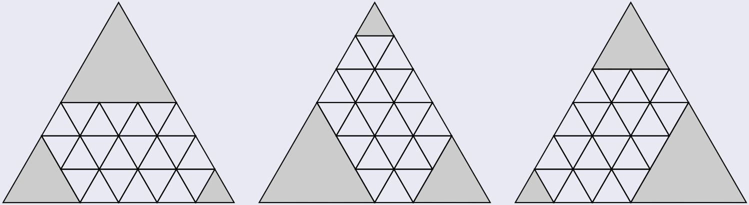
Theorem (Lindström, Gessel & Viennot)

If T is balanced, then $\displaystyle\sum_{\tau \text{ tiling of } T} \operatorname{lpsgn} \tau := \det N(T).$

Lattice paths

Example

$T = T_6(x^3, y^4, z^5)$ and its rotations.



$$\begin{bmatrix} 10 \end{bmatrix} \qquad \begin{bmatrix} 4 & 6 \\ 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 & 0 \\ 3 & 3 & 1 \\ 1 & 3 & 3 \end{bmatrix}$$

Comparisons

$T = T_d(I) \subset \mathcal{T}_d$

τ lozenge tiling of T:

- perfect matching sign $\operatorname{msgn} \tau$ - enumerated by $\det Z(T)$
- lattice path sign $\operatorname{lpsgn} \tau$ - enumerated by $\det N(T)$

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Theorem

(a) Let τ and τ' be two lozenge tilings of T. Then

$$\operatorname{msgn}(\tau) \cdot \operatorname{lpsgn}(\tau) = \operatorname{msgn}(\tau') \cdot \operatorname{lpsgn}(\tau').$$

(b)

$$|\det Z(T)| = |\det N(T)|.$$

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$$|\det Z(T)| = |\det N(T)|.$$

Corollary

If T is tileable and simply connected, then

$$|\det Z(T)| = \operatorname{perm} Z(T) > 0.$$

Comparisons

Example

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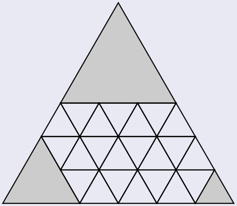


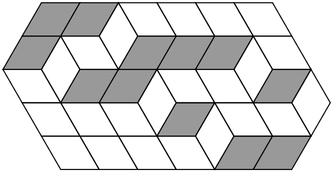
Then

$$10 = |\det N(T)| = |\det Z(T)| = \mathrm{perm}(T).$$

Mahonian determinants

3	3	2	2	2	1
3	2	2	1	0	0

A $2 \times 6 \times 3$ plane partition.



The associated lozenge tiling.

Theorem (MacMahon)

The number of plane partitions in an $a \times b \times c$ box is

$$\mathrm{Mac}(a,b,c) := \frac{\mathcal{H}(a)\mathcal{H}(b)\mathcal{H}(c)\mathcal{H}(a+b+c)}{\mathcal{H}(a+b)\mathcal{H}(a+c)\mathcal{H}(b+c)},$$

where $\mathcal{H}(n) := \prod_{i=0}^{n-1} i!$ is the hyperfactorial of n.

Mahonian determinants

Proposition

If $T = T_d(x^a, y^b, z^c)$ is balanced, that is, $d = \frac{1}{2}(a+b+c)$ is an integer, then

$$|\det Z(T)| = \operatorname{perm} Z(T) = \operatorname{Mac}(d-a, d-b, d-c).$$

Mahonian determinants

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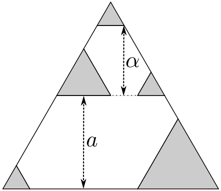
Proposition

If $T = T_d(x^{a+\alpha}, y^b, z^c, x^a y^\beta, x^a z^\gamma)$ is balanced, then

$$\begin{aligned} |\det Z(T)| &= \operatorname{perm} Z(T) \\ &= \operatorname{Mac}(d-a, d-b, d-c)\operatorname{Mac}(d-a-\alpha, d-a-\beta, d-a-\gamma). \end{aligned}$$

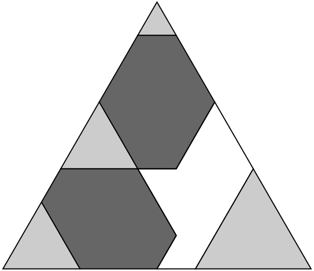


Mahonian determinants

Proposition

If $T = T_d(x^a, y^b, z^c, x^\alpha y^\beta)$ is balanced (as below), then $|\det Z(T)| = \operatorname{perm} Z(T)$ is

$$\operatorname{Mac}(a+\beta-d, d-a, d-(\alpha+\beta))\operatorname{Mac}(\alpha+b-d, d-b, d-(\alpha+\beta))$$
$$\times\frac{\mathcal{H}(d-a+d-(\alpha+\beta))\mathcal{H}(d-b+d-(\alpha+\beta))\mathcal{H}(d-c+d-(\alpha+\beta))\mathcal{H}(d)}{\mathcal{H}(a)\mathcal{H}(b)\mathcal{H}(c)\mathcal{H}(d-(\alpha+\beta))}.$$



Relation to WLP

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If K is infinite, then R/I has the WLP iff multiplications by $\ell = x + y + z$ have maximal rank.

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Theorem

- *For each $d \geq 1$, the coordinate matrix of $[R/I]_{d-2} \xrightarrow{x+y+z} [R/I]_{d-1}$ with respect to monomial bases in revlex order is $Z(T_d(I))$.*
- $\dim_K [R/(I, x + y + z)]_{d-1} = \dim_K(\ker N(T_d(I))^T)$.

Relation to WLP

$I \subset R = K[x, y, z]$ artinian monomial ideal.

If K is infinite, then R/I has the WLP iff multiplications by $\ell = x + y + z$ have maximal rank.

Theorem

- *For each $d \geq 1$, the coordinate matrix of $[R/I]_{d-2} \xrightarrow{x+y+z} [R/I]_{d-1}$ with respect to monomial bases in revlex order is $Z(T_d(I))$.*
- $\dim_K [R/(I, x + y + z)]_{d-1} = \dim_K(\ker N(T_d(I))^T)$.

Corollary

Assume $T = T_d(I)$ is balanced and the socle elements of R/I have degrees $\geq d - 1$. TFAE:

- R/I has the WLP.
- $\det Z(T_d(I)) \bmod (\operatorname{char} K) \neq 0$.
- $\det N(T_d(I)) \bmod (\operatorname{char} K) \neq 0$.

Type two algebras

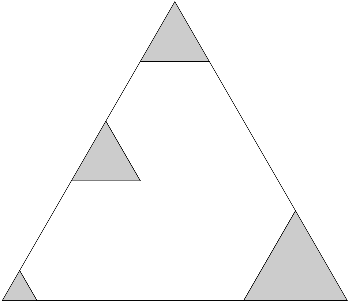
Proposition

If R/I has type two, then I has one of the following two forms:

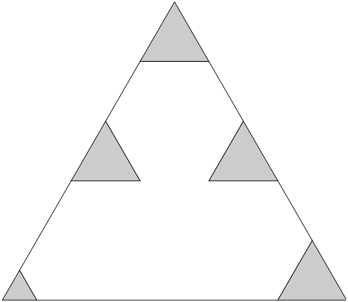
(i) $I = (x^a, y^b, z^c, x^\alpha y^\beta)$,

(ii) $I = (x^a, y^b, z^c, x^\alpha y^\beta, x^\alpha z^\gamma)$,

where $0 < \alpha < a$, $0 < \beta < b$, and $0 < \gamma < c$.



$T_d(x^a, y^b, z^c, x^\alpha y^\beta)$



$T_d(x^a, y^b, z^c, x^\alpha y^\beta, x^\alpha z^\gamma)$

Type two algebras

Assume char $K = 0$.

Theorem

If R/I has type two, then R/I fails to have the WLP if and only if $I = (x^a, y^b, z^c, x^\alpha y^\beta, x^\alpha z^\gamma)$ and there exists an integer d such that

$$\max\left\{a, \alpha+\beta, \alpha+\gamma, \frac{a+\alpha+\beta+\gamma}{2}\right\} < d$$

$$< \min\left\{a+\beta+\gamma, \frac{\alpha+b+c}{2}, b+c, \alpha+c, \alpha+b\right\}.$$

Type two algebras

Assume char $K = 0$.

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Corollary (BMMNZ, 2012)

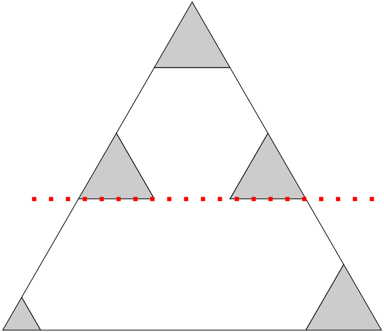
If R/I has type two and is level, then R/I has the WLP.

New proof: R/I is level if and only if $a-\alpha = b-\beta+c-\gamma$. Then

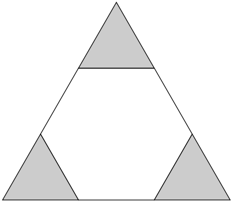
$$\frac{a+\alpha+\beta+\gamma}{2} = \frac{2\alpha+b+c}{2} \geq \alpha + \min\{b, c\}.$$

Proof of the Theorem (sketch)

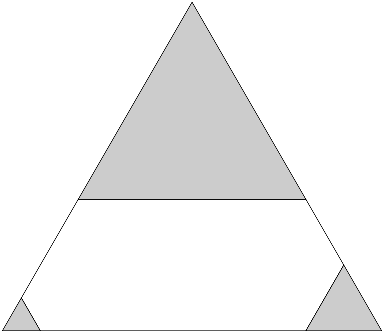
Decompose $T_d(I)$:



$T_d(x^a, y^b, z^c, x^\alpha y^\beta, x^\alpha z^\gamma)$



$T^u = T_{d-\alpha}(x^{a-\alpha}, y^\beta, z^\gamma)$



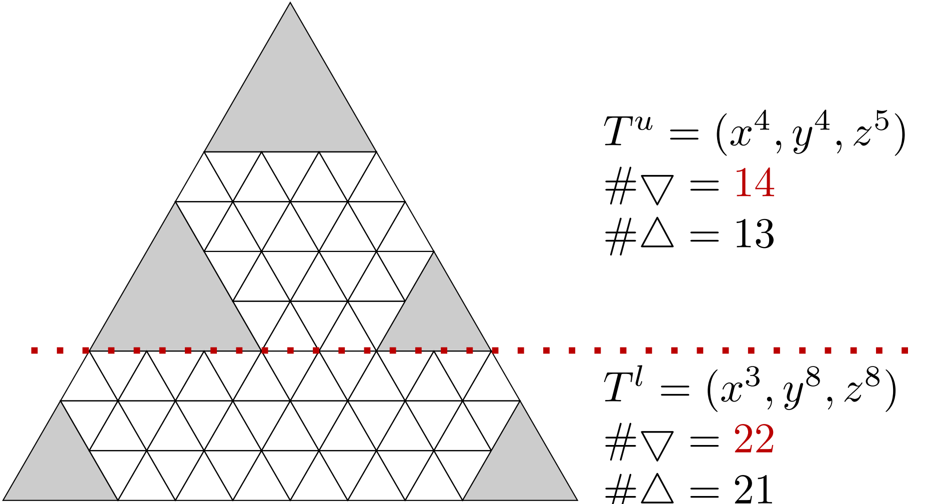
$T^l = T_d(x^\alpha, y^b, z^c)$

Proof (sketch)

		T^u		
		$\triangle$	Balanced	$\triangledown$
T^l	$\triangle$	1	2	9
	Balanced	3	4	5
	$\triangledown$	8	6	7

Proof (sketch)

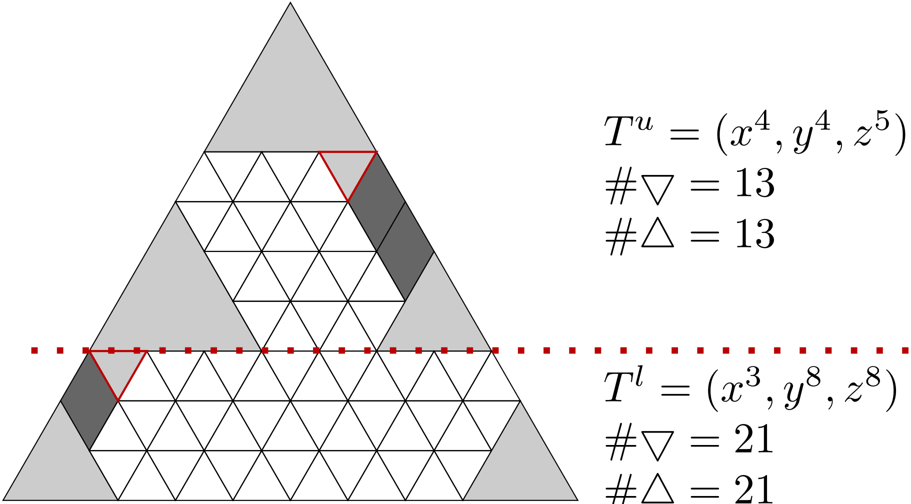
Cases 1 – 7:



$T_{10}(x^7, y^8, z^8, x^3y^4, x^3z^5)$

Proof (sketch)

Cases 1 – 7:

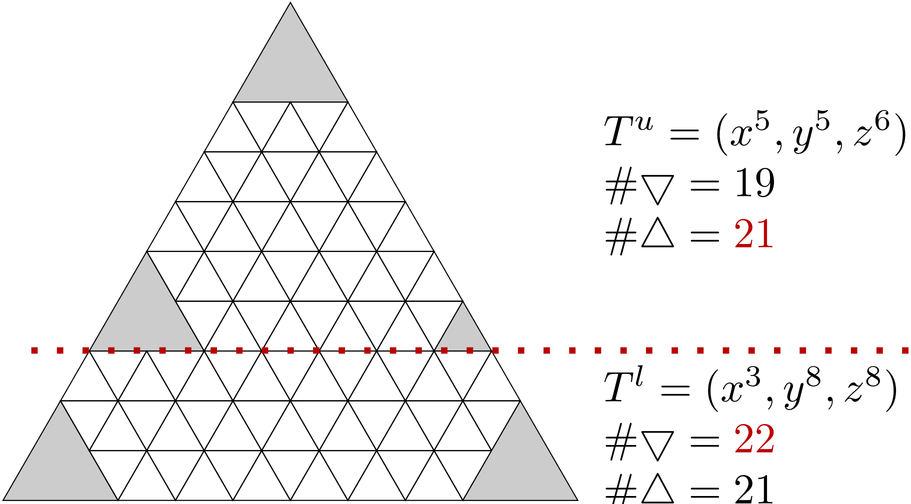


$$T^u = (x^4, y^4, z^5)$$
$$\#\bigtriangledown = 13$$
$$\#\bigtriangleup = 13$$

$$T^l = (x^3, y^8, z^8)$$
$$\#\bigtriangledown = 21$$
$$\#\bigtriangleup = 21$$

$$T_{10}(x^7, y^8, z^8, x^3y^4, x^3z^5)$$

Case 8:

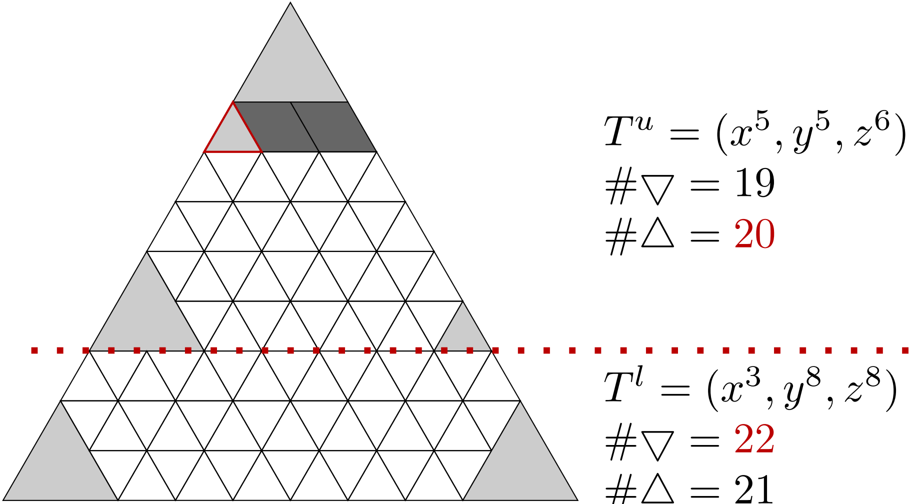


$T_{10}(x^8, y^8, z^8, x^3y^5, x^3z^6)$

$T^u = (x^5, y^5, z^6)$
$\#\triangledown = 19$
$\#\triangle = 21$

$T^l = (x^3, y^8, z^8)$
$\#\triangledown = 22$
$\#\triangle = 21$

Case 8:



$T^u = (x^5, y^5, z^6)$
$\#\bigtriangledown = 19$
$\#\bigtriangleup = 20$

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$\#\bigtriangledown = 22$
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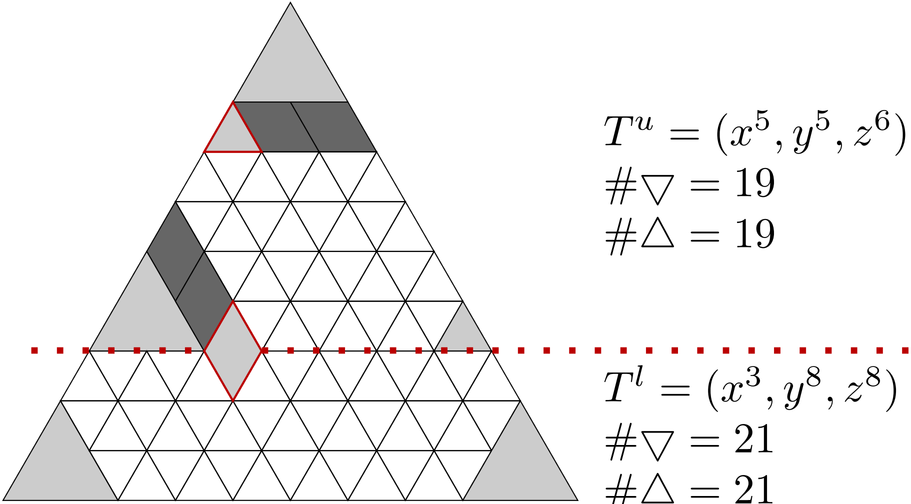
Proof (sketch)

Case 8:



$T_{10}(x^8, y^8, z^8, x^3y^5, x^3z^6)$

$T^u = (x^5, y^5, z^6)$

$\# \bigtriangledown = 19$

$\# \bigtriangleup = 19$

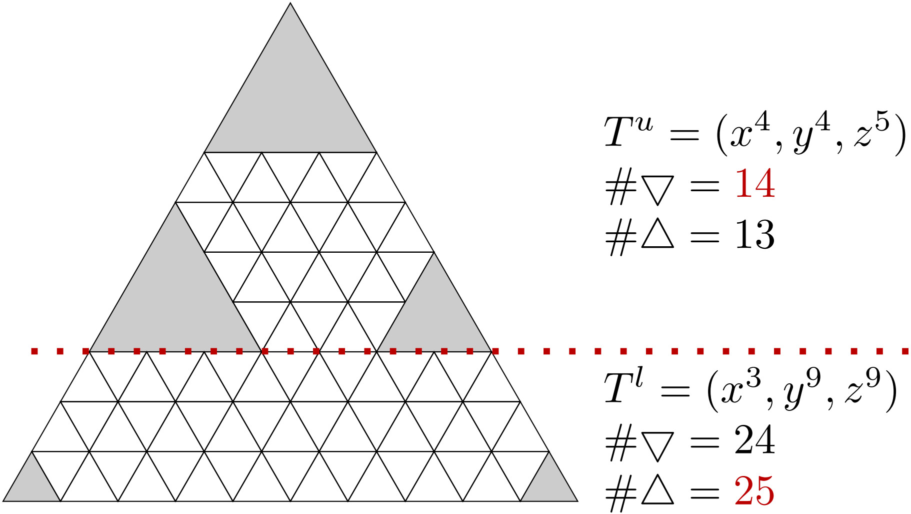
$T^l = (x^3, y^8, z^8)$

$\# \bigtriangledown = 21$

$\# \bigtriangleup = 21$

Proof (sketch)

Case 9:



$T_{10}(x^7, y^9, z^9, x^3y^4, x^3z^5)$

$T^u = (x^4, y^4, z^5)$

$\# \bigtriangledown = 14$

$\# \bigtriangleup = 13$

$T^l = (x^3, y^9, z^9)$

$\# \bigtriangledown = 24$

$\# \bigtriangleup = 25$

Laplace equations

$X \subset \mathbb{P}^N = \mathbb{P}^N_K$ n-dim proj. variety, $K = \overline{K}$, $\mathrm{char}\, K = 0$

$P \in X$ a smooth point, φ a local parametrization around P

$T_P^{(s)} X = \mathbb{P}$(span of partial derivatives of φ of order at most s)

s-th osculating space to X at P

Expected dimension is $\binom{n+s}{s} - 1$.

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Definition

X is said to satisfy δ Laplace equations of order s if, for a general point P of X,

$$\dim T_P^{(s)} X = \binom{n+s}{s} - 1 - \delta.$$

Interesting only if $N \geq \binom{n+s}{s} - 1$.

Togliatti, 1929, 1946

Perkinson, 2000

Mezzetti, Miró-Roig, Ottaviani, 2012

Di Genaro, Ilardi, Vallès, 2012

Laplace equations

$I = (f_1, \dots, f_r) \subset S = K[x_0, \dots, x_n]$, where $\deg f_i = d$

$\varphi_I : \mathbb{P}^n \dashrightarrow \mathbb{P}^{r-1}$ induced rational map with image $X_{n,[I]_d}$

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Example (Togliatti)

Let $n = 2, \quad J = (x^2y, x^2z, xy^2, xz^2, y^2z, yz^2)$. Then $X_{2,[J]_3} \subset \mathbb{P}^5$ is a toric surface satisfying one Laplace equation of order 2.

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Remark

(i) If $I \subset S$ is an artinian monomial ideal, then I^{-1} is generated by monomials in $S \setminus I$.

(ii) $\dim_K [I^{-1}]_d = \dim_K [S/I]_d$.

(iii) If $I = (x^3, y^3, z^3, xyz)$, then $I^{-1} = J$.

Laplace equations

Mezzetti, Miró-Roig, Ottaviani, 2012: connection to WLP

Theorem

$I \subset S$ artinian ideal with $r \leq \binom{n+d}{n}$ minimal generators of degree d, $\ell \in [S]_1$ general. TFAE:

(a) *Multiplication map $[S/I]_{d-1} \xrightarrow{\ell} [S/I]_d$ has a δ-dim kernel.*

(b) *$X_{n,[I^{-1}]_d} = \varphi_{I^{-1}}(\mathbb{P}^n)$ satisfies δ Laplace equations of order $d-1$.*

If $\delta > 0$, then I is said to define a Togliatti system.

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Assume $n = 2$, $I \subset R = K[x, y, z]$ monomial.

Example

Togliatti systems with few generators:

(i) (Franco, Ilardi, 2002; Vallès, 2006) 4 generators:
$$I = (x^3, y^3, z^3, xyz).$$

(ii) 5 generators: $I = (x^4, y^4, z^4, x^2yz, y^2z^2)$ or
$$I = (x^d, y^d, z^d, x^{d-1}y, x^{d-1}z).$$

Laplace equations

Proposition

Let $U \subset T_{d+1}(I)$ be a tileable monomial subregion such that $\det Z(U) \neq 0$. Let J be a monomial ideal such that $T \setminus U = T_{d+1}(J)$.
Then $[R/I]_{d-1} \xrightarrow{x+y+z} [R/I]_d$ and $[R/J]_{d-1} \xrightarrow{x+y+z} [R/J]_d$ fail to have maximal rank by the same margin.

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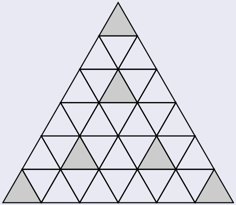
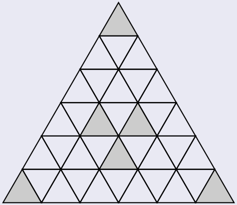
Example

Togliatti systems obtained from $T_6(x^5, y^5, z^5, xyz)$.



$(x^5, y^5, z^5, xyz^3, xy^3z, x^3yz)$



$(x^5, y^5, z^5, xy^2z^2, x^2yz^2, x^2y^2z)$

Laplace equations

Proposition

Assume, $[R/I]_{d-1} \stackrel{x+y+z}{\longrightarrow} [R/I]_d$ is not injective although it is expected ($\dim_K [R/I]_{d-1} \le \dim_K [R/I]_d$), $T = T_{d+1}(I)$ has no overlapping punctures, and $x^d, y^d, z^d \in I$.
For each puncture, in each row fill in all triangles, but one $\triangle$-triangle. Call the result T', and let J be the smallest ideal such that $T' = T_{d+1}(J)$. Then J defines a Togliatti system.

Laplace equations

Theorem

Let j be an integer such that $1 \le j \le \frac{d-1}{4}$ and

$$I_j = (y^d) + z^{4j+1}(y,z)^{d-1-4j} + (x^3, y^3, z^3, xyz) \cdot x^{d+1-4j} \cdot (x^4, y^4)^{j-1}.$$

Then:

(a) $[R/I_j]_{k-1} \xrightarrow{x+y+z} [R/I]_k$ *has maximal rank for all* $k \neq d$.

(b) $Z_{d+1}(I_j)$ *is balanced.*

(c) $X_{n,[(I_j)^{-1}]_d}$ *satisfies exactly* j *Laplace equations of order* $d-1$.

Example

$T_{14}(I_2)$