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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Elizabeth Gross **Email/Phone:** egross7@uic.edu

Speaker's Name: Matteo Varbaro

Talk Title: Relations between Minors

Date: 12 / 6 / 12 **Time:** 2 : 00 am / (pm) (circle one)

List 6-12 key words for the talk: determinental ideals, syzygies, young diagrams, Schur modules, minimal relations, t-admissable, Grassmanian, Plücker relations

Please summarize the lecture in 5 or fewer sentances:

Explores the problem of determining the algebraic relations between minors of a generic matrix.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
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RELATIONS BETWEEN MINORS

Matteo Varbaro

Università di Genova (Italy)
MSRI, Berkeley

Joint work with Winfried Bruns and Aldo Conca

Problem

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & \cdots & x_{mn} \end{pmatrix}$$

Which algebraic relations do occur between the t-minors of X???

Notation and maximal minors (Grassmannian)

- K is a field of characteristic 0;
- $t \leq m \leq n$ are positive integers;
- X is an $m \times n$-matrix of indeterminates over K;
- $A_t(X)$ is the subalgebra of $K[X]$ generated by the t-minors.

If $t = m$, then $A_t(X)$ is the coord. ring of a Grassmannian. So the minimal relations between t-minors of X are the Plücker relations.

EXAMPLE (Simplest Plücker relation). $t = m = 2, n = 4$:

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \end{pmatrix}.$$

Then $[12][34] - [13][24] + [14][23] = 0, \quad [ij] = \det \begin{pmatrix} X_{1i} & X_{1j} \\ X_{2i} & X_{2j} \end{pmatrix}.$

What if $t < m$?

Bruns and Conca started the study of $A_t(X)$ in 2001. They proved a lot (from now on $t < m$ and $2 \leq t \leq n-2$):

- $A_t(X)$ is a normal Cohen-Macaulay domain.
- $A_t(X)$ is Gorenstein if and only if $1/t = 1/m + 1/n$.
- Description of the singular locus of $\mathrm{Spec}(A_t(X))$.
- Much more .....

But what about the relations???

What if $t < m$?

First of all we need a notation for the t-minors:

$$[i_1, \ldots, i_t | j_1, \ldots, j_t] = \det \begin{pmatrix} X_{i_1,j_1} & \cdots & X_{i_1,j_t} \\ \vdots & & \vdots \\ X_{i_t,j_1} & \cdots & X_{i_t,j_t} \end{pmatrix}$$

Already if $t = 2$, $m = 3$ and $n = 4$ degree 2 is not anymore enough. The following is a minimal cubic relation:

$$(*) \quad \det \begin{pmatrix} [12|12] & [12|13] & [12|14] \\ [13|12] & [13|13] & [13|14] \\ [23|12] & [23|13] & [23|14] \end{pmatrix} = 0$$

This was noticed by Bruns already in 1991. The goal of the first part of the talk will be to introduce the necessary representation theoretic tools to understand why (*) must be there.

Representation theory of $\mathrm{GL}(V)$

V is a finite dimensional K-vector space. There is a bijection:

$$\{\text{polynomial irreducible } \mathrm{GL}(V)\text{-representations}\}$$

$$\updownarrow$$

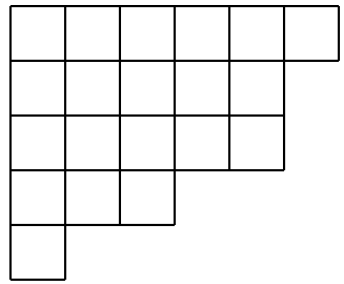
$$\{\lambda = (\lambda_1, \ldots, \lambda_k) \text{ partitions } (\lambda_1 \geq \ldots \geq \lambda_k > 0) \text{ with } \lambda_1 \leq \dim_K V\}$$

For all such partitions λ, the Schur functors L_λ associate a representation to any representation.

Thm: $L_\lambda V$ is a nonzero irreducible representation for every $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \leq \dim_K V$. Moreover, all polynomial representations decompose as direct sum of $L_\lambda V$'s.

Young diagrams

It is useful to figure out a partition as a diagram. For example:

$(6,5,5,3,1) \; =$ 

In our (unusual) convention:

$\bigwedge^t V \leftrightarrow (t) \leftrightarrow$ 

We write $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash e$ if λ has e boxes ($\lambda_1 + \ldots + \lambda_k = e$).

Examples

We all know that $V \otimes V$ decomposes as:

$$V \otimes V \cong \mathrm{Sym}^2 V \oplus \bigwedge^2 V \cong L_{(1,1)} V \oplus L_{(2)} V$$

Such a decomposition is available for all tensor powers:

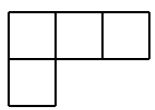
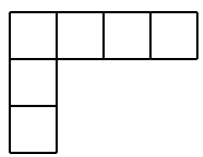
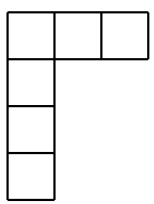
$$\begin{aligned} V \otimes V \otimes V &\cong \mathrm{Sym}^3 V \oplus (L_{(2,1)} V)^2 \oplus \bigwedge^3 V \\ &\cong L_{(1,1,1)} V \oplus (L_{(2,1)} V)^2 \oplus L_{(3)} V \end{aligned}$$

Pieri's rule

Pieri's rule determines for all λ the decomposition in irreducible representations of $L_\lambda V \otimes \wedge^t V$. It says:

$$L_\lambda V \otimes \bigwedge^t V \cong \bigoplus_\mu L_\mu V,$$

where μ is gotten adding t boxes to different columns of λ. In such a case we say that μ is a (t-)successor of λ (and λ is a (t-)predecessor of μ).

For example, if $t = 2$ and $\lambda =$  , then $\mu =$  is a successor of λ, whereas $\gamma =$  is not.

The action on our objects

- V is a K-vector space of dimension m;
- W is a K-vector space of dimension n;
- $G = \mathrm{GL}(V) \times \mathrm{GL}(W)$.

G acts on our algebra of minors $A_t(X)$, so we have to deal with the representation theory of G. Luckily, the irreducible polynomial representations of G are of the form:

$$L_\gamma V \otimes L_\lambda W,$$

so we can use the information coming from the representation theory of $\mathrm{GL}(V)$. Therefore we will speak of bi-diagrams $(\gamma|\lambda)$, bi-predecessors, bi-successors ...

The action on our objects

We say $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash e$ is (t-)admissible if $e = dt$ and $k \leq d$.

(DeConcini, Eisenbud and Procesi):

$$A_t(X) \cong \bigoplus_{\lambda} L_\lambda V \otimes L_\lambda W^*$$

where λ is t-admissible with $\lambda_1 \leq m$.
Calling $E = \wedge^t V$ and $F = \wedge^t W$, we are interested in the kernel of the following G-equivariant map:

$$\phi : \mathrm{Sym}(E \otimes F^*) \longrightarrow A_t(X).$$

To find a decomposition in G-irreducibles of $\mathrm{Sym}(E \otimes F^*)$ is out of reach, so it may be convenient to go one step more to the left:

$$\psi : (\bigotimes E) \otimes (\bigotimes F^*) \to \mathrm{Sym}(E \otimes F^*) \to A_t(X)$$

The first cubic minimal relation

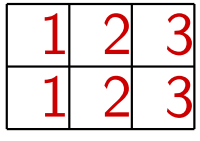
The decomposition of $\left(\bigotimes E\right) \otimes \left(\bigotimes F^*\right)$ follows by Pieri's rule:

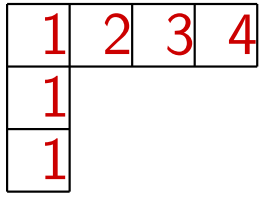
$$\left(\bigotimes E\right) \otimes \left(\bigotimes F^*\right) \cong \bigoplus_{\gamma,\lambda} (L_\gamma V \otimes L_\lambda W^*)^{m(\gamma,\lambda)}$$

where γ and λ are t-admissible with $\gamma_1 \leq m$ and $\lambda_1 \leq n$. The cubic of the beginning ($t = 2$):

$$\det \begin{pmatrix} [12|12] & [12|13] & [12|14] \\ [13|12] & [13|13] & [13|14] \\ [23|12] & [23|13] & [23|14] \end{pmatrix} = 0$$

corresponds to $L_\gamma V \otimes L_\lambda W^*$ where:

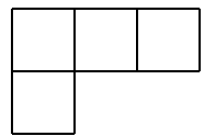
$\gamma =$ 

and $\lambda =$ 

The first cubic minimal relation

If $(\gamma|\lambda)$ were not minimal in $\ker(\psi)$, then there would be a 2-admissible bi-predecessor of $(\gamma|\lambda)$ in $\ker(\psi)$.

The only 2-admissible bi-predecessor of $(\gamma|\lambda)$ is the pair $(\alpha|\alpha)$,

$\alpha \; =$ 

$L_\alpha V \otimes L_\alpha W^*$ has multiplicity 1 both in $\left(\bigotimes E\right) \otimes \left(\bigotimes F^*\right)$ and in $A_t(X)$. So it cannot be in $\ker(\psi)$. In particular

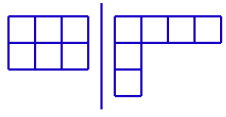
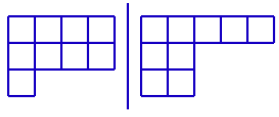
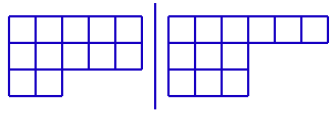
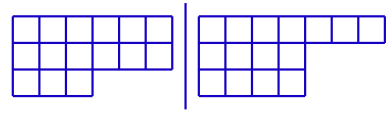
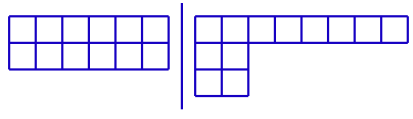
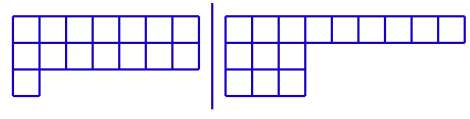
$$\det \begin{pmatrix} [12|12] & [12|13] & [12|14] \\ [13|12] & [13|13] & [13|14] \\ [23|12] & [23|13] & [23|14] \end{pmatrix}$$

is a minimal cubic relation between 2-minors.

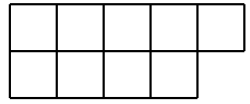
T-shape relations

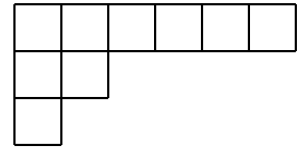
In this way we can find other minimal cubic relations, namely:

$$\gamma_u = (t+u,\ t+u,\ t-2u),$$
$$\lambda_u = (t+2u,\ t-u,\ t-u).$$

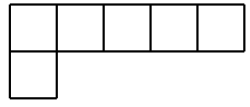
	$t=2$	$t=3$	$t=4$	$t=5$	
$(\gamma_1\|\lambda_1)$					
$(\gamma_2\|\lambda_2)$					

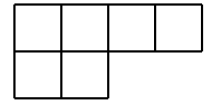
A minimal cubic for 3-minors of different nature

$\rho =$ 

$\sigma =$ 

Let us look at the 3-admissible predecessors of ρ:

$\alpha =$ 

$\beta =$ 

They are the same 3-admissible predecessors of σ. So the 3-admissible bi-predecessors of $(\rho|\sigma)$ are:

$$(\alpha|\alpha), \quad (\beta|\beta), \quad (\alpha|\beta), \quad (\beta|\alpha)$$

We have asymmetric friends, we cannot use the previous argument.

A minimal cubic for 3-minors of different nature

This time we have to think in $\mathrm{Sym}(\wedge^3 V \otimes \wedge^3 W^*)$. To do this we have to introduce to the game the bigger group

$$H = \mathrm{GL}(E) \times \mathrm{GL}(F),$$

where $E = \wedge^3 V$ and $F = \wedge^3 W$. The Cauchy decomposition says:

$$\mathrm{Sym}(E \otimes F^*) \cong \bigoplus L_\mu E \otimes L_\mu F^*$$

where $\mu_1 \leq \dim_K E = \binom{m}{3}$.

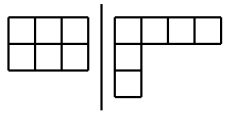
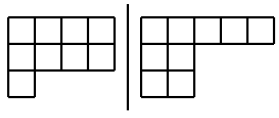
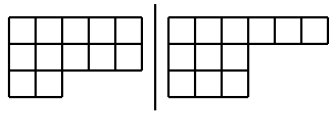
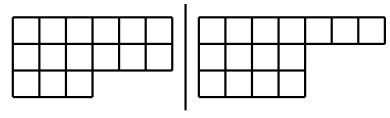
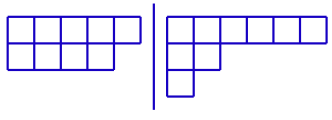
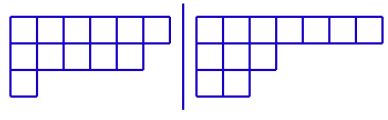
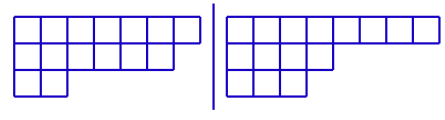
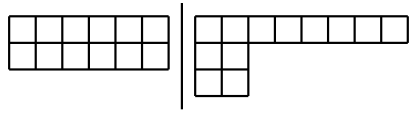
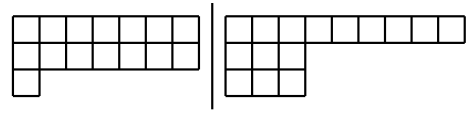
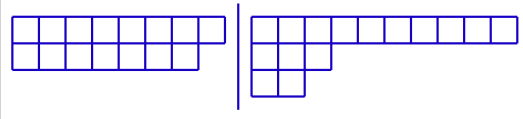
Exploiting it one can show that $(\rho|\sigma)$ occurs in $\mathrm{Sym}(E \otimes F^*)$ and has only symmetric bi-predecessors in $\mathrm{Sym}(E \otimes F^*)$.

So $((5,4)|(6,2,1))$ gives a minimal relation between 3-minors.

Shape relations

With this technique we can find all the following minimal cubics:

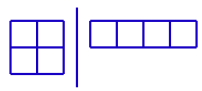
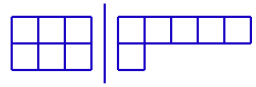
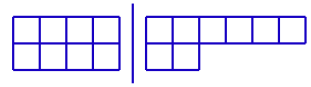
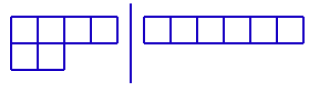
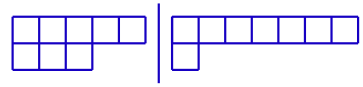
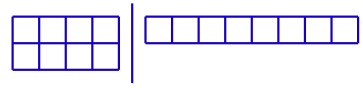
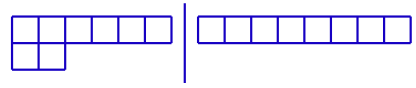
$$\rho_u = (t+u,\ t+u-1,\ t-2u+1),$$
$$\sigma_u = (t+2u-1,\ t-u+1,\ t-u).$$

	$t=2$	$t=3$	$t=4$	$t=5$	
$(\gamma_1\|\lambda_1)$					
$(\rho_2\|\sigma_2)$					
$(\gamma_2\|\lambda_2)$					
$(\rho_3\|\sigma_3)$					

The conjecture

It is easy to describe in a representation-theoretic fashion the minimal quadratic relations:

$(\tau_u|\tau_v)$, where $\tau_u = (t+u, t-u)$, $u \neq v$, $u+v$ even.

	$t=2$	$t=3$	$t=4$
$(\tau_0\|\tau_2)$			
$(\tau_1\|\tau_3)$			
$(\tau_0\|\tau_4)$			
$(\tau_2\|\tau_4)$			

Conjecture: $(\tau_u|\tau_v)$, $(\gamma_u|\lambda_u)$ and $(\rho_u|\sigma_u)$ (and their mirror bi-diagrams) generate the ideal of relations between t-minors. In particular, such minimal relations are at most cubic.

Evidence

Based on a mixture of theoretical and computational tools:

- The conjecture is true for 2-minors and $m \leq 4$.
- No further cubic minimal relations for $t = 2, 3$.
- No degree 4 minimal relations between 2-minors.

Regularity does not help: $\operatorname{reg}(A_t(X)) \approx mn - mn/t$.

Single $\wedge^t$-type

All the minimal relations we found have a common, nice, feature:

Fixed $\lambda \vdash td$, the multiplicity of $L_\lambda V$ in $L_\mu(\wedge^t V)$, where $\mu \vdash d$, is denoted by $m_\lambda(\mu)$.

We say that $\lambda \vdash td$ is of single $\wedge^t$-type if m_λ does not vanish only at one $\mu \vdash d$ and $m_\lambda(\mu) = 1$.

Fact: τ_u, γ_u, λ_u, ρ_u and σ_u are of single $\wedge^t$-type.

Single $\wedge^t$-type

Theorem (Bruns,-): A t-admissible diagram $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash td$ is of single $\wedge^t$-type if and only if one of the following holds:

- $k = d$ and $(\lambda_1 - 1, \ldots, \lambda_d - 1)$ is of single $\wedge^{t-1}$-type.
- $\lambda_1 \leq t + 1$.
- $\lambda_2 \leq 1$ (hooks).
- $k = d - 1$ and $\lambda_{d-1} \geq \lambda_1 - 1$.

We can also describe the $\mu \vdash d$ where each of the above λ's occurs.

As a consequence, one can prove that there are no further minimal relations $(\gamma|\lambda)$ between t-minors with γ and λ of single $\wedge^t$-type .....