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17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Elizabeth Gross **Email/Phone:** egross7@uic.edu

Speaker's Name: Thomas Kahle

Talk Title: The combinatorics of binomial ideals

Date: 12/6/12 **Time:** 9:00 (am)/ pm (circle one)

List 6-12 key words for the talk: binomial ideals, mesoprimary decomposition, primary decomposition, commutative monoids, monoid congruences, monomial localization

Please summarize the lecture in 5 or fewer sentances:

Introduces mesoprimary decomposition for commutative monoid congruences. Explains how the theory can be adapted for decomposing binomial ideals.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

The combinatorics of binomial ideals

Thomas Kahle (MSRI)
(joint with Ezra Miller, Duke)

December 6, 2012

No Theorem

Let $\Bbbk$ be a field. Every binomial ideal in $\Bbbk[x_1, \ldots, x_n]$ is an intersection of primary binomial ideals.

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Theorem (Lasker/Noether 1905/1921)

Let $\Bbbk$ be a field. Every binomial ideal in $\Bbbk[x_1, \ldots, x_n]$ is an intersection of primary ideals.

Theorem (Eisenbud/Sturmfels 1996)

Let $\Bbbk$ be an algebraically closed field. Every binomial ideal in $\Bbbk[x_1, \ldots, x_n]$ is an intersection of primary binomial ideals.

"If one has never done any calculations, one would be inclined to say – let's extend $\mathbb{k}$ as far as needed to split our algebraic set. *That is a very bad idea!*"

Bayer / Mumford
"What can be computed in algebraic geometry"

Monoids

- In this talk $(Q, +)$ is a commutative Noetherian monoid.

Monoid Algebra

Let $\Bbbk$ be a field. The monoid algebra over Q is the $\Bbbk$-vector space

$$\Bbbk[Q] := \bigoplus_{q \in Q} \Bbbk \{\mathbf{t}^q\} \qquad \text{with} \qquad \mathbf{t}^q \mathbf{t}^u := \mathbf{t}^{q+u}.$$

A binomial ideal is an ideal generated by binomials

$$\mathbf{t}^q - \lambda \mathbf{t}^u, \quad q, u \in Q, \lambda \in \Bbbk.$$

Polynomial rings

$\Bbbk[x_1, \ldots, x_n] = \Bbbk[\mathbb{N}^n]$

$\rightarrow$ Fix the no theorem combinatorially.

Congruence basics

Definition

A congruence on Q is an equivalence relation $\sim$ such that

$$a \sim b \Rightarrow a + q \sim b + q \quad \forall q \in Q$$

The quotient $\overline{Q} := Q/\sim$ is a monoid again.

Congruences from binomial ideals

Each binomial ideal $I \subseteq \Bbbk[Q]$ induces a congruence $\sim_I$ on Q:

$$a \sim_I b \Leftrightarrow \exists \lambda \neq 0 : \mathbf{t}^a - \lambda \mathbf{t}^b \in I$$

Monomial ideals?

Let $T \subseteq Q$ be a monoid ideal. The monomial ideal $\langle \mathbf{t}^q : q \in T \rangle$ induces the Rees congruence identifying all elements of T.

Congruence basics

Special congruences

- The identity congruence: $\{(q,q) : q \in Q\}$: $\overline{Q} = Q$
- The universal congruences: $Q \times Q : \overline{Q} = \{0\}$.

Lemma

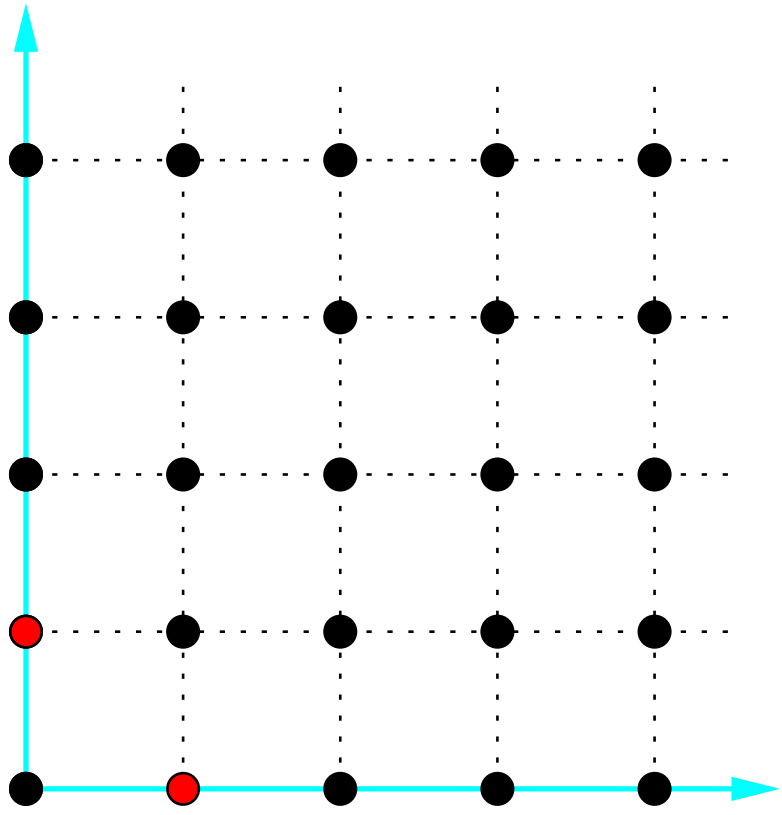
- Congruences are exactly the kernels of monoid morphisms:

$$\ker \phi = \left\{(u,v) \in Q^2 : \phi(u) = \phi(v)\right\}$$

- Every commutative Noetherian monoid is presented as $Q = \mathbb{N}^n/\sim$.

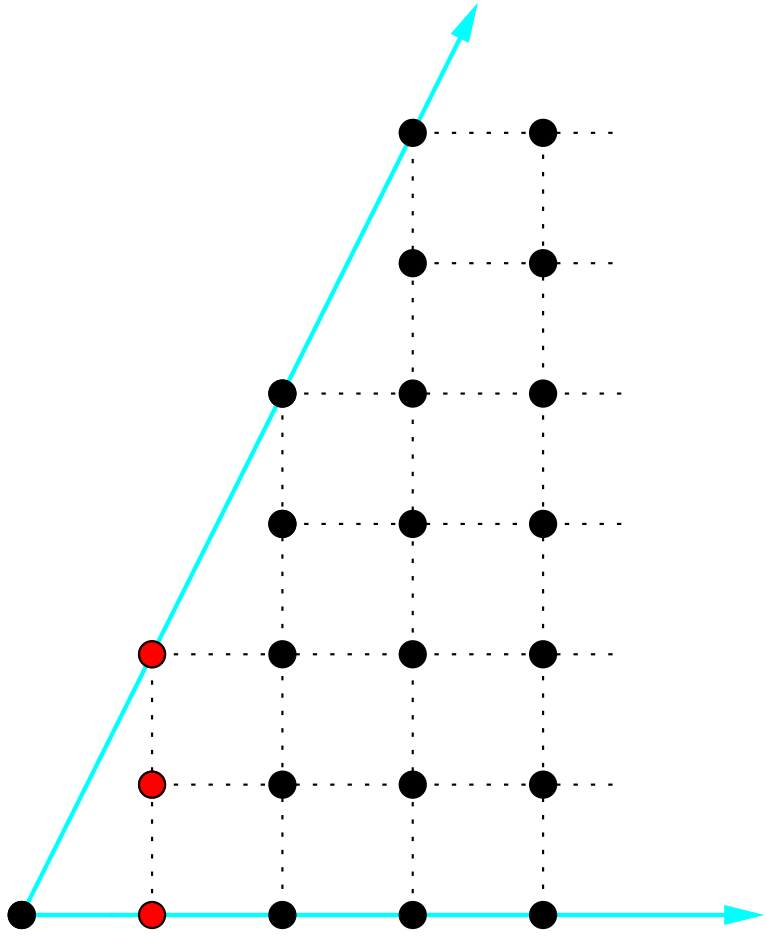
$\rightarrow$ understand monoids really well

The mother of all monoids: $\mathbb{N}^n$

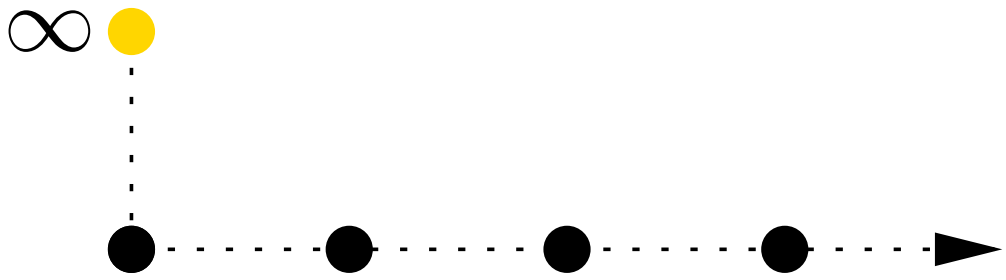


$\mathbb{N}^2$

Affine semigroups



Weird monoids

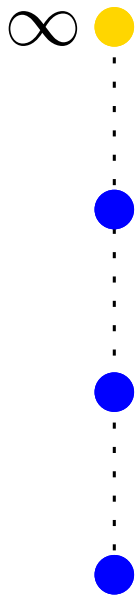


$\mathbb{N} \cup \{\infty\}$

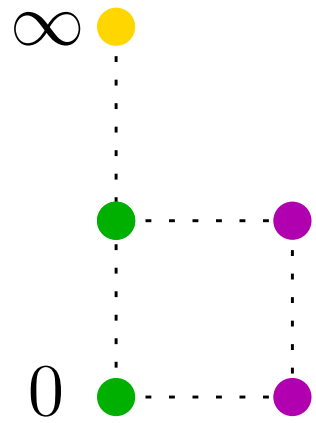
Nil

An element $q \in Q$ is nil if $q + a = q$ for all $a \in Q$.

Weird monoids

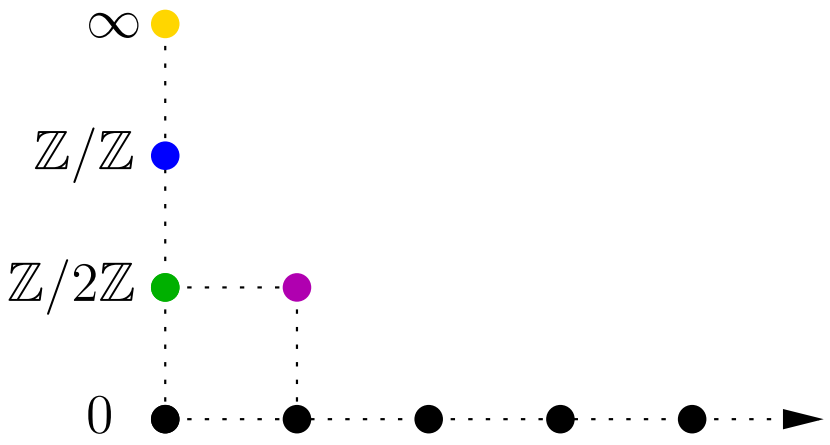


$\mathbb{N}/(3+\mathbb{N})$

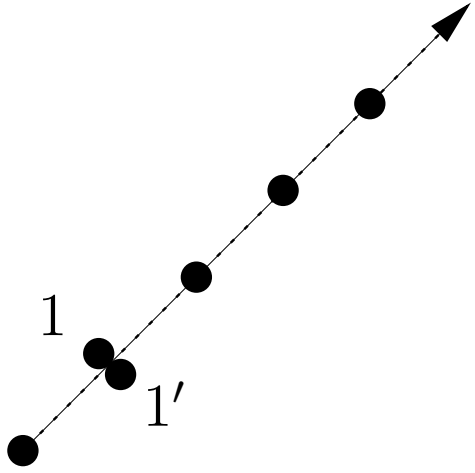


$(\mathbb{Z}/2\mathbb{Z}\times\mathbb{N})/(2+\mathbb{N})$

Weird monoids

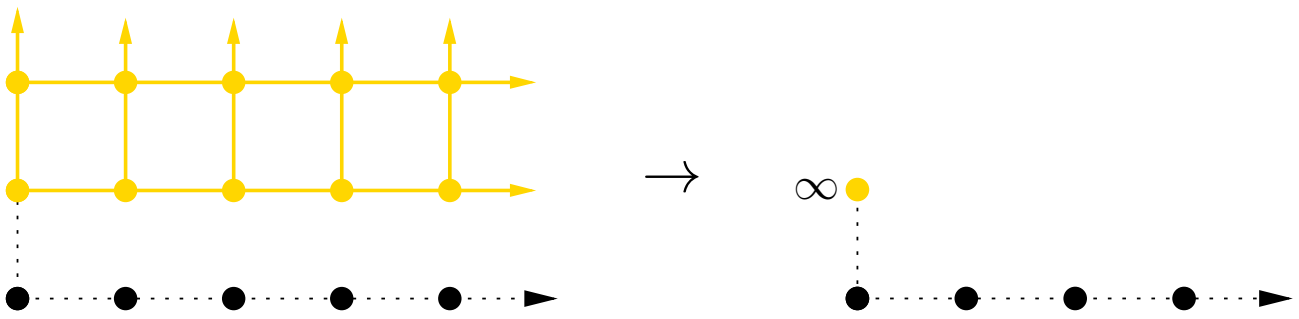


A weird monoid

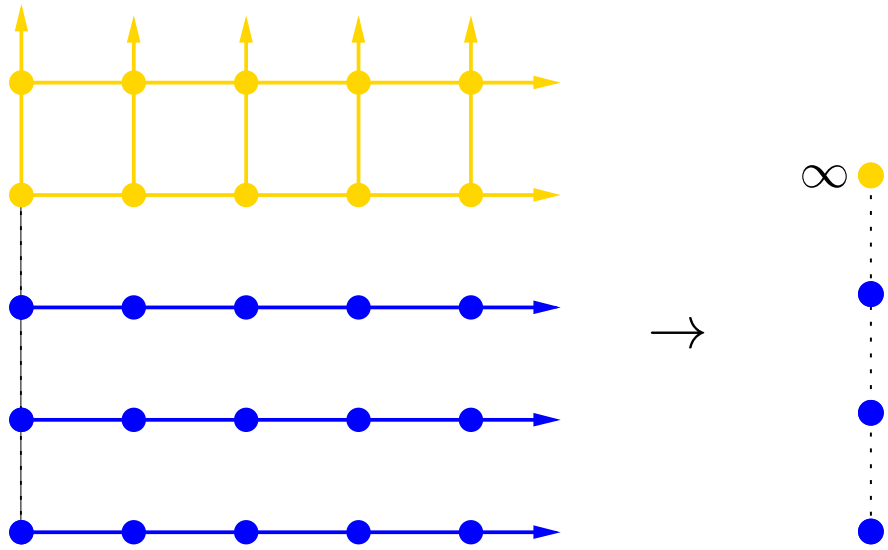


$\mathbb{N}$ with one doubled

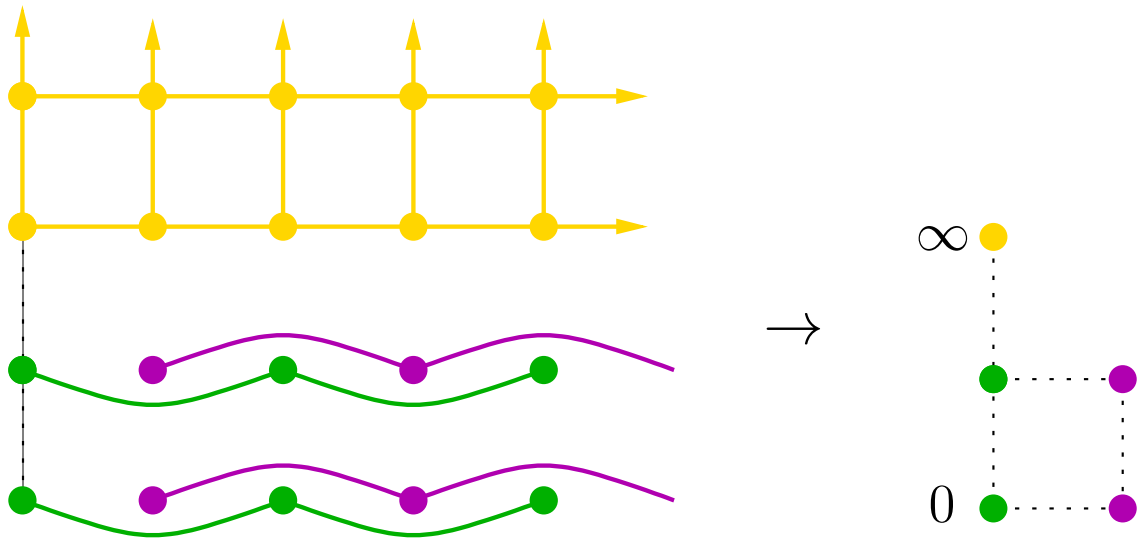
Presenting weird monoids



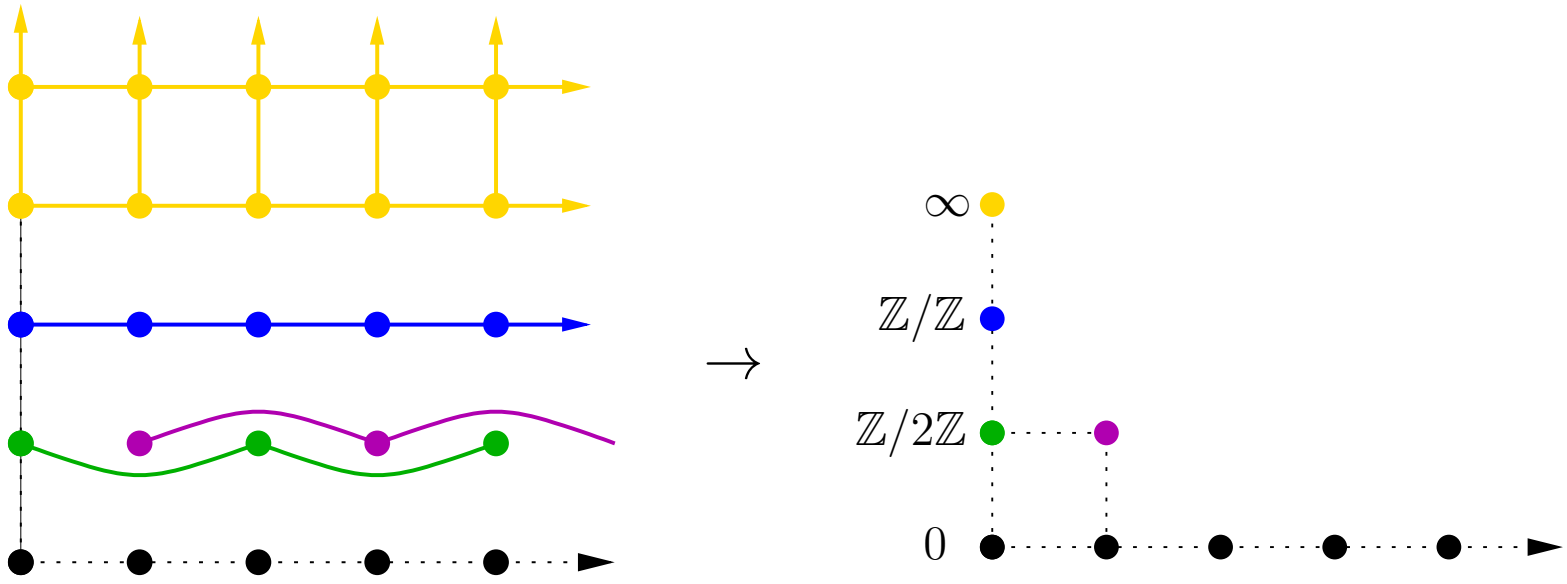
Presenting weird monoids



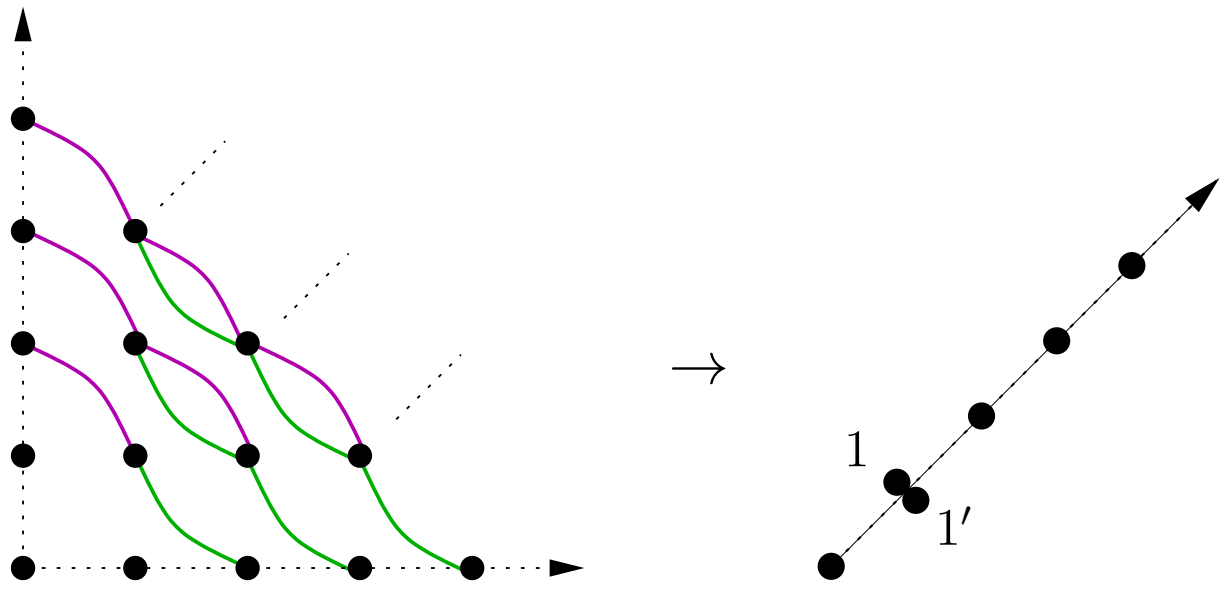
Presenting weird monoids



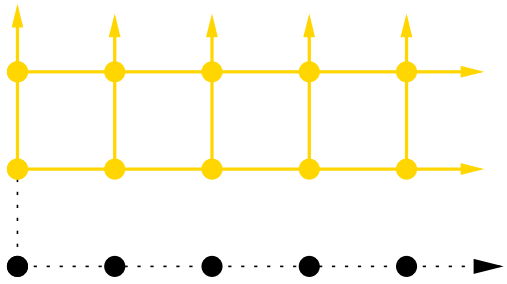
Presenting weird monoids



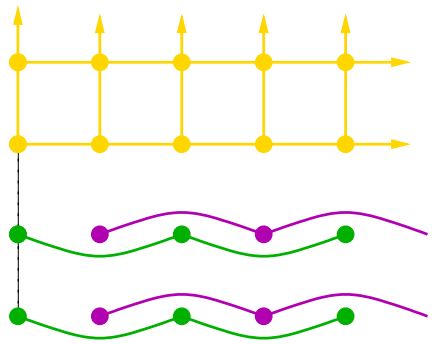
Presenting weird monoids



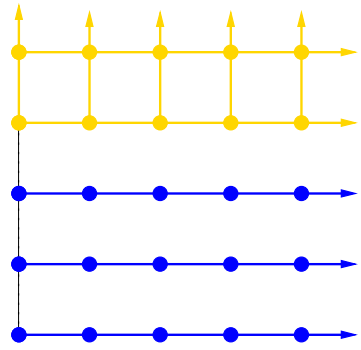
Some binomial ideals



$\langle y \rangle$

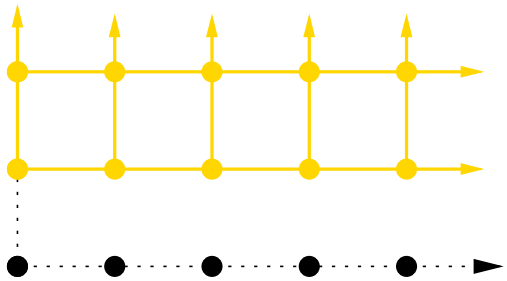


$\langle y^2, x^2 - 1 \rangle$

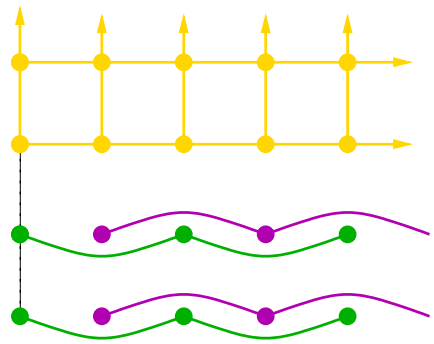


$\langle y^3, x - 1 \rangle$

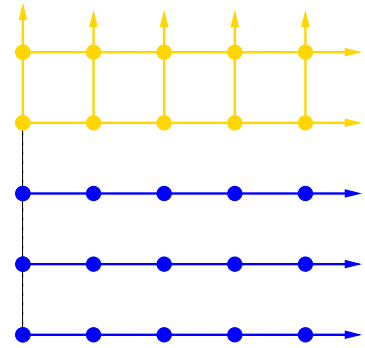
Some binomial ideals



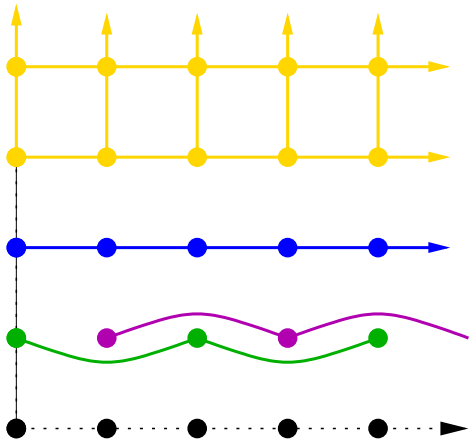
$\langle y \rangle$



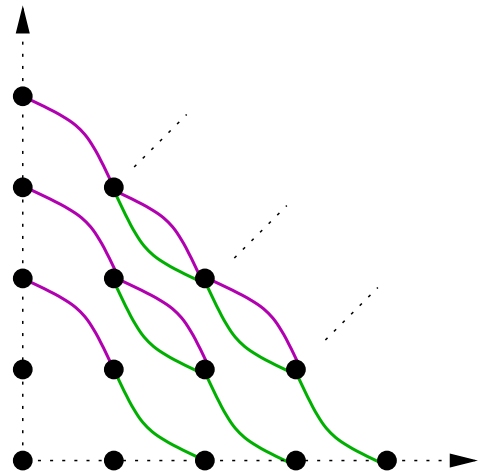
$\langle y^2, x^2 - 1 \rangle$



$\langle y^3, x - 1 \rangle$



$\langle y^3, y^2(x-1), y(x^2-1) \rangle$



$\langle x^2 - xy, xy - y^2 \rangle$

Monoid elements

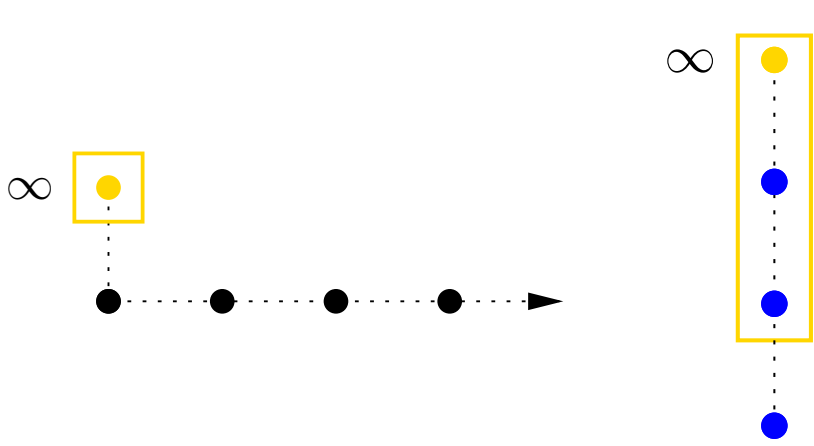
Monoid elements

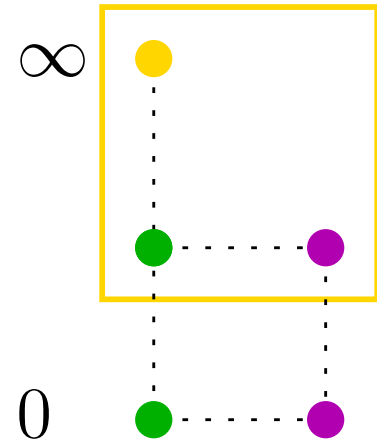
An element $q \in Q$ is

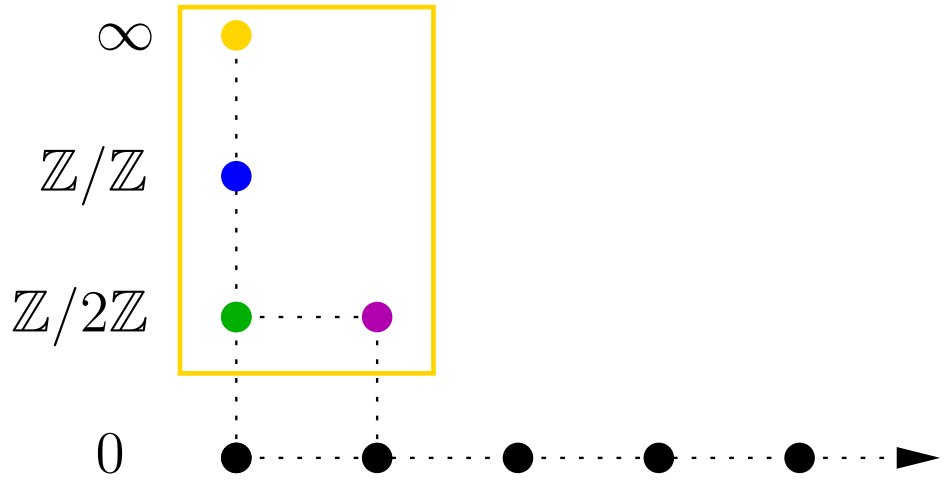
- cancellative if $a + q = b + q \Rightarrow a = b$, for all $a, b \in Q$.
- nilpotent if a multiple is nil.

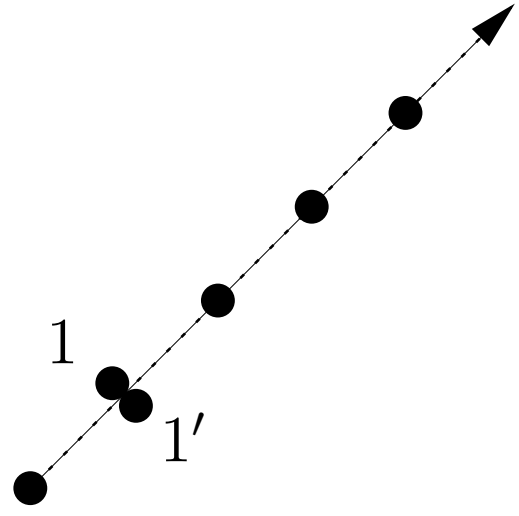
monomial $\mathbf{t}^q \in \Bbbk[Q]$	monoid element $q \in Q$
nonzerodivisor	cancellative
"nilpotent"	nilpotent

Nilpotents









Formal analogy

Definition

An ideal $I \subseteq \Bbbk[Q]$ is

- prime if in $\Bbbk[Q]/I$ every element is either zero or a nonzerodivisor.
- primary if in $\Bbbk[Q]/I$ every element is either nilpotent or a nonzerodivisor.
- irreducible if it is not the intersection of two strictly larger ideals.

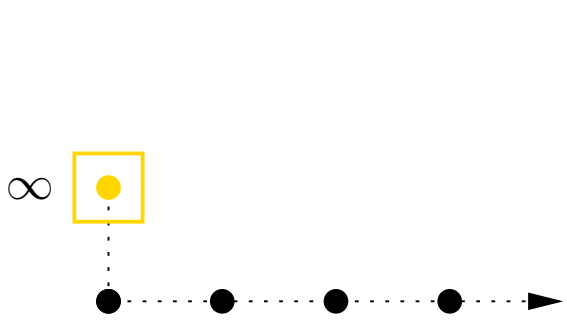
Definition (Drbohlav, 1963)

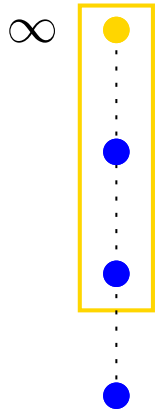
A congruence $\sim$ on Q is

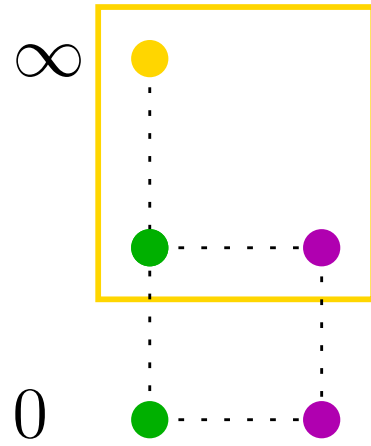
- prime if in $Q/\!\sim$ every element is either nil or cancellative.
- primary if in $Q/\!\sim$ every element is either nilpotent or cancellative.
- irreducible if it is not the common refinement of two strictly coarser congruences

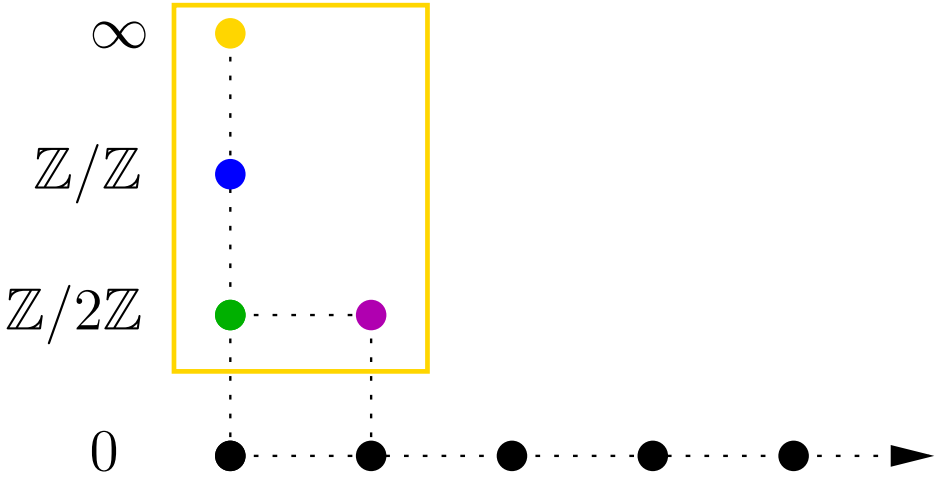
$\Rightarrow$ Congruences have primary decompositions.

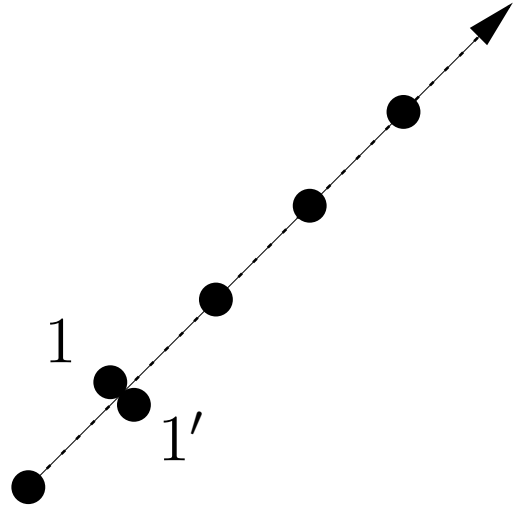
Primary quotients



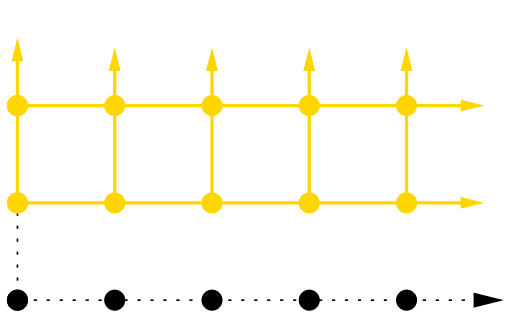


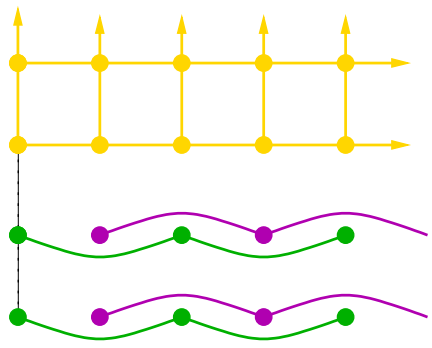


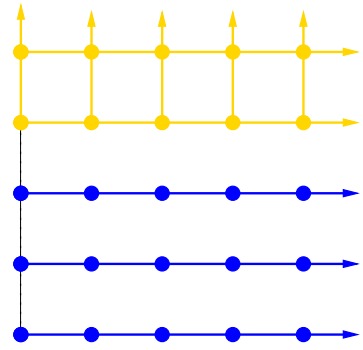




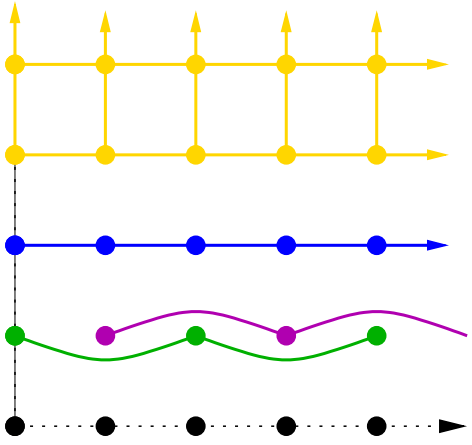
Primary congruences







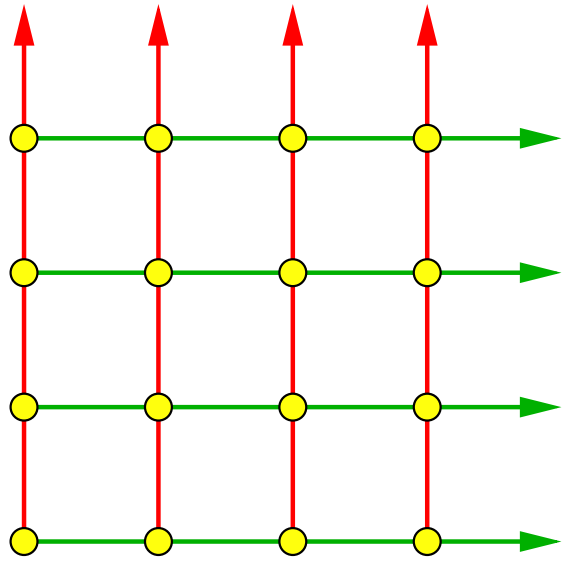
Wants to be decomposed



Conclusion

- Primary decomposition of congruences is too coarse.

Prime congruences may be reducible



The identity congruence on $\mathbb{N}^2$ has a primary decomposition

Conclusion

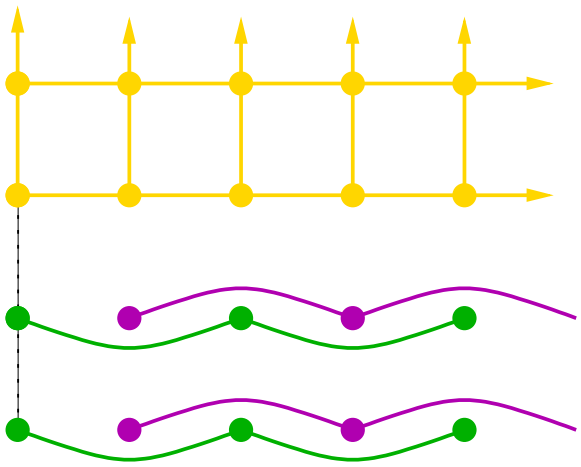
- Primary decomposition of congruences is too fine

Mesoprimary congruences

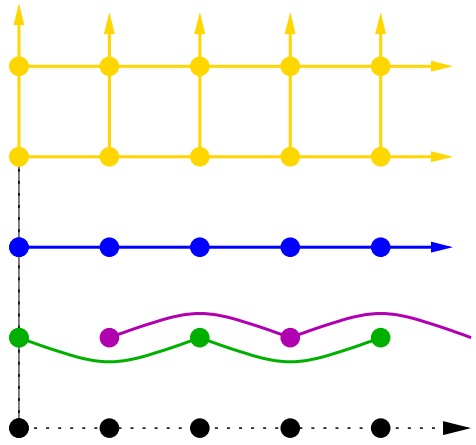
Definition

- An element $q \in Q$ is partly cancellative if $a + q = b + q$ implies $a = b$ or $a + q = \infty$, whenever a and b differ by a cancellative.
- A congruence is mesoprimary if it is primary and in $Q/\sim$ every element is partly cancellative.

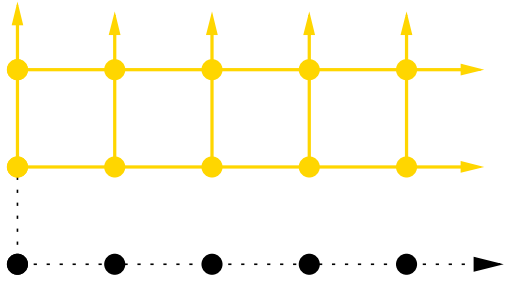
mesoprimary



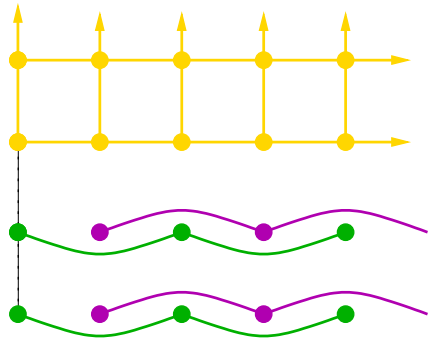
not mesoprimary



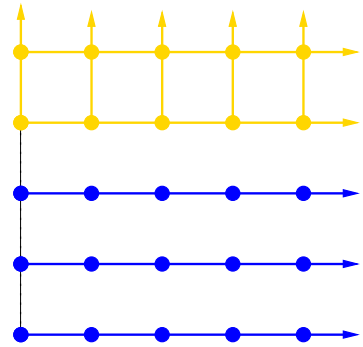
Mesoprimary decomposition



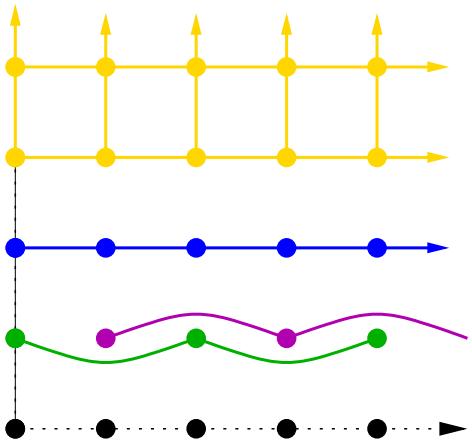
$\langle y \rangle$



$\langle x^2 - 1, y^2 \rangle$



$\langle x - 1, y^3 \rangle$



$\langle y(x^2 - 1), y^2(x - 1), y^3 \rangle$

How to find components?

What are their associated objects?

Localizations of monoids at prime ideals

Prime ideals in monoids

- Q has only finitely many prime ideals.
- $\varnothing \subseteq Q$ is an ideal (think: $\langle 0 \rangle \subseteq \Bbbk[Q]$)

Localization

Let $P \subseteq Q$ be a prime ideal, and $\sim$ a congruence on Q.

- The localization Q_P of Q at P is the monoid arising from adjoining inverses for all elements not in P.
- The induced congruence on Q is also denoted $\sim$, and $\overline{Q}_P := Q_P/\sim$.

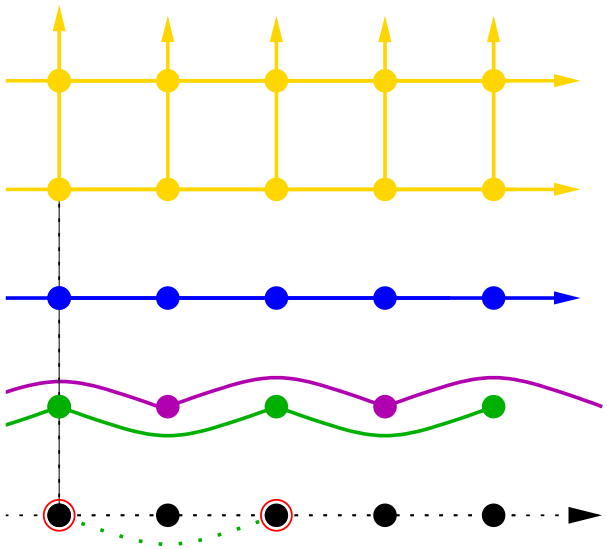
Example

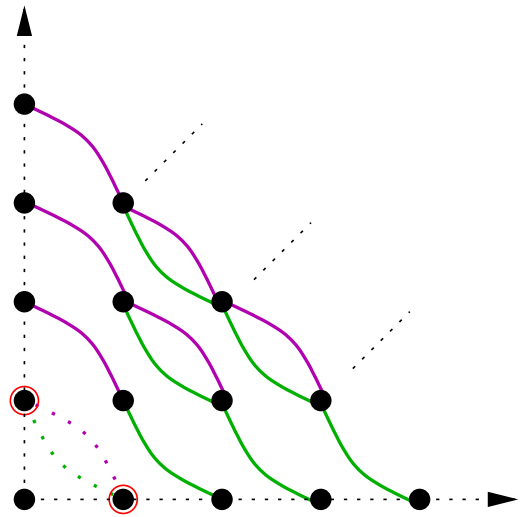
- Localizing a monoid Q without nil at $\varnothing$ gives its universal group $Q_{\varnothing}$.

Witnesses

Detecting combinatorial changes

1. Localize at a prime $P \subseteq Q$
2. Detect witnesses: socle elements in Q_P
   - non-trivial kernels of addition morphisms $\phi_p : q \mapsto q + p,\ p \in P_P$



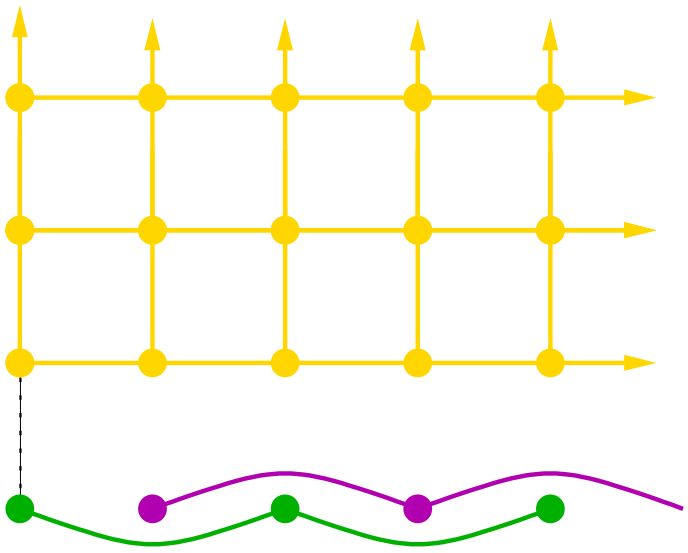


Associated prime congruences

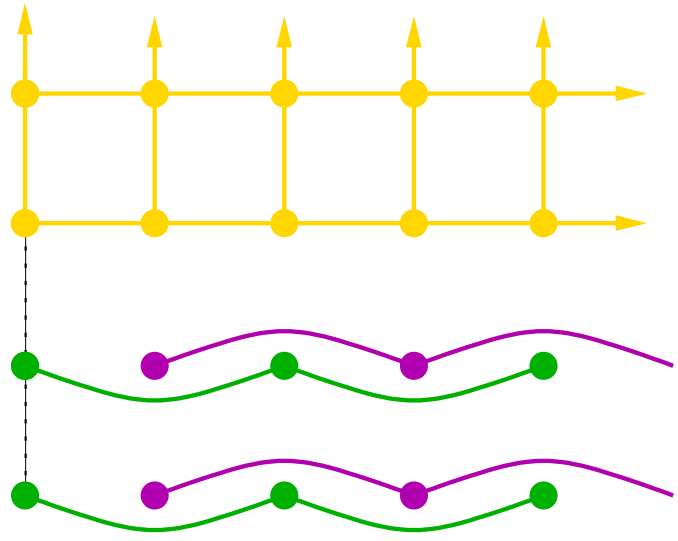
Prime congruences

A congruence $\sim$ is prime if in $Q/\sim$ every element is either nil or cancellative.

prime



mesoprimary



A prime congruence is associated if its non-nils look like a witness class.

Coprincipal and mesoprimary components

Witnessed decompositions

- The congruence $\sim$ defines a set of witnesses (P, w)
- For each witness (P, w):
  1. Localize at P.
  2. Lemma: The nilpotent quotient $\overline{Q}_P$ is partially ordered.
  3. Make every class below w look like w.
  4. Join to ∞ every class not below w.

Coprincipal component: the coarsest mesoprimary congruence such that

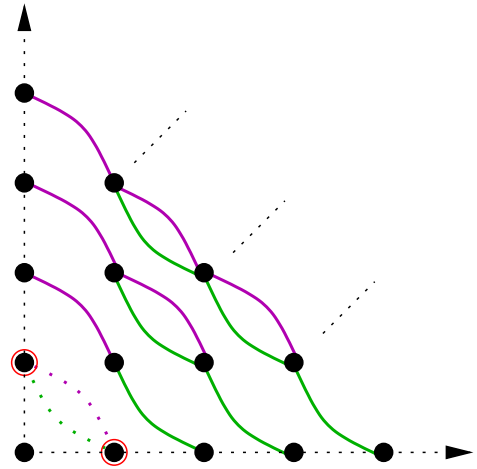
- The classes of w under $\sim$ and the component are identical.

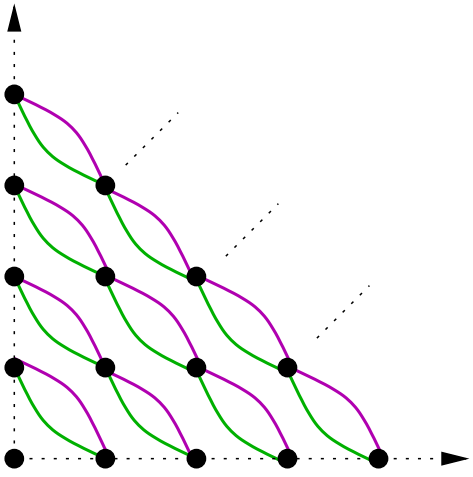
Coprincipal mesoprimary decomposition

- Decompose using coprincipal components
- Components at key witnesses suffice

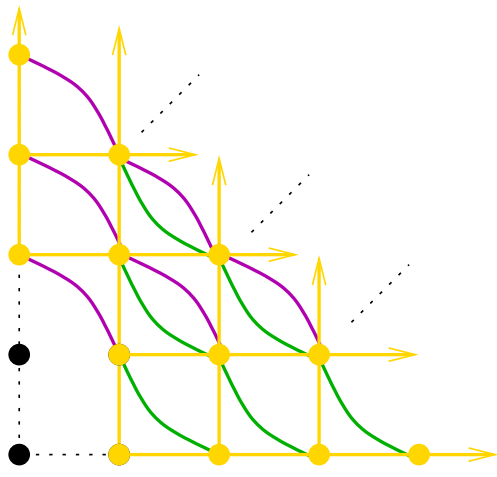
Coprincipal mesoprimary decomposition

$\langle x^2 - xy, xy - y^2 \rangle$

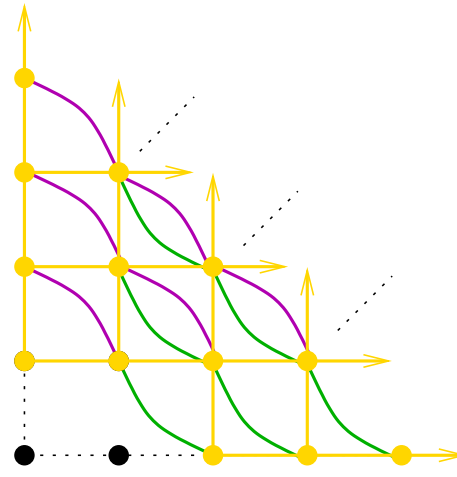




$\langle x - y \rangle$



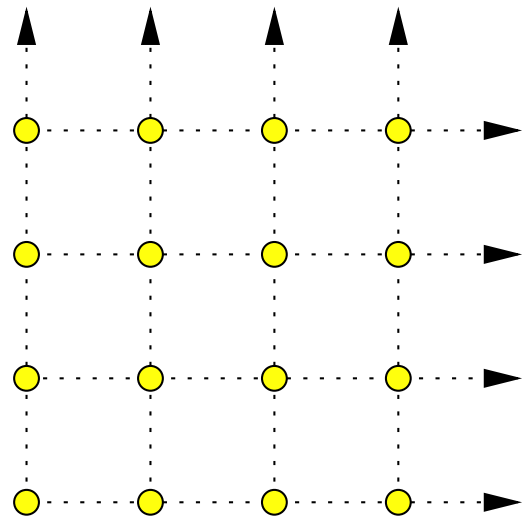
$\langle x, y^2 \rangle$



$\langle x^2, y \rangle$

No coprincipal decomposition

Nothing to decompose:



Mesoprimary decomposition of congruences

Theorem

Every congruence $\sim$ on Q is the common refinement of mesoprimary congruences that are coprincipal components at key witnesses.

Mesoprimary decomposition of congruences

- is canonical
- need not be irredundant
- fixes deficiencies of irreducible decomposition

Lifting decompositions to $\Bbbk[Q]$

Mesoprimary decompositions of binomial ideals

- Determine witnesses of $\sim_I$.
- Construct coprincipal components of I:
  - induce coprincipal components of $\sim_I$,
  - inherit their coefficients from I.
- Decompose using coprincipal components for character witnesses.

Subtleties in the lifting procedure

- $\langle x-1\rangle \cap \langle y-1\rangle$ is not binomial.
  - There are character witnesses that are not key.
- $\langle z-1\rangle \cap \langle z+1\rangle \neq \langle z-1\rangle \cap \langle z-1\rangle$
  - There are false witnesses among the key witnesses.

Mesoprimary decomposition of binomial ideals

Theorem

Let $\Bbbk$ be any field. Every binomial ideal $I \subseteq \Bbbk[Q]$ has a mesoprimary decomposition into binomial ideals that are coprincipal components of I at character witnesses.

Mesoprimary decomposition of binomial ideals

- is canonical
- need not be irredundant
- yields irreducible decomposition of binomial ideals

Even if $\mathbb{k} = \mathbb{C}$

Do you really...

want your computer to decompose $\langle x^{17} - 1, y^2 \rangle$?

```
{ideal(x-1,y^2), ideal(x-ww_17,y^2), ideal(x-ww_17^2,y^2), ideal(x-ww_17^3,y^2),
ideal(x-ww_17^4,y^2), ideal(x-ww_17^5,y^2),ideal(x-ww_17^6,y^2), ideal(x-ww_17^7,y^2),
ideal(x-ww_17^8,y^2), ideal(x-ww_17^9,y^2), ideal(x-ww_17^10,y^2), ideal(x-ww_17^11,y^2),
ideal(x-ww_17^12,y^2), ideal(x-ww_17^13,y^2), ideal(x-ww_17^14,y^2), ideal(x-ww_17^15,y^2)
ideal(x+ww_17^15+ww_17^14+ww_17^13+ww_17^12+ww_17^11+ww_17^10+ww_17^9+ww_17^8+
                ww_17^7+ww_17^6+ww_17^5+ww_17^4+ww_17^3+ww_17^2+ ww_17+1,y^2)}
```

Even if $\mathbb{k} = \mathbb{C}$

Do you really...

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```
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ideal(x-ww_17^8,y^2), ideal(x-ww_17^9,y^2), ideal(x-ww_17^10,y^2), ideal(x-ww_17^11,y^2),
ideal(x-ww_17^12,y^2), ideal(x-ww_17^13,y^2), ideal(x-ww_17^14,y^2), ideal(x-ww_17^15,y^2)
ideal(x+ww_17^15+ww_17^14+ww_17^13+ww_17^12+ww_17^11+ww_17^10+ww_17^9+ww_17^8+
                ww_17^7+ww_17^6+ww_17^5+ww_17^4+ww_17^3+ww_17^2+ ww_17+1,y^2)}
```

Thank you for your attention.