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Name: Elizabeth Gross **Email/Phone:** egross7@uic.edu

Speaker's Name: Tim Roemer

Talk Title: Absolutely Koszul algebras

Date: 12 / 5 / 12   **Time:** 9 : 00 [✓ am] / pm (circle one)

List 6-12 key words for the talk: Koszul algebras, monomial ideals, minimal graded free resolution, componentwise linear, Poincaré series, absolutely Koszul, Koszul rings

Please summarize the lecture in 5 or fewer sentances:

Presents recent results on the class of absolutely Koszul rings.

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Absolutely Koszul algebras

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Joint work with

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Outline

(1) Koszul algebras
(2) The linear part of a minimal free resolution
(3) Absolutely Koszul algebras
(4) Standard operations and related invariants

Notation

- K is a field,
- R is a standard graded K-algebra,
- $R = S/I$ where $S = K[X_1, \ldots, X_n]$ is a standard graded polynomial ring and $I = \bigoplus_{i \geq 2} I_i \subset S$ is a graded ideal,
- $\mathfrak{m}_R$ the maximal graded ideal of R,
- M is always a finitely generated graded R-module with graded Betti numbers

$$\beta_{ij}^R(M) = \dim_K \operatorname{Tor}_i^R(M, K)_j$$

and total Betti numbers

$$\beta_i^R(M) = \sum_{j \in \mathbb{Z}} \beta_{ij}^R(M).$$

Koszul algebras

Definition

R is called Koszul if K has linear free resolution (as an R-module), i. e. $\beta_{ij}^R(K) = 0$ for $i \neq j$.

- So R is Koszul if and only if the minimal graded free resolution of K is of the form:

$$\cdots \to R(-i)^{\beta_i^R(K)} \to \cdots \to R(-1)^{\beta_1^R(K)} \to R \to K \to 0$$

- For example, if R is Koszul, then its defining ideal I is generated by quadrics. But the converse does not hold.

Finite projective dimension

To see the importance of Koszul algebras, recall the following famous result. At first let

$$\mathrm{pd}_R(M) = \sup\{i \in \mathbb{N} : \beta_i^R(M) \neq 0\}$$

be the projective dimension of a f. g. graded R-module M. Then:

Theorem (Auslander-Buchsbaum-Serre)

The following statements are equivalent:

(i) *R is regular, i. e. R is a standard graded polynomial ring;*

(ii) $\mathrm{pd}_R(M) < \infty$ *for every f.g. graded R-module M;*

(iii) $\mathrm{pd}_R(K) < \infty$.

Finite regularity

Let

$$\operatorname{reg}_R(M) = \sup\{j - i : \beta_{i\,j}^R(M) \neq 0\}$$

be the Castelnuovo-Mumford regularity of a f. g. graded *R*-module *M*. Then:

Theorem (Avramov-Eisenbud-Peeva)

The following statements are equivalent:

(i) *R is Koszul;*

(ii) $\operatorname{reg}_R(M) < \infty$ *for every f.g. graded R-module M;*

(iii) $\operatorname{reg}_R(K) < \infty$.

Questions

- Usually it is rather difficult to prove that a given algebra is Koszul, if there is no additional information available.
- It is an interesting problem to find and study special classes of Koszul algebras (e. g. G-quadratic algebras).
- Then one would like to decide whether an algebra belongs to such a special class, or how algebra operations behave within in this class.
- It is not totally obvious in which way Koszulness can be defined for a noetherian local ring, but interesting to do so.

The linear part of a minimal free resolution

We recall the following construction/definition due to Herzog and Iyengar ('05). Let

$$F\colon \cdots \to F_i \to \cdots \to F_1 \to F_0 \to 0$$

be the minimal graded free resolution of a f.g. graded module M. The (standard) filtration of F is given by the subcomplexes

$$\mathscr{F}^j(F) \text{ with } \mathscr{F}^j(F)_i = \mathfrak{m}^{j-i}F_i \text{ where } \mathfrak{m}^j = R \text{ for } j \leq 0, i.e.$$

$$\mathscr{F}^j(F)\colon \cdots \to F_{j+1} \to F_j \to \mathfrak{m}^1 F_{j-1} \to \cdots \to \mathfrak{m}^{j-1}F_1 \to \mathfrak{m}^j F_0 \to 0.$$

The associated graded complex $\operatorname{lin}(F)$ is called the linear part of F. This is a complex of $\operatorname{gr}_{\mathfrak{m}}(R)$-modules with

$$\operatorname{lin}_i(F) = \operatorname{gr}_{\mathfrak{m}}(F_i)(-i).$$

The linear part of a minimal free resolution

- Note that this construction and its consequences are also possible for local rings.
- In this talk we restrict ourself to the situation of a standard graded algebra R as considered above.
- Since here $R = \operatorname{gr}_{\mathfrak{m}}(R)$ the complex $\operatorname{lin}(F)$ can be described in an easy way:
  Just replace all entries in the matrices of the complex representing the homomorphisms of degree > 1 by 0.

Definition

We call

$$\operatorname{ld}_R(M) = \sup\{i \in \mathbb{N} : H_i(\operatorname{lin}(F)) \neq 0\}$$

the linearity defect of M.

The exterior algebra case

- The study of $\operatorname{ld}_R(M)$ was in particular motivated by results of Eisenbud, Fløystad and Schreyer ('03) about graded modules over the exterior algebra. For example:

- Let $E = K\langle e_1, \dots, e_n\rangle$ be a standard graded exterior algebra over K. Analogously one can define $\operatorname{ld}_E(M)$ for a f.g. graded E-module M. Then

$$\operatorname{ld}_E(M) < \infty$$

- But be careful! There is no global bound on $\operatorname{ld}_E(M)$.

- An analogous "commutative" algebra result for complete intersections was proved by Herzog and Iyengar.

Exterior face rings

- Let Δ be a non-trivial simplicial complex on $\{1,\ldots,n\}$. Analogous to a Stanley-Reisner ring one can define the exterior face ring $K\{\Delta\} = E/J_\Delta$. A result of Herzog-R shows that
$$\operatorname{ld}_E(K\{\Delta\}) \leq n-1.$$
- This was improved by Okazaki and Yanagawa ('07) who showed that the bound can be improved by 1 (for $n \geq 4$) and they also characterized when we have equality.
- Let Δ^* be the Alexander dual of Δ. Examples show that
$$\operatorname{ld}_E(K\{\Delta\}) + \operatorname{ld}_E(J_{\Delta^*})$$
is rather small compared to the trivial bound induced from above. An interesting question is what is here an optimal bound.

$\operatorname{ld}_R(M) = 0$

Back to the case of a standard graded algebra R. We have the following characterization for the case $\operatorname{ld}_R(M) = 0$.

Theorem (Herzog-Iyengar, Şega, R)

The following statements are equivalent:

- (i) $\operatorname{ld}_R(M) = 0$;
- (ii) $\operatorname{lin}(F)$ *is the minimal graded free resolution of* $\operatorname{gr}_{\mathfrak{m}}(M)$.

If R is Koszul, then (i) and (ii) are equivalent to

- (iii) *M is componentwise linear.*

Note that the equivalence of (i) and (ii) also holds in the local setting. For (iii) recall that M is componentwise linear iff

$$M_{\langle d \rangle} \text{ has a } d\text{-linear resolution for all } d.$$

Here $M_{\langle d \rangle}$ is the submodule of M generated by M_d.

$\operatorname{ld}_R(M) < \infty$

If $\operatorname{ld}_R(M) < \infty$, then we know the following (He-Iy'05):

- $\operatorname{reg}_R(M) < \infty$,
- The Poincaré series

$$P_M^R(t) = \sum_i \beta_i^R(M) t^i$$

  of M is rational and the denominator depends only (on the Hilbert series) of R.

Koszul algebras

Combining the results of Herzog-Iyengar and Avramov-Eisenbud-Peeva one gets:

Theorem

The following statements are equivalent:

(i) *R is Koszul;*

(ii) $\operatorname{ld}_R(K) = 0$*;*

(iii) $\operatorname{ld}_R(K) < \infty$*.*

Be careful. There exists Koszul algebras (Roos '05) where

$$\operatorname{ld}_R(M) = \infty \text{ for some } M$$

with non-rational Poincaré series. In the local case it is an interesting question whether $\operatorname{ld}_R(K) < \infty$ implies $\operatorname{ld}_R(K) = 0$.

Absolutely Koszul algebras

We saw that even Koszul algebras may fail the property that $\operatorname{ld}_R(M) < \infty$ for all M. This is one motivation for:

Definition

We say that R is absolutely Koszul if

$$\operatorname{ld}_R(M) < \infty \text{ for all f.g. graded } R\text{-modules } M.$$

We see immediately from above:

Corollary

If R is absolutely Koszul, then R is Koszul.

Questions

- Which classes of algebras are absolutely Koszul?
- Relate this property to other “Koszul” properties.
- What is the behavior of the property “absolutely Koszul” with respect to standard operations in algebra?

Classes of absolutely Koszul algebras I

Theorem (Herzog-Iyengar '05)

If $R = S/I$ is either

- *a complete intersection of quadrics, or*
- $\operatorname{reg}_S(R) = 1$,

then R is absolutely Koszul.
In the second case we have

$$\operatorname{gl}\operatorname{ld}(R) = \sup\{\operatorname{ld}_R(M)\} < \infty.$$

In the first case $\operatorname{gl}\operatorname{ld}(R)$ *is finite if and only if R is a hypersurface.*

Certain artinian Gorenstein algebras with short Hilbert-function are also known to be absolutely Koszul by recent results of Henriques and Şega ('11). But very little is known beside this.

Classes of absolutely Koszul algebras II

In some cases we can say a little bit more:

Theorem (Conca-Iyengar-Nguyen-R)

Let $I \subset S$ be a squarefree monomial ideal with a 2-linear resolution. Let $L \subset S$ be a quadratic monomial c.i. such that I does not contain any minimal generator of L. Then $S/(I+L)$ is an absolutely Koszul algebra.

Sketch of the proof.

The key step of the proof is to show that

$$\operatorname{reg}_{S/L} S/(I+L) \leq \operatorname{reg}_S S/I = 1.$$

Hence $(I+L)/L$ has an 2-linear resolution over the complete intersection S/L. This implies that $S/L \to S/(I+L)$ is Golod and $S/(I+L)$ is Koszul (He-Iy '05). This concludes the proof by another result of Herzog-Iyengar ('05). □

Absolutely- and universally Koszul algebras I

The last result can be used in the following situation. Recall the following definition of Conca ('00):

Definition

R is called universally Koszul, if every ideal generated by linear forms has a linear resolution over R.

Universally Koszul algebras are very special. So one could hope that there is a relationship to absolutely Koszul algebras.

Absolutely- and universally Koszul algebras II

Theorem (Conca-Iyengar-Nguyen-R)

Let R be a universally Koszul algebra defined by quadratic monomials. Then R is absolutely Koszul.

Sketch of the proof.

The key fact is that Conca ('02) classified monomial universally Koszul algebras. From

$$K[X_1,\dots,X_m]/((X_1,\dots,X_{m-1})^2+X_m^2)$$

they are all obtained by taking polynomial extensions and fiber products and by one more operation (in char(K)=2). Now one can conclude the proof by checking that we can write the defining ideal in each step as $I+L$ where L is a complete intersection of squares and I is a squarefree monomial ideal with 2-linear resolution as needed above. $\square$

Absolutely- and universally Koszul algebras III

The last result indicates that the following might be true.

Question: Is it always true that an universally Koszul algebra is absolutely Koszul?

Tensor products

The property “absolutely Koszul” is not well behaved with respect to all operations in algebra as the following example shows:

Let

$$R = K[X, Y]/(X, Y)^2.$$

Then R is absolutely Koszul. But the tensor product $R \otimes_k R$ appears in the list of bad Koszul algebras of Roos (’05). So tensor products in general do not respect absolutely Koszulness.

Other algebra operations

But we have:

Theorem (Conca-Iyengar-Nguyen-R)

Let R and T be standard graded K-algebras. Then:

(i) *The fiber product $R \times_K T$ is absolutely Koszul if and only if R and T are absolutely Koszul.*

(ii) *A polynomial ring extension $R[X]$ is absolutely Koszul if and only if R is absolutely Koszul.*

(iii) *If there exists a surjective homomorphism of rings $R \to T$ such that $\operatorname{pd}_R(T) < \infty$ and T is absolutely Koszul, then also R is absolutely Koszul.*

Veronese subalgebras

Question: Let R be absolutely Koszul and $d \geq 1$. Is the Veronese subring $R^{(d)}$ absolutely Koszul ?

We have only answers in special cases. For example, for $d = 2$ and $n \leq 6$ this question has a positive answer. This proof is already complicated, but the method does not work for larger n.

The same question might be asked for example, for Segre products.

Bounds for $\operatorname{gl\,ld}(R)$

- We saw that sometimes $\operatorname{gl\,ld}(R) = \infty$ even if $\operatorname{ld}_R(K) < \infty$.
- $\operatorname{gl\,ld}(R) < \infty$ should be rather restrictive. For example, if R is Gorenstein, then this can only happen, if R is a hypersurface (He-Iy '05).
- A lower bound is given by

  $$\operatorname{gl\,ld}(R) \geq \operatorname{ld}_R(M) + \operatorname{depth}(M)$$

  for any finitely generated graded R-module M. Choosing $M = R$ yields

  $$\operatorname{gl\,ld}(R) \geq \operatorname{depth}(R)$$

  We expect that $\dim(R)$ is also a lower bound which we can prove in the monomial situation.
- At least for polynomial ring extensions we can prove

  $$\operatorname{gl\,ld}(R[X]) = \operatorname{gl\,ld}(R) + 1.$$

Summary

- Absolutely Koszul algebras are interesting to study and there are many open problems to consider also in the context of rings defined by monomial ideals.
- These algebras and more generally, the invariant “ld” are a good way to consider the Koszul property for noetherian local rings.

References

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- J. Herzog and S. Iyengar, *Koszul modules*. J. Pure Appl. Algebra **201** (2005), 154–188.
- S.B. Iyengar and T. Römer, *Linearity defects of modules over commutative rings*. J. Algebra **322** (2009), 3212–3237.