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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Elizabeth Gross Email/Phone: egross7@uic.edu

Speaker's Name: Christine Berkesch

Talk Title: Euler - Koszul homology for hypergeometric systems

Date: 12 / 5 / 2012 Time: 11 : 30 am / pm (circle one)

List 6-12 key words for the talk: A-hypergeometric systems, Euler - Koszul homology, D-modules, toric varieties, Euler operators, resonance, rank jumps, GKZ - hypergeometric system

Please summarize the lecture in 5 or fewer sentences:

Introduces A-hypergeometric systems & their connection to toric ideals. Discusses the parametric behavior of their solution spaces.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Euler-Koszul homology for hypergeometric systems

$$\left.\begin{array}{l} A \in \mathbb{Z}^{d\times n} \\ \beta \in \mathbb{C}^d \end{array}\right\} \longrightarrow M_A(\beta) \quad \text{GKZ-system at } \beta$$

<u>Question</u> How do solutions vary with β?

① Systems

$$\mathbb{C}[\underline{x}] = \mathbb{C}[x_1, \dots, x_n], \quad \partial_i := \partial/\partial x_i$$

$$D = \frac{\mathbb{C}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle}{[x_i, x_j] = 0 = [\partial_i, \partial_j],\ [\partial_i, x_j] = \delta_{ij}} \quad \text{Weyl algebra}$$

J left D-ideal

$f \in \mathcal{O}^{an}$ is in $\mathrm{Sol}(D/J)$ if $P \cdot f = 0 \ \forall P \in J$

As $\mathbb{C}$-vs

$$\mathrm{Sol}(D/J) \cong \mathrm{Hom}_D(D/J, \mathcal{O}_x^{an})$$

$$\phi(1) \longleftarrow\!\shortmid\ \phi$$

$\therefore$ $\mathrm{Sol}(-)$ is contravariant

$A = (a_1, \dots, a_n) \in \mathbb{Z}^{d\times n}$, $\mathbb{Z}A = \mathbb{Z}^d$, $\mathbb{R}_{\geq 0}A$ pointed

<u>toric ideal</u>

$$I_A = \langle \partial^u - \partial^v \mid Au = Av \rangle \subseteq \mathbb{C}[\partial]$$

A-grading: $\deg(\partial_i) = a_i = -\deg(x_i)$

Euler operators

$$\left.\begin{array}{l} \underset{1\le i\le d}{E_i} := \sum_{j=1}^{n} a_{ij} x_j \partial_j \\ \beta \in \mathbb{C}^d \end{array}\right\} \to E-\beta = E_1-\beta_1, \ldots, E_d-\beta_d$$

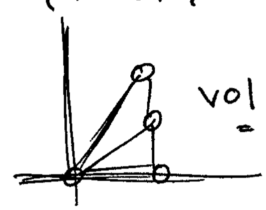
$$x_j \partial_j \cdot x^k = k_j x^k$$

$$\mathrm{Sol}\left(D/\langle E-\beta\rangle\right) \supseteq \{ x^k \mid Ak = \beta \}$$

GKZ-system at β:

$$M_A(\beta) = D/D\cdot I_A + \langle E-\beta\rangle$$

faces: A, $\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$, $\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$, $\emptyset$



$\mathrm{vol}(A) = 2$

Ex:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$M_A(\beta) = D/D\cdot\langle \partial_1\partial_3 - \partial_2^2,\ x_1\partial_1 + x_2\partial_2 + x_3\partial_3 - 0,\ x_2\partial_2 + 2x_3\partial_3 + 1 \rangle$$

$$\mathrm{Sol}(M_A(\beta)) = \mathrm{span}_{\mathbb{C}} \left\{ \frac{-x_2 \pm \sqrt{x_2^2 - 4x_1x_3}}{2x_3} \right\}$$

$$0 = x_3 T^2 + x_2 T + x_1$$

$$\mathrm{rank}\ D/J := \dim_{\mathbb{C}} \mathrm{Sol}(D/J)$$

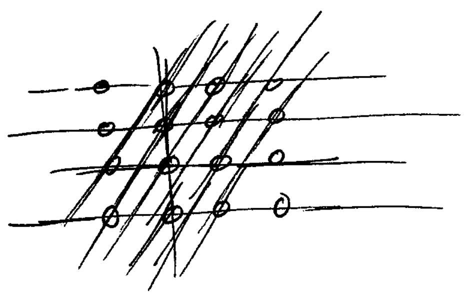
[Gelfand - Kapranov - Zelevinsky, Adolphson
Matusevich - Miller - Walther]:

$$\mathrm{vol}(A) \le \mathrm{rank}\, M_A(\beta) < \infty \qquad \forall A, \beta$$
$$= \text{generic } \beta$$

A <u>face</u> of A is a subset $F \subseteq \mathrm{col}(A)$
st $\mathbb{R}_{\geq 0} F \leq \mathbb{R}_{\geq 0} A$ & $\mathbb{R}F \cap A = F$

<u>Generic</u> β:

$$\beta \notin \mathrm{Res}(A) := \bigcup_{F \lneq A} (\mathbb{Z}^d + \mathbb{C}F)$$

$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ 

② Euler - Koszul homology

$$M_A(\beta) = D / D \cdot I_A + \langle E - \beta \rangle = H_0\left(\frac{\mathbb{C}[\partial]}{I_A}, \beta\right)$$

P A-homogeneous, $P \in$ some left D-module

$$(E_i - \beta_i) \circ P := (E_i - \beta_i + \deg_i(P))P$$
$$(\text{If } P \in D: = P(E_i - \beta_i))$$

Euler - Koszul homology:

N A-graded $\mathbb{C}[\partial]$-mod

$$\mathcal{H}_\bullet(N, \beta) := H_\bullet \mathrm{Kos}_\bullet(D \otimes_{\mathbb{C}[\partial]} N ; E - \beta)$$

N finitely generated

$$\deg(N) := \{\alpha \in \mathbb{Z}^d \mid N_\alpha \neq 0\}$$

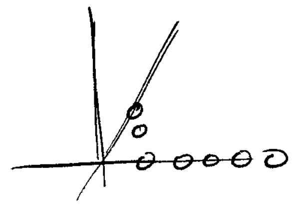
Ex $\quad S_A := \dfrac{\mathbb{C}[\partial]}{I_A} \cong \mathbb{C}[\mathbb{N}A] \qquad \deg(S_A) = \mathbb{N}A$

toric algebra

$F \leq A \quad S_F := \mathbb{C}[\partial]/I_F + \langle \partial_i \mid a_i \notin F\rangle \cong \mathbb{C}[\mathbb{N}F]$

$$\deg(S_F) = \mathbb{N}F$$

$$\operatorname{qdeg}(N) := \overline{\{\alpha \in \mathbb{Z}^d \mid N_\alpha \neq 0\}}^{\mathrm{Zar}} \subseteq \mathbb{C}^d$$



$$\operatorname{qdeg}(S_F) = \mathbb{C}F$$

[MMW]:

(i) $H_\bullet(N,\beta) = 0 \iff \beta \notin \operatorname{qdeg}(N)$

(ii) $H_{>0}(N,\beta) = 0 \quad \forall \beta$

$\iff$ N CM S_A-mod of dim d

$\iff \mathcal{E}_A(N) := -\operatorname{qdeg}\left(\bigoplus_{i=0}^{d-1} \operatorname{Ext}^{n-i}_{\mathbb{C}[\partial]}(N, \mathbb{C}[\partial])(-\sum a_i)\right)$
$= \emptyset$

(3) Resonance

[Schulze - Walther]

$\exists F$ st $\beta \in \mathbb{Z}^d + \mathbb{C}F$ $\iff$ $M_A(\beta)$ has reducible monodromy

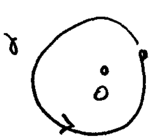
w/ $\mathrm{val}(F) < \mathrm{val}(A)$

$\mathbb{C}^n$:

$f_1, \dots, f_r \in \operatorname{Sol}(M_A(\beta))$

near 0

γ



$f_i \mapsto \sum_j m_{ij} f_j$

$$0 \to P_F \to S_A \to S_F \to 0$$

$$\cdots \to H_0(S_A, \beta) \to H_0(S_F, \beta) \to 0$$

$$\mathrm{Sol}(M_A(\beta)) \hookleftarrow \mathrm{Sol}(H_0 S_F)$$

"usually" $\mathrm{vol}(A) > \mathrm{vol}(F)$

(4) Rank jumps

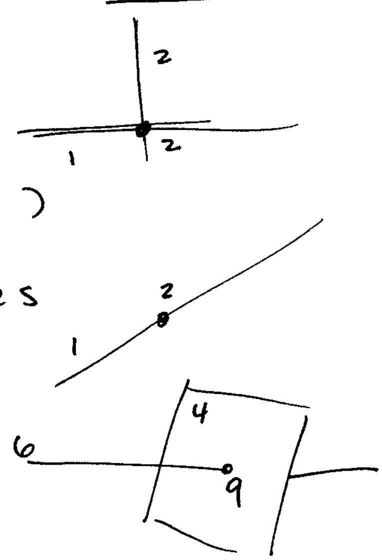
$$j(\beta) := \mathrm{rank}\, M_A(\beta) - \mathrm{vol}(A)$$

[MMW]: $\{\beta \mid j(\beta) > 0\} = \mathcal{E}_A(S_A)$

[Cattani, D'Andrea, Dickenstein]:

$$A = \begin{pmatrix} 1 & \text{—} & 1 \\ * & \text{—} & * \end{pmatrix} \Rightarrow j(\beta) \leq 1$$

some exs



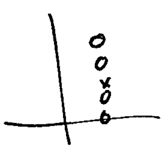
[Okuyama] $d = 3$, computed $j(\beta)$ from certain graphs

[B–]: $\forall A, \beta$, $\exists$ formula for $j(\beta)$ in terms of certain lattice translates

(vol(F), $[\mathbb{R}F \cap \mathbb{Z}^d : \mathbb{Z}F]$)

Ex

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$$



$$\mathcal{E}_A(S_A) = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

$$0 \to S_A \to \bar{S}_A \to Q \to 0, \quad Q = \frac{\mathbb{C}[\partial]}{\langle \partial \rangle} \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$0 \to H_1(Q, \beta) \to M_A(\beta) \to H_0(\bar{S}_A, \beta) \to H_0(Q, \beta) \to 0$$

$\beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$: 2 (new) $\leftarrow \cdots$ 5 $\leftarrow \cdots$ 4 ("generic") $\leftarrow \cdots$ 1 (disappears)

$$0 \to S_A \to \bar{S}_A \to Q \to 0$$

$$j(\beta) = \overset{\text{rank}}{\mathcal{H}_1}(Q, \beta) - \operatorname{rank} H_0(Q, \beta)$$

$$\text{"}Q\text{"} \leftrightarrow \bigoplus_{F_i} S_{F_i}(-\alpha_i)$$

⑤ Euler - Mellin integrals

$$A = \begin{pmatrix} 1 & \text{---} & 1 \\ \alpha_1 & \text{---} & \alpha_n \end{pmatrix} \qquad \alpha_i \in \mathbb{Z}^{d-1}$$

$$\Phi := \Phi(x, \beta) = \begin{pmatrix} \text{Gamma} \\ \text{functions} \\ \text{in } \beta \end{pmatrix} \cdot \int_{\mathbb{R}^{d-1}_{\geq 0}} \frac{z_1^{-\beta_2} \cdots z_{d-1}^{-\beta_d}}{\left(\sum x_i z^{\alpha_i}\right)^{-\beta_0}} \frac{dz_1 \wedge \cdots \wedge dz_{d-1}}{z_1 \cdots z_{d-1}}$$

[B - , Forsgård, Passare]

Φ is an entire function in β.

* $d = 2$: basis of EM integrals for generic β
* Φ captures resonance
* Φ 〃 rank-jumping special solutions

<u>Ex</u>

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Phi = (2 - 4\beta_1 + \beta_2) \frac{x_2^2}{x_1} + (2 - \beta_2) \frac{x_3^2}{x_4} +$$
$$(2 - 4\beta_1 + \beta_2)(2 - \beta_2)\bar{\Psi}$$