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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: ~~[illegible]~~ Elizabeth Gross Email/Phone: egross@uic.edu

Speaker's Name: Josephine Yu

Talk Title: Tropical geometry for computational algebra

Date: 12 / 6 / 2012 Time: 10 : 30 am / pm (circle one)

List 6-12 key words for the talk: Tropical geometry, elimination, implicitization, discriminat, resultant, polyhedral complex, polytopes

Please summarize the lecture in 5 or fewer sentances:

Shows how tropical geometry can ~~[illegible]~~ be used to find the defining equations of a parameterized variety & to compute resultants

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
    - **Computer Presentations**: Obtain a copy of their presentation
    - **Overhead**: Obtain a copy or use the originals and scan them
    - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
    - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Tropical Geometry for Computational Algebra

A. Implicitization : Find defining equations of a parameterized variety.

Ex :

$$x = t + t^2 + t^3$$
$$y = 2t^3 + 5t^7$$

(input)

$125x^7 + 300x^5 - 35x^4y + 120x^4 - 590x^3y - 35x^2y^2 + 38x^3 - 387x^2y - 29xy^2 - y^3 - 117xy - y^2 - 19y$

(output)

B. Resultants : Find the defining equations of the locus of polynomials for which a system of polynomials have a solution in $(\mathbb{C}^*)^n$

Ex

$$f_1 = a_{10} + a_{11}x + a_{12}x^2$$
$$f_2 = a_{20} + a_{21}x + a_{22}x^2$$

(input)

$$\det\begin{pmatrix} a_{10} & a_{11} & a_{12} & 0 \\ 0 & a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} & 0 \\ 0 & a_{20} & a_{21} & a_{22} \end{pmatrix}$$

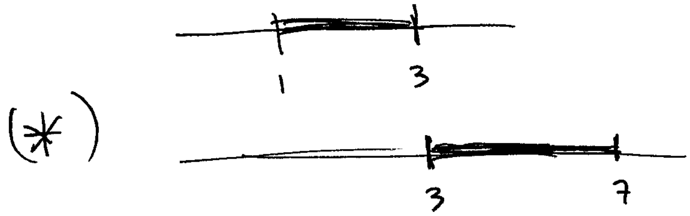
= polynomial ~~[illegible]~~ with 6 terms

(output)

Combinatorial Versions

Newton polytope of a polynomial is the convex hull of the exponents.

A.



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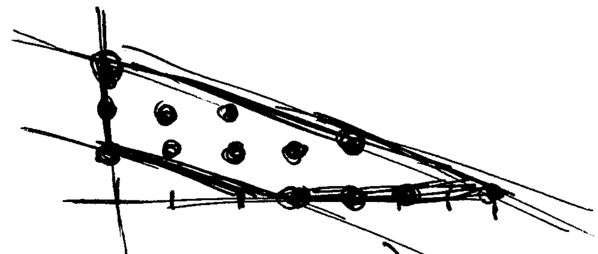
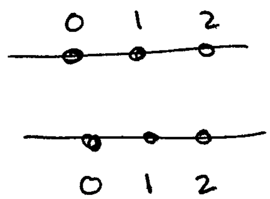


(output)

B. Find Newton polytope of resultant



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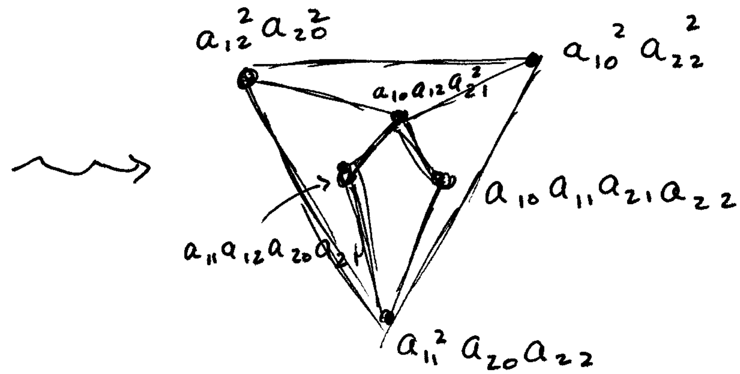

(output)

How to describe a polytope in $\mathbb{R}^n$

$\mathcal{V}$ convex hull of finite set of points

$\mathcal{H}$ bound intersection of finitely many half spaces

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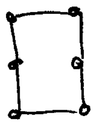
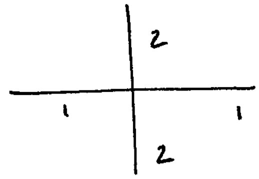
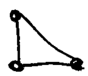
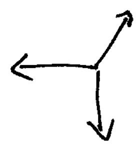
Oracle $w \in \mathbb{R}^n \longmapsto \text{vertex}_w(P)$

Tropical $\tau(P) = \{ w : \dim \text{face}_w P \geq 1 \}$ + multiplicities

τ () = 

τ (zonotope) = $\bigcup$ zones (dual hyperplanes)

τ (permutohedron) = $\bigcup \{ x_i = x_j \}$

τ () = 

Conversion Algorithms are important!

$\mathcal{V} \longleftrightarrow \mathcal{H}$ dual-description methods

$\mathcal{H} \longleftrightarrow \Theta$ linear programming

$\Theta \longrightarrow \mathcal{V}/\mathcal{H}$ beneath-beyond, gift-wrapping

$\tau \longrightarrow \Theta$ [Dickenstein - Feichtner - Sturmfels 2007]

$\tau \rightsquigarrow \mathcal{V}$ "ray-shooting"

[Jensen - Y., 2011]

new

implemented in Gfan

Higher codim?

Tropical variety of an ideal I is

$$\tau(I) := \bigcap_{f \in I} \tau(\mathrm{New}(f))$$

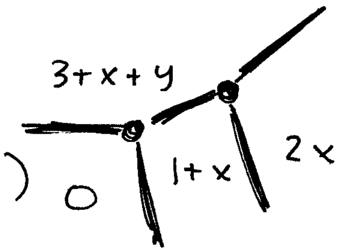
$$= \{ w : \mathrm{in}_w I \not\ni \text{monomials} \}$$

Tropical Algebra $(\mathbb{R}, \underset{\max}{\oplus}, \underset{+}{\odot})$

<u>Ex</u> $x^{\odot 2} \oplus 3 \odot x \odot y \oplus 1 \odot x \oplus 0$

Tropical hypersurface - maximum is attained twice

If coefficients are all the same $\mathcal{T}(F) = \mathcal{T}(\mathrm{New}(F))$



<u>A</u>. $x = t + t^2 + t^3$
$y = 2t^3 + 5t^7$

$\mathrm{trop}(x) = T \oplus T^{\odot 2} \oplus T^{\odot 3}$
$\mathrm{trop}(y) = T^{\odot 3} \oplus T^{\odot 7}$ $: \mathbb{R} \to \mathbb{R}^2$

<u>Thm</u> (Sturmfels - Tevelev - Y. 2007)

$$\mathcal{T}\left(\begin{array}{c}\text{parameterized}\\ \text{variety}\end{array}\right) = \bigcup \mathrm{trop}(f)(\sigma) + \mathbb{R}_+\{-e_j : j \in J\}$$

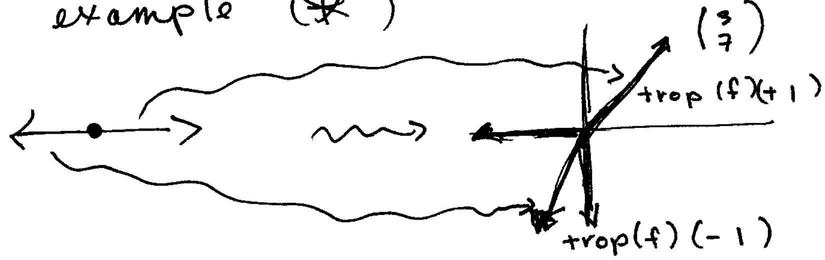
$\mathbb{C}^k \to \mathbb{C}^n$
$t \mapsto (f_1, \dots, f_n)$
$P_1, \dots, P_n$
$\mathrm{trop}\, f : \mathbb{R}^k \to \mathbb{R}^n$

σ is a cone in common refinement of normal fan of $P_1, \dots, P_n$

st $MV(\mathrm{face}_\sigma\, P_j)_{j \in J} > 0$

mixed volume

For example $(*)$



B. Resultants

Def: The tropical resultant is the locus of coefficients for which a system of tropical polynomials have a common solution.

e.g. $a_{10} \oplus a_{11} \odot x \oplus a_{12} \odot x^{\odot 2}$

$a_{20} \oplus a_{21} \odot x \oplus a_{22} \odot x^{\odot 2}$

$(0, 1, 0, -1, -1, 1)$ is in tropical resultant

$$\left.\begin{array}{l} \max(0,\ 1+x,\ 2x) \\ \max(-1,\ -1+x,\ 1+2x) \end{array}\right\} \ -1 \text{ is a common solution}$$

Thm (Jensen-Yu 2011)

Trop variety of the resultant

$=$ tropical resultant

$= \bigcup \mathbb{R}_+ \{-e_\alpha : \alpha \notin E\}$

E consists of 2 pts from each A_i

$+$ rowspace $\left(\begin{array}{cccc} 1111 & & & \\ & 1111 & & \\ & & & 1111 \\ \hline A_1 & A_2 & \cdots & A_2 \end{array}\right)$ ← Cayley configuration

Ex

9 cones
$$\left|\begin{array}{l} \mathbb{R}_+\{-e_{12}, -e_{22}\} \\ \cup\ \mathbb{R}_+\{-e_{12}, -e_{21}\} \\ \quad \vdots \end{array}\right. + \text{row sp} \begin{array}{c} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 1 & 2 & 0 & 1 & 2 \end{array}\right) \\ e_{10}\ e_{11}\ e_{12}\ \ e_{20}\ e_{21}\ e_{22} \end{array}$$