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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Elizabeth Gross **Email/Phone:** egross7@uic.edu

Speaker's Name: Daniel Erman

Talk Title: Duality in Boij-Soederberg Theory

Date: 12/5/12 **Time:** 10:30 am/ pm (circle one)

List 6-12 key words for the talk: Boij-Soederberg Theory, Hilbert polynomial, Betti table, cohomology table, decomposition of Betti tables, Eisenbud-Schreyer functionals

Please summarize the lecture in 5 or fewer sentances:

Describes connection between the Betti table of the free complex & the cohomology table of a coherent sheaf.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Question:

1. What is the precise nature of the duality between syzygies & vector bundles?
2. What happens for complexes of modules?
3. What about other graded rings & varieties?

$S = k[x_0, \dots, x_n]$

F bounded graded free S-complex:

$$F = [F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_p \leftarrow 0]$$

- F_i is a free finitely generated S-module
- $\phi_i \cdot \phi_{i+1} = 0$

$F \in D^b_{gr}(S)$ bounded derived category of graded S-mod

$\beta_{ij}(F) = \dim \operatorname{Tor}_i(F, k)_j$

$\beta(F) = (\beta_{ij}(F)) \in \bigoplus_{i,j} \mathbb{Q}$

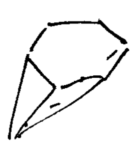
$B^c(S) = \mathbb{Q}_{\geq 0}\{\beta(F) \mid \operatorname{codim} F \geq c\} \subseteq \bigoplus_{i,j} \mathbb{Q}$

$\operatorname{codim} F := \operatorname{codim}(\bigoplus_{i \in \mathbb{Z}} H_i(F))$

$B^c(S)$

$\uparrow$

cone of betti tables



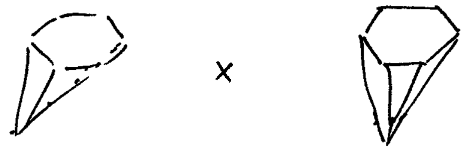
$\mathbb{P}^n$. $\mathcal{E}$ coherent sheaf on $\mathbb{P}^n$

$$\gamma_{ij}(\mathcal{E}) := \dim H^i(\mathbb{P}^n, \mathcal{E}(j))$$ ②

$$\gamma(\mathcal{E}) = (\gamma_{ij}(\mathcal{E})) \in \prod_{i,j} \mathbb{Q}$$ cohomology table of $\mathcal{E}$.

$$C^{n+1-c}(\mathbb{P}^n) = \mathbb{Q}_{\geq 0}\{\gamma(\mathcal{E}) \mid \operatorname{codim} \operatorname{supp} \mathcal{E} \geq n+1-c\} \subseteq \prod_{i,j} \mathbb{Q}$$

$B^c(S)$ $\qquad\qquad$ $C^{n+1-c}(\mathbb{P}^n)$



the study of these cones should be closely related

Ex $\quad S = k[x,y] \qquad \mathcal{E}$ sheaf on $\mathbb{P}^1$

$$F = [\, S^1 \xleftarrow{(x,y)} S^2(-1) \xleftarrow{\binom{y}{-x}} S^1(-2) \leftarrow 0 \,]$$

$$\tilde{F} = [\, 0 \leftarrow \mathcal{O}_{\mathbb{P}^1} \leftarrow \mathcal{O}^2(-1) \leftarrow \mathcal{O}(-2) \leftarrow 0 \,]$$ short exact sequence

$$\tilde{F} \otimes \mathcal{E} = [\, 0 \leftarrow \mathcal{E} \leftarrow \mathcal{E}^2(-1) \leftarrow \mathcal{E}(-2) \leftarrow 0 \,]$$

$$0 \to H^0(\mathcal{E}(-2)) \to H^0(\mathcal{E}^2(-1)) \to H^0(\mathcal{E})$$
$$\to H^1(\mathcal{E}(-2)) \to H^1(\mathcal{E}^2(-1)) \to H^1(\mathcal{E}) \to 0$$

we get the following information

- $\gamma_{0,-2} \geq 0$
- $2 \cdot \gamma_{0,-1} - \gamma_{0,-2} \geq 0$
- $\gamma_{00} - 2\gamma_{0,-1} + \gamma_{0,-2} \geq 0$

$\vdots$

One Betti table $\beta(F)$, $\dim F = 0$ $\longrightarrow$ many inequalities on $\gamma(\mathcal{E})$, any $\mathcal{E}$.

other direction

many inequalities on Betti tables $\beta(F)$, $\dim F = 0$ $\longleftarrow$ 1 cohomology table

Duality Pairs

$$\Phi : D^b_{gr}(S) \times D^b(\mathbb{P}^n) \to D^b_{gr}(k[t])$$

F

$$\begin{array}{ccccc} \mathbb{P}^n & \xleftarrow{p_1} & \mathbb{P}^n \times \operatorname{Spec}(k[t]) & \xrightarrow{\text{“}\Sigma\text{''}} & \mathbb{A}^{n+1} \\ & & \downarrow p_2 & & \\ & & \operatorname{Spec}(k[t]) & & \end{array}$$

$$\Phi(F, \mathcal{E}) = R_{p_{2*}}(\Sigma^* F \overset{b}{\otimes} p_1^* \mathcal{E})$$

Ex

$$\Sigma^*(S \xleftarrow{f} S(-\deg f))$$

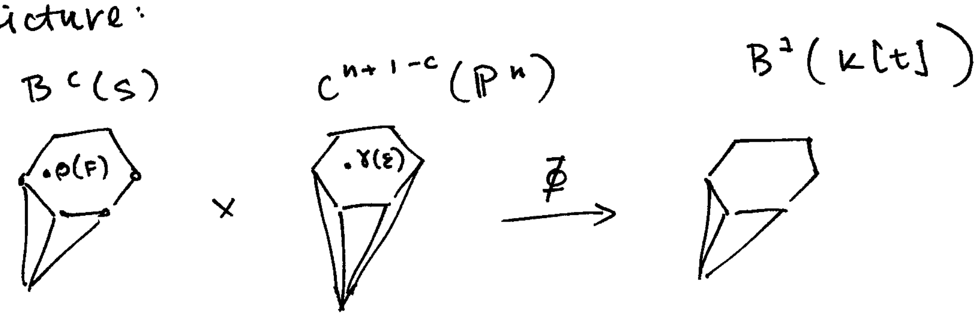
$$= \mathcal{O}_{\mathbb{P}^n} \boxtimes k[t] \xleftarrow{t^{\deg f} \cdot f} \mathcal{O}(-\deg f) \boxtimes k[t](-\deg f)$$

Thm (1) The $k[t]$-Betti table of $\Phi(F, \mathcal{E})$ only depends on $B(F)$ & $\gamma(\mathcal{E})$

(2) If $\operatorname{codim} F + \operatorname{codim} \mathcal{E} \geq n+1$, then the Betti table of $\Phi(F, \mathcal{E})$ lies in $B'(k[t])$

(3) Fix $v \in \bigoplus_{i,j} \mathbb{Q}$. v lies in $B^c(S)$ iff $\Phi(v, \mathcal{E})$ has the Betti table of a codim 1 complex $\forall \mathcal{E}$ with $\operatorname{codim} \mathcal{E} = n+1-c$.

New picture:



Decomposition Betti tables of complexes

Defn (shifted) pure diagram of type

$(e,d) = (e, (d_0, \dots d_p)) \in \mathbb{Z} \oplus \mathbb{Z}^{p+1}$ is a

Betti table of the form $\beta(F^d[e])$ where

$F^d = [S^\bullet(-d_0) \leftarrow S^\bullet(-d_1) \leftarrow \cdots \leftarrow S^\bullet(-d_p) \leftarrow 0]$

is a pure resolution of a C-M module.

Input
- a sequence $(\cdots, c_{-1}, c_0, c_1, \dots) \in \bigoplus_{-\infty}^{\infty} \mathbb{Z}$ where $c_i \le c_{i+1}$
- a complex F

Output A positive, rational decomposition:

$$\beta(F) = \sum_{(e,d)} a_{(e,d)} \cdot \beta(F^d[e]) \quad \text{where}$$

$a_{(e,d)} \in \mathbb{Q}_{\ge 0}$ and $c_\ell \le \operatorname{codim} H_\ell(F^d[e]) \le c_{\ell+1}$.

Ex $S = k[x,y,z]$ $I = (x^2, xy, y^2, xz)$, $J = (xy)$

F' = min'l free resolution of S/I F'' = min'l free res of S/J

$F = F' \otimes F'' \leftarrow$ computes $\operatorname{Tor}^\bullet(S/I, S/J)$

$$\operatorname{codim} H_i(F) = \begin{cases} 2 & \text{if } i = 0,1 \\ \infty & \text{else} \end{cases}$$

5

$$B(F) = \begin{pmatrix} 1 & - & - & - & - \\ - & \underline{5} & \underline{4} & \underline{1} & - \\ - & - & 4 & 4 & \underline{1} \end{pmatrix}$$

$e = 1 \quad d = (2, 3, 4, 6)$

$$= \begin{pmatrix} - & - & - & - & - \\ - & 1/2 & 4/5 & 1 & - \\ - & - & - & - & 1/6 \end{pmatrix} + \begin{bmatrix} - & - & - & - & - \\ - & 5/6 & 5/3 & - & - \\ - & - & - & 5/3 & 5/6 \end{bmatrix}$$

$$+ \begin{bmatrix} - & - & - & - & - \\ - & 2/3 & 1 & - & - \\ - & - & - & 1/3 & - \end{bmatrix} + \begin{bmatrix} - & - & - & - & - \\ - & 1 & - & - & - \\ - & - & 3 & 2 & - \end{bmatrix} + \begin{bmatrix} 1 & - & - & - & - \\ - & 2 & - & - & - \\ - & - & 1 & - & - \end{bmatrix}$$