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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Elizabeth Gross Email/Phone: egross7@uic.edu

Speaker's Name: Mateusz Michalek

Talk Title: Derived Categories of Toric Varieties

Date: 12 / 4 / 12 Time: 2 : 00 am / [✓ pm] (circle one)

List 6-12 key words for the talk: derived category, toric varieties, coherent sheaves, exceptional collections, line bundles, Fano variety

Please summarize the lecture in 5 or fewer sentances: Introduces bounded derived categories of coherent sheaves for smooth algebraic varieties. Presents known results on the existence of strongly exceptional collections for toric varieties.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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Derived categories of toric varieties

Mateusz Michałek

Max Planck Institute for Mathematics

Combinatorial Commutative Algebra and Applications
MSRI

Outline

1. Derived categories
   - Motivations
   - Definition
   - Exceptional collections
2. Derived categories of toric varieties
   - General results
   - Example of the projective space
   - Further results

Motivations to study derived categories

X – algebraic variety (smooth, complete, over $\mathbb{C}$)

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Two noetherian schemes have equivalent categories of coherent sheaves if and only if they are isomorphic.

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but... many functors are not exact.

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$$0 \to P \to I_1 \to I_2 \to \dots$$

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- The resolution may be not bounded and the objects may be not coherent

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After applying the functor we obtain a *complex*. From algebraic topology we know that the information about cohomologies may be not sufficient. There exist topological spaces with isomorphic homology groups, but not homotopy equivalent.

Objects

The bounded derived category of coherent sheaves

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We may identify coherent sheaves with complexes (with cohomology) concentrated in degree 0.

$$\dots \to 0 \to P \to 0 \to \dots$$

Morphisms

Morphisms of complexes:

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Consider a graded morphism (f_i) of degree n of complexes A_i and B_i (squares do not have to commute):

$$\begin{tikzcd}
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A morphism of complexes is called a quasi-isomorphism if it induces an isomorphism of cohomologies.

As homotopy equivalent morphisms define the same morphism on the level of cohomologies the notion of quasi-isomorphism makes sense in the homotopy category.

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Each rational number is a (class of) pair of two integers (a, b) what we represent as $\frac{a}{b} = b^{-1}a$. The number b must be different from 0.

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Each morphism in the derived category is a (class of) pair of morphisms (f, g) what we represent as $g^{-1}f$. The morphism g must be a quasi-isomorphism. More formally a morphism from an object A to B is a (class of) pair of morphisms as in the diagram (roof):

$$\begin{tikzcd}
 & C \arrow[dl, Rightarrow, "g"'] \arrow[dr, "f"] & \\
A & & B
\end{tikzcd}$$

where C is any object and g is a quasi-isomorphism.

Morphisms

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Two roofs define the same morphism if they can be dominated by a common roof:

$$\begin{tikzcd}
 & E \arrow[dl, Rightarrow] \arrow[dr] & \\
C \arrow[d, Rightarrow] \arrow[drr] & & D \arrow[dll, Rightarrow] \arrow[d] \\
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For any two roofs, there exists a dominating roof:

$$\begin{tikzcd}
 & & H \arrow[dl, Rightarrow] \arrow[dr] & & \\
 & F \arrow[dl, Rightarrow] \arrow[dr] & & G \arrow[dl, Rightarrow] \arrow[dr] & \\
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in *the homotopy category*!

General remarks on the derived category

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As before we can identify objects of the abelian category with complexes (with cohomology) concentrated in degree 0.

Derived category of a point

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The derived category is very simple: we can consider only complexes with trivial differentials.
We can see that there is an *exceptional* object E in this category: the one dimensional vector space in degree 0. All other objects can be obtained by shifts and sums.
Note that $\mathrm{Hom}(E, E) = \mathbb{C}$ and $\mathrm{Hom}(E, E[k]) = 0$, where $[\cdot]$ is the shift and $k \neq 0$.

Derived category of an algebraic variety

In general exceptional objects

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are very helpful for understanding the structure of the derived category. If we can find an exceptional object then it generates a subcategory equivalent to $D^b(pt)$.
If we can find sufficiently many of them (with additional conditions on orthogonality) we have a much better understanding of the derived category.

Exceptional collections

Definition

A sequence of exceptional objects $(E_1, \dots, E_n)$ *is called an exceptional collection if* $\mathrm{Hom}_D(E_i, E_j[k]) = 0$ *for* $i > j$*. It is called strong if* $\mathrm{Hom}_D(E_i, E_j[k]) = 0$ *always when* $k \neq 0$*. It is called full if it generates the derived category.*

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Do full (strong) exceptional collections exist? If yes, what are the objects E_i?
In general full exceptional collections do not have to exist.
For a given variety these questions are hard to answer. It is an interesting problem to try to answer them for...

Our beloved toric varieties

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Theorem (-)

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Theorem (Efimov)

There exist smooth, toric Fano varieties with no full exceptional collections of line bundles.

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Why not?

By Efimov's result it is not enough to consider collections of line bundles.

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- Orthogonality
  For two coherent sheaves A and B we have $\mathrm{Hom}_{Der}(A, B[i]) = Ext^i(A, B)$.
- Generation
  If all elements, but one of an exact sequence are generated, then we can generate also the missing one. For toric varieties it is enough to generate line bundles.

Beilinson's theorem

Theorem (Beilinson)

The sequence $(\mathcal{O}(a), \mathcal{O}(a+1), \ldots, \mathcal{O}(a+n))$ *is a full strong exceptional collection on* $\mathbb{P}^n$ *for any integer* a*.*

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Moreover for $i < j$ we have $H^n(\mathcal{O}(j-i)) = 0$, thus the collection is strong. □

Beilinson's theorem continued

Proof.

Consider the Koszul exact sequence:

$$0 \to \mathcal{O}(-\sum D_i) \to \cdots \to \oplus\mathcal{O}(-D_i - D_j) \to \oplus_{i=1}^{n+1}\mathcal{O}(-D_i) \to \mathcal{O} \to 0.$$

Tensor by $\mathcal{O}(a+n+1)$, generating $\mathcal{O}(a+n+1)$. Analogously we generate all other line bundles. □

Beilinson's theorem continued

Proof.

Consider the Koszul exact sequence:

$$0 \to \mathcal{O}(-\sum D_i) \to \cdots \to \oplus \mathcal{O}(-D_i - D_j) \to \oplus_{i=1}^{n+1} \mathcal{O}(-D_i) \to \mathcal{O} \to 0.$$

Tensor by $\mathcal{O}(a+n+1)$, generating $\mathcal{O}(a+n+1)$. Analogously we generate all other line bundles. ☐

Theorem (Bondal, Costa, Efimov, Hille, Lasoń, Miró-Roig, Perling, -)

For many smooth complete toric varieties there exist full strongly exceptional collections of line bundles.

Positive results

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Theorem (Kawamata)

For any smooth, complete toric variety there exists a full, exceptional collection.

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For any smooth, complete toric variety there exists a full, exceptional collection.

Conjecture

Does a smooth, complete toric variety admit a full, strong exceptional collection?

Two important facts

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Fact 1:'Before Serre, just a few maestri who had spent all their lives contemplating the intricacies of the black arts could say when some restriction map was surjective, and all you could do was to believe them; after Serre, any idiot could write down exact sequences and deduce any number of such'
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Fact 2: Christmas is coming

Two important facts

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'Before <*your name*>, just a few maestri could describe derived categories of some algebraic varieties, and all you could do was to believe them; after <*your name*>, any idiot can deduce any number of such'
M. Michalek

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