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Name: Elizabeth Gross Email/Phone: egross7@uic.edu

Speaker's Name: Jenna Rajchgot

Talk Title: Compatibly split subvarieties of the Hilbert scheme of points in the plane

Date: 12 / 7 / 12 Time: 10 : 30 am / pm (circle one)

List 6-12 key words for the talk: Frobenius splittings, Hilbert schemes, compatibly split subvarieties, Stanley-Reisner ideals, monomial ideals, shellability, Poisson tensor

Please summarize the lecture in 5 or fewer sentances:

Introduces a Frobenius splitting of the Hilbert scheme of n points in the plane. Describes all compatibly split subvarieties of $Hilb^n(A_k^2)$ for small n. Explains a Gröbner degeneration to Stanley-Reisner schemes that is helpful for understanding arbitrary n.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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Compatibly split subvarieties of $\mathrm{Hilb}^n(\mathbb{A}^2_k)$

Jenna Rajchgot
MSRI

Combinatorial Commutative Algebra
December 3-7, 2012

Throughout this talk, let k be an algebraically closed field of characteristic $p > 2$.

Definition: $\mathrm{Hilb}^n(\mathbb{A}_k^2)$ is the scheme parametrizing dimension-0, degree-n subschemes of $\mathbb{A}_k^2$. So, set theoretically,

$$\mathrm{Hilb}^n(\mathbb{A}_k^2) := \{I \subset k[x,y] : \dim(k[x,y]/I) = n \text{ as a vector space over } k\}.$$

Properties:

- [Hartshorne] $\mathrm{Hilb}^n(\mathbb{A}_k^2)$ is connected.
- [Fogarty] $\mathrm{Hilb}^n(\mathbb{A}_k^2)$ is non-singular.
- This $2n$-dimensional scheme has a $T^2 = (k^*)^2$ action induced by the standard action of T^2 on $\mathbb{A}_k^2$ (i.e. $(t_1, t_2) \cdot (x, y) = (t_1 x, t_2 y)$). The T^2-fixed points of $\mathrm{Hilb}^n(\mathbb{A}_k^2)$ are the colength-n monomial ideals.

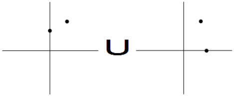
Goal: To understand a particular stratification of $\mathrm{Hilb}^n(\mathbb{A}_k^2)$. This stratification will consist of finitely many (locally closed) strata which are automatically reduced, regular in codimension 1, and stable under the T^2-action.

We begin by stratifying $\mathrm{Hilb}^2(\mathbb{A}_k^2)$ by reduced, T^2-invariant subvarieties such that the open strata are regular in codimension 1.

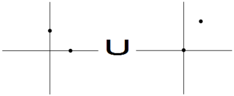
Consider $\mathrm{Hilb}^2(\mathbb{A}_k^2)$ and the reduced, T^2-invariant divisor D where

$$D = \text{“at least one point is on a coordinate axis”}.$$

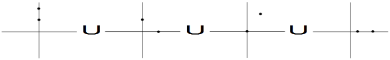
The two components of D will be the codimension 1 subvarieties in our stratification.



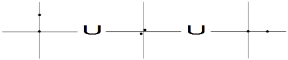
We can intersect the irreducible components of this divisor and decompose the intersection to obtain some new subvarieties. These subvarieties are reduced!



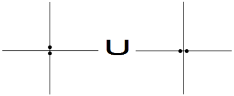
Neither irreducible component of D is regular in codimension 1. So, we include the codimension 1 component of the singular loci in the union of codimension 2 subvarieties to appear in the stratification of $\mathrm{Hilb}^2(\mathbb{A}_k^2)$.



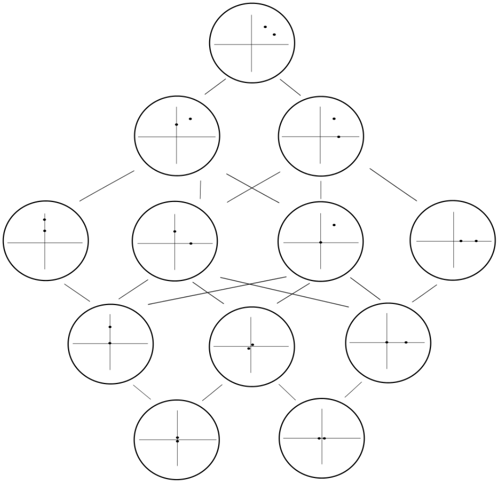
Intersecting each one of these subvarieties with the union of the others and then decomposing each intersection yields the following codimension 3 subvarieties:



Repeating this procedure once more produces the T^2-fixed points.



This sequence of intersecting, decomposing, and including non-R1 loci produces the following stratification by reduced subvarieties:



This is precisely the collection of "compatibly Frobenius split" subvarieties of $\mathrm{Hilb}^2(\mathbb{A}^2_k)$.

Definition: Let R be a (commutative) k-algebra and let $X = \mathrm{Spec}(R)$. Say that R (or X) is **Frobenius split** by $\phi : R \to R$ if:

$$\phi(a+b) = \phi(a) + \phi(b), \quad \phi(a^p b) = a\phi(b), \quad \phi(1) = 1$$

for any $a, b \in R$.
(Notice that ϕ is an R-module map which "splits" the Frobenius endomorphism $F : R \to R, r \mapsto r^p$. That is, $\phi \circ F = \mathrm{Id}.$)

It immediately follows from the definition that if R is Frobenius split then R has no nilpotents. So, $X = \mathrm{Spec}(R)$ is reduced.

Definition: Let $I \subset R$ be an ideal. We say that I (or $V(I)$) is **compatibly Frobenius split** if $\phi(I) \subset I$.

In this case, there is an induced splitting, $\overline{\phi} : R/I \to R/I$ and we get that I is a radical ideal.

The following are some consequences which we have already used:

1. Intersections, unions and components of compatibly split subschemes are compatibly split.
2. The non-$R1$ locus of any compatibly split subvariety is compatibly split.

Note: The above definitions and results generalize to schemes $(X, \mathcal{O}_X)$.

Theorem:

1. For X regular, $\mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \cong H^0(X, F_*(\omega_X^{1-p}))$, where ω_X is the canonical bundle on X. Thus, certain anticanonical sections determine Frobenius splittings.
2. [Kumar-Mehta] Let X be an irreducible, normal variety which is Frobenius split by $\sigma \in H^0(X, F_*(\omega_X^{1-p}))$. If Y is compatibly split then $Y \subseteq V(\sigma)$ or $Y \subseteq \mathrm{sing}(X)$.

Example: The divisor $\{x_1x_2\cdots x_n = 0\}$ determines a Frobenius splitting of $\mathbb{A}^n$. This splitting of $\mathbb{A}^n$ is called the **standard splitting**. By intersecting the components of the divisor, decomposing the intersections, intersecting the new components, etc., we obtain the collection of coordinate subspaces. This is precisely the set of compatibly split subvarieties of $\mathbb{A}^n$ with the standard splitting.

Theorem: [Lakshmibai-Mehta-Parameswaran] Let $f \in k[x_1, \ldots, x_n]$. If there is a term order on $k[x_1, \ldots, x_n]$ such that $\mathrm{init}(f) = x_1x_2\cdots x_n$ then $\{f = 0\}$ determines a splitting of $\mathbb{A}^n$ that compatibly splits $\{f = 0\}$.

Theorem: [Kumar-Thomsen] The anticanonical divisor described by "at least one point is on an axis" determines a Frobenius splitting of $\mathrm{Hilb}^n(\mathbb{A}_k^2)$.

Algorithm: [Knutson-Lam-Speyer]
Input: $(X, \partial X)$ where X is Frobenius split and ∂X is the anticanonical divisor which induces the splitting.

Output: Suppose that $\partial X = D_1 \cup \cdots \cup D_r$. Let $E_i = D_1 \cup \cdots \cup \hat{D}_i \cup \cdots \cup D_n$. There are two cases.

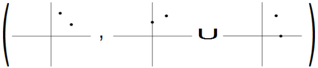
1. If X is normal, then return $(D_1, D_1 \cap E_1)$,...,$(D_n, D_n \cap E_n)$.
2. If X is not normal, return $(\widetilde{X}, \nu^{-1}(\partial X \cup X_{\text{non-}R1}))$ where $\nu : \widetilde{X} \to X$ is the normalization of X.

Repeat until neither 1. nor 2. can be applied. When finished, map all subvarieties back to the original Frobenius split variety to obtain a list of many (for large p) compatibly split subvarieties.

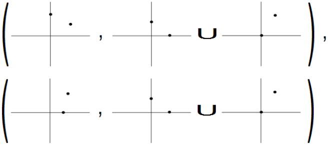
At each stage of the algorithm, check if $\exists$ a component of the singular locus that is both compatibly split and of codimension ≥ 2. (Hard!)
If so, add it (and its compatibly split subvarieties) to the list.

In certain cases, the final list consists of all compatibly split subvarieties of $(X, \partial X)$.

As an example of the algorithm, we consider (again) the case of $\mathrm{Hilb}^2(\mathbb{A}_k^2)$. Start with $(\mathrm{Hilb}^2(\mathbb{A}_k^2), D)$ where D is as before.

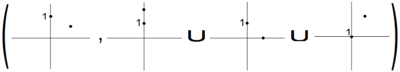


Apply 1.

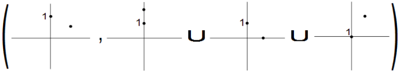


Due to the symmetry, continue with just the first of the two pairs.

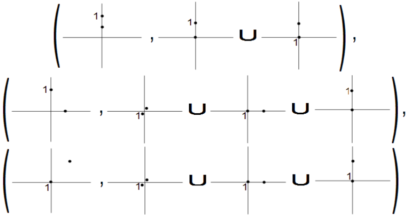
Next, recall that the components of D are not regular in codimension 1. Apply 2.



From the previous slide, we have:

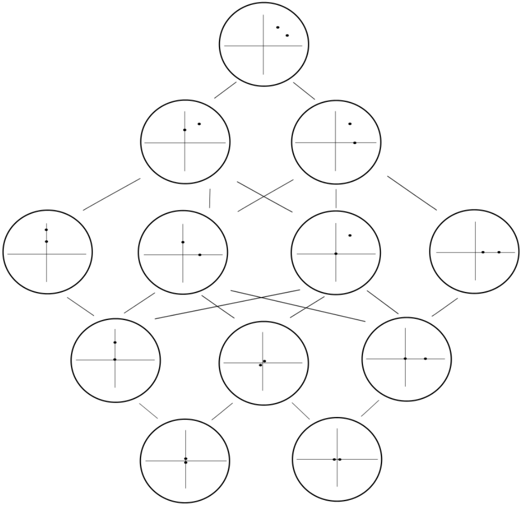


Apply 1.

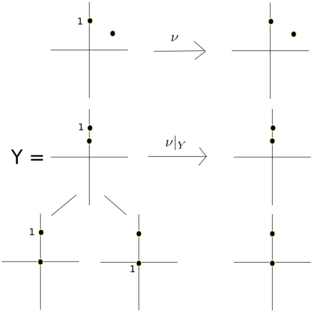


Applying 1. once more obtains the preimage of the T^2-fixed points under map $\pi : X_n \to \mathrm{Hilb}^2(\mathbb{A}_k^2)$ where X_n denotes the isospectral Hilbert scheme (i.e. the scheme of *labelled* points in the affine plane).

The algorithm produces the following stratification of $\mathrm{Hilb}^2(\mathbb{A}^2_k)$:

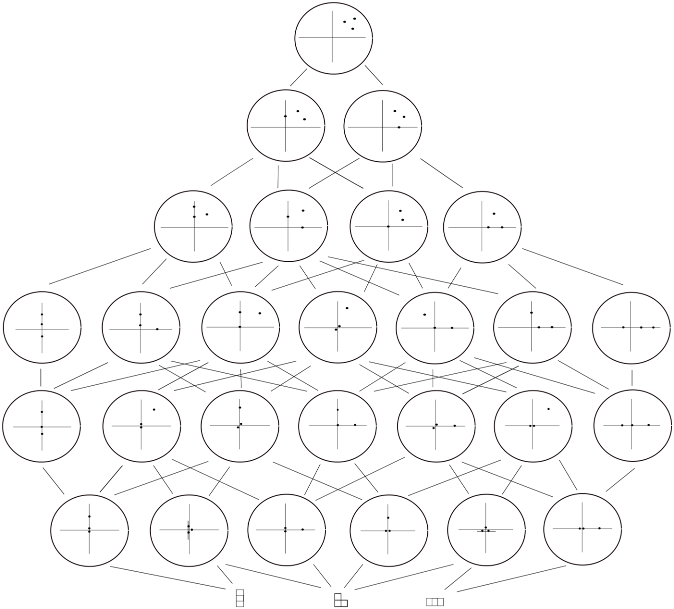


In contrast to previously studied cases (eg. the flag variety), not every compatibly split subvariety of $\mathrm{Hilb}^n(\mathbb{A}^2_k)$ is normal. As a result, additional non-split subvarieties naturally arise.

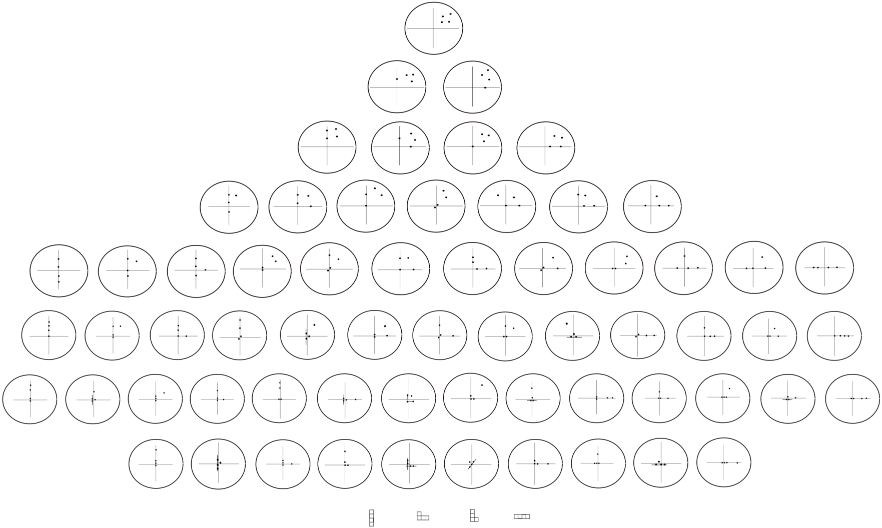


Notice that $\nu|_Y$ is generically 2:1 but is ramified along the locus where the two points collide. Letting $s = y_1 + y_2$ and $m = y_1 y_2$ be the two coordinates on $\nu(Y) \cong \mathbb{A}^2_k/S_2$, we can check that the splitting of $\nu(Y)$ is given by the section $(s^2 - 4m)^{(p-1)/2} m^{p-1}$. From this we see that $\{m = 0\}$ is compatibly split. It would be nice to be able to say that $\{s^2 = 4m\}$ (which agrees with the ramification locus of $\nu|_Y$) is "half split".

The stratification of $\mathrm{Hilb}^3(\mathbb{A}^2_k)$ by all compatibly split subvarieties:



The compatibly split subvarieties of $\mathrm{Hilb}^4(\mathbb{A}_k^2)$:



Proposition: $Y \subseteq \mathrm{Hilb}^n(\mathbb{A}^2_k)$ is a compatibly split subvariety if and only if Y is the closure of the image of the morphism

$$i : \mathrm{Hilb}^a(\text{"punctured y-axis"}) \times \mathrm{Hilb}^b(\text{"punctured x-axis"}) \times \mathrm{Hilb}^c(\mathbb{A}^2_k \setminus \{xy = 0\}) \times Z \to \mathrm{Hilb}^n(\mathbb{A}^2_k)$$

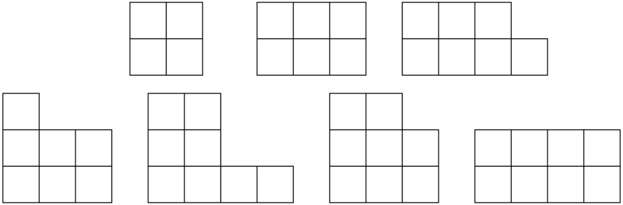
$$(I_1, I_2, I_3, I_4) \mapsto I_1 \cap I_2 \cap I_3 \cap I_4$$

for some $a, b, c \geq 0$ with $a + b + c \leq n$ and for some compatibly split $Z \subseteq \mathrm{Hilb}^{n-a-b-c}(\mathbb{A}^2_k)$, where Z is contained inside of the punctual Hilbert scheme of $n - a - b - c$ points all supported at the origin.

Thus, the problem of finding all compatibly split subvarieties of $\mathrm{Hilb}^n(\mathbb{A}^2_k)$ is equivalent to the problem of finding all compatibly split subvarieties of $\mathrm{Hilb}^m(\mathbb{A}^2_k)$, $m \leq n$, where all points are at the origin.

Notice that for $n = 1, 2, 3, 5$, all torus fixed points are indeed 0-dimensional compatibly split subvarieties of $\mathrm{Hilb}^n(\mathbb{A}^2_k)$. However, when $n = 4$, $\{\langle x^2, y^2 \rangle\}$ is not compatibly split.

For $n \leq 8$, the colength-n monomial ideals which **do not** correspond to any of the following standard sets (or their 'transposes') are compatibly split:



Conjecturally, a torus fixed point is a 0-dimensional compatibly split subvariety of $\mathrm{Hilb}^n(\mathbb{A}^2_k)$ if and only if the associated monomial ideal is integrally closed.

For the remainder of the talk, we will restrict to a specific open patch of $\mathrm{Hilb}^n(\mathbb{A}^2_k)$ (for arbitrary n).

Definition: Let λ be a colength-n monomial ideal. U_λ is the set of all $I \in \mathrm{Hilb}^n(\mathbb{A}^2_k)$ such that the monomials outside λ form a vector space basis of $k[x,y]/I$.

Example: Let $\lambda = \langle x, y^2 \rangle \in \mathrm{Hilb}^2(\mathbb{A}^2_k)$. Then $I = \langle y^2 + y, x + 2 \rangle \in U_\lambda$ as $\{1, y\}$ spans the vector space $k[x,y]/I$.

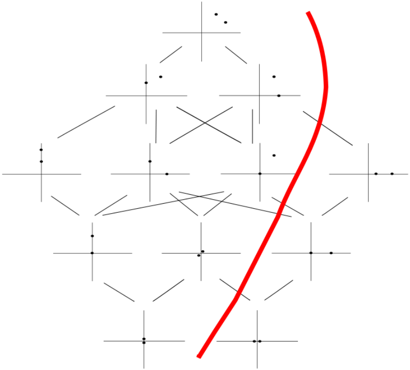
Unless otherwise indicated, we consider $U_{\langle x, y^n \rangle}$ from now on. We'll study the simpler stratification of $U_{\langle x, y^n \rangle}$ by all of its compatibly split subvarieties.

All colength-n ideals in $U_{\langle x, y^n \rangle}$ have a Gröbner basis of the form:

$$\begin{array}{lllllllllll}
y^n & - & b_1 y^{n-1} & - & b_2 y^{n-2} & - & \cdots & - & b_{n-1} y & - & b_n \\
xy^{n-1} & - & a_1 y^{n-1} & - & c_{12} y^{n-2} & - & \cdots & - & c_{1(n-1)} y & - & c_{1n} \\
xy^{n-2} & - & a_2 y^{n-1} & - & c_{22} y^{n-2} & - & \cdots & - & c_{2(n-1)} y & - & c_{2n} \\
 & & & & \vdots & & & & & & \\
xy & - & a_{n-1} y^{n-1} & - & c_{(n-1)2} y^{n-2} & - & \cdots & - & c_{(n-1)(n-1)} y & - & c_{(n-1)n} \\
x & - & a_n y^{n-1} & - & c_{n2} y^{n-2} & - & \cdots & - & c_{n(n-1)} y & - & c_{nn}
\end{array}$$

where each c_{ij} is a polynomial in $a_1, \ldots, a_n, b_1, \ldots, b_n$.
We see that $U_{\langle x, y^n \rangle} \cong \mathbb{A}^{2n} = \mathrm{Spec}\ k[a_1, b_1, \ldots, a_n, b_n]$.

As before, we begin with the $n = 2$ case. The compatibly split subvarieties of $U_{\langle x,y^2\rangle}$ are the non-empty $Y \cap U_{\langle x,y^2\rangle}$ such that $Y \subseteq \mathrm{Hilb}^2(\mathbb{A}^2_k)$ is compatibly split.

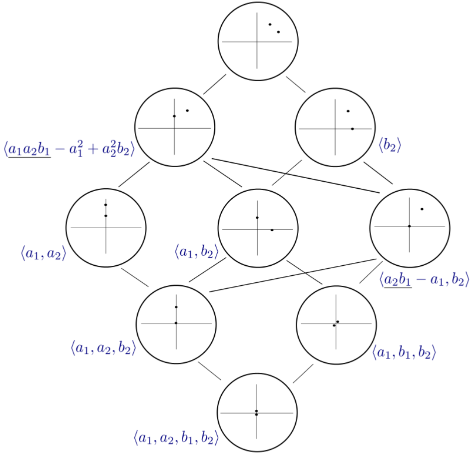


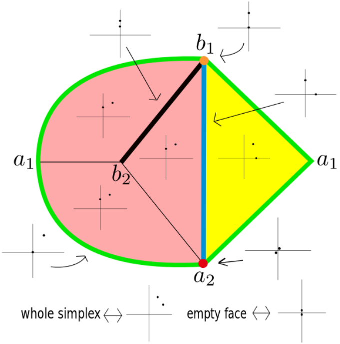
The subvarieties to the left of the red curve have non-trivial intersection with $U_{\langle x,y^2\rangle} \subset \mathrm{Hilb}^2(\mathbb{A}^2_k)$.

Recall that $U_{\langle x,y^2\rangle} \cong \mathrm{Spec}(k[a_1, b_1, a_2, b_2])$. By imposing the condition "at least one point is on an axis", we obtain the divisor $\{f_2 = 0\}$ where $f_2 = a_1 b_1 a_2 b_2 - a_1^2 b_2 + a_2^2 b_2^2$. Under the term order

$$\mathrm{Revlex}_{b_2},\ \mathrm{Lex}_{a_2},\ \mathrm{Revlex}_{b_1},\ \mathrm{Lex}_{a_1},$$

$\mathrm{init}(f_2) = a_1 b_1 a_2 b_2$. In fact, for *any* compatibly split ideal I, $\mathrm{init}(I)$ (under the same term order) is a squarefree monomial ideal. We may therefore associate a simplicial complex to each $\mathrm{init}(I)$.

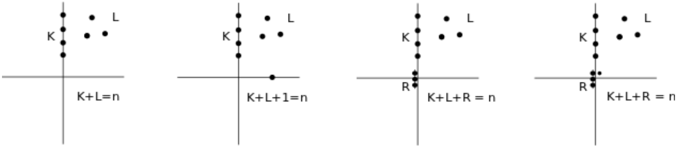




The $n = 2$ case generalizes.

Proposition:

1. With respect to the term order $\mathrm{Revlex}_{b_n}, \mathrm{Lex}_{a_n}, \ldots, \mathrm{Revlex}_{b_1}, \mathrm{Lex}_{a_1}$, $\mathrm{init}(f_n) = a_1 b_1 \ldots a_n b_n$ and (by a theorem of Knutson) *all* compatibly split subvarieties of $U_{\langle x, y^n \rangle}$ degenerate to Stanley-Reisner schemes.
2. More precisely, if $Y \subseteq U_{\langle x, y^n \rangle}$ is compatibly split, then $\mathrm{Lex}_{a_n} \mathrm{Revlex}_{b_n} Y$ is a compatibly split subvariety of $U_{\langle x, y^{n-1} \rangle} \times \mathbb{A}^2_k$.
3. $Y \subseteq U_{\langle x, y^n \rangle}$ is compatibly split if and only if it is of one of the following four types:



4. Each compatibly split subvariety of $U_{\langle x, y^n \rangle}$ degenerates to the Stanley-Reisner scheme of a shellable simplicial complex. Thus, each compatibly split subvariety of $U_{\langle x, y^n \rangle}$ is Cohen-Macaulay.
5. Suppose that Y is compatibly split and that a stratum representative of Y either
   - has no points in $\mathbb{A}^2_k \setminus \{xy = 0\}$ or
   - has at most one point on the punctured y-axis,

   then $\mathrm{init}(Y)$ is the Stanley-Reisner scheme of a simplicial ball.

We now present the ideas in the proof of the proposition. To begin, consider the following theorem.

Theorem: Let $f \in k[x_1, \ldots, x_n]$ be a degree n polynomial such that, under some term order, $\mathrm{init}(f) = \prod_i x_i$.

1. [LMP] $\{f = 0\}$ determines a Frobenius splitting of $\mathbb{A}^n$.
2. [Knutson] If I is compatibly split with respect to this splitting, then $\mathrm{init}(I)$ is compatibly split with respect to the splitting determined by $\{\mathrm{init}(f) = 0\}$.

This theorem applies to our situation:

All elements of $U_{\langle x, y^n \rangle}$ are ideals generated by polynomials of the form:

$$\begin{array}{lllllllllll}
y^n & - & b_1 y^{n-1} & - & b_2 y^{n-2} & - & \cdots & - & b_{n-1} y & - & b_n \\
xy^{n-1} & - & a_1 y^{n-1} & - & c_{12} y^{n-2} & - & \cdots & - & c_{1(n-1)} y & - & c_{1n} \\
xy^{n-2} & - & a_2 y^{n-1} & - & c_{22} y^{n-2} & - & \cdots & - & c_{2(n-1)} y & - & c_{2n} \\
 & & & & \vdots & & & & & & \\
xy & - & a_{n-1} y^{n-1} & - & c_{(n-1)2} y^{n-2} & - & \cdots & - & c_{(n-1)(n-1)} y & - & c_{(n-1)n} \\
x & - & a_n y^{n-1} & - & c_{n2} y^{n-2} & - & \cdots & - & c_{n(n-1)} y & - & c_{nn}
\end{array}$$

where each c_{ij} is a polynomial in $a_1, \ldots, a_n, b_1, \ldots, b_n$. Let M_n be the matrix of coefficients $(-c_{ij})_{1 \leq i,j \leq n}$ where $c_{i1} = a_i$. The divisor that determines the splitting on $U_{\langle x, y^n \rangle}$ is given by $\{f_n = 0\}$ where $f_n = -b_n(\det M_n)$. Under the term order Revlex_{b_n}, $\mathrm{Lex}_{a_n}, \ldots,$ Revlex_{b_1}, Lex_{a_1}, $\mathrm{init}(f_n)$ is $a_1 b_1 \cdots a_n b_n$.

For example, we have:

$$M_2 = \begin{pmatrix} -a_1 & -a_2 b_2 \\ -a_2 & -(a_1 - b_1 a_2) \end{pmatrix}, \quad M_3 = \begin{pmatrix} -a_1 & -(a_2 b_2 + a_3 b_3) & -a_2 b_3 \\ -a_2 & -(a_1 - b_1 a_2) & -a_3 b_3 \\ -a_3 & -(a_2 - b_1 a_3) & -(a_1 - b_1 a_2 - b_2 a_3) \end{pmatrix}$$

$$M_4 = \begin{pmatrix} -a_1 & -(a_2 b_2 + a_3 b_3 + a_4 b_4) & -(a_2 b_3 + a_3 b_4) & -a_2 b_4 \\ -a_2 & -(a_1 - b_1 a_2) & -(a_3 b_3 + a_4 b_4) & -a_3 b_4 \\ -a_3 & -(a_2 - b_1 a_3) & -(a_1 - b_1 a_2 - b_2 a_3) & -a_4 b_4 \\ -a_4 & -(a_3 - b_1 a_4) & -(a_2 - b_1 a_3 - b_2 a_4) & -(a_1 - b_1 a_2 - b_2 a_3 - b_3 a_4) \end{pmatrix}$$

Computing the determinant of M_4 using cofactors along the last column, we get:

$$\det\ M_4 = (M_4)_{44}(\det\ M_3) + b_4(\cdots).$$

Taking the terms with the smallest power of b_4 (i.e. computing $\mathrm{Revlex}_{b_4}(\det\ M_4)$) yields:

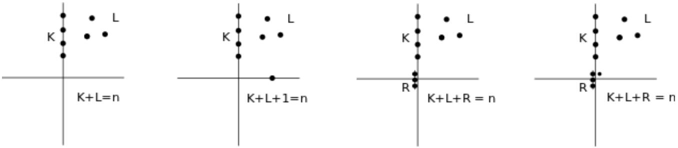
$$(M_4)_{44}(\det\ M_3).$$

Taking Lex_{a_4} of this polynomial yields:

$$a_4 b_3(\det\ M_3).$$

Proposition:

1. With respect to the term order Revlex_{b_n}, $\mathrm{Lex}_{a_n}, \ldots,$ Revlex_{b_1}, Lex_{a_1}, $\mathrm{init}(f_n) = a_1 b_1 \ldots a_n b_n$ and (by a theorem of Knutson) *all* compatibly split subvarieties of $U_{\langle x, y^n \rangle}$ degenerate to Stanley-Reisner schemes.
2. More precisely, if $Y \subseteq U_{\langle x, y^n \rangle}$ is compatibly split, then $\mathrm{Lex}_{a_n}\mathrm{Revlex}_{b_n} Y$ is a compatibly split subvariety of $U_{\langle x, y^{n-1} \rangle} \times \mathbb{A}^2_k$.
3. $Y \subseteq U_{\langle x, y^n \rangle}$ is compatibly split if and only if it is of one of the following four types:



4. Each compatibly split subvariety of $U_{\langle x, y^n \rangle}$ degenerates to the Stanley-Reisner scheme of a shellable simplicial complex. Thus, each compatibly split subvariety of $U_{\langle x, y^n \rangle}$ is Cohen-Macaulay.
5. Suppose that Y is compatibly split and that a stratum representative of Y either
   - has no points in $\mathbb{A}^2_k \setminus \{xy = 0\}$ or
   - has at most one point on the punctured y-axis,

   then $\mathrm{init}(Y)$ is the Stanley-Reisner scheme of a simplicial ball.

We do not sketch the proof of 3. here.

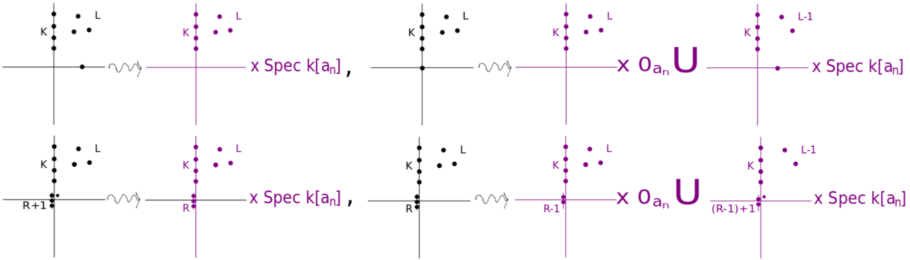
For 4., we first determine $\mathrm{init}(Y)$ for each compatibly split subvariety Y by understanding the degenerations given by Revlex_{b_n} and Lex_{a_n}.

Lemma: Let Y be a subvariety of $\mathrm{Spec}\, k[x_1, ..., x_n]$. Let H be the hyperplane $\{x_1 = 0\}$.

- If $Y \subseteq H$ then $\mathrm{Revlex}_{x_1}(Y) = Y \times O_{x_1} \subseteq H \times \mathrm{Spec}\, k[x_1]$.
- If $Y \not\subseteq H$ then $\mathrm{Revlex}_{x_1}(Y) = (Y \cap H) \times \mathrm{Spec}\, k[x_1] \subseteq H \times \mathrm{Spec}\, k[x_1]$.

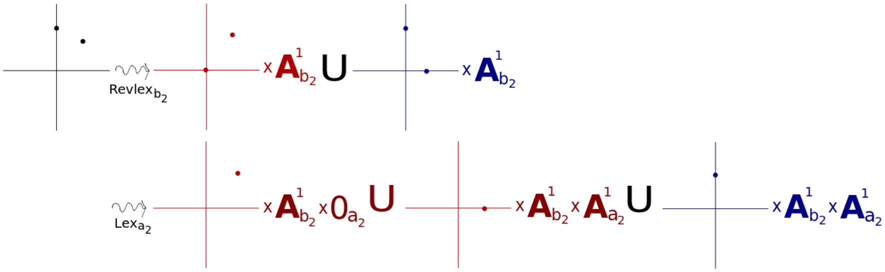
In our case, $H = \{b_n = 0\}$, which is the subvariety "one point is on the x-axis".

The Lex_{a_n} degenerations can be described by the following pictures:

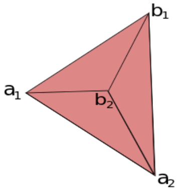


Using the step by step degenerations, we can compute the initial schemes of a compatibly split ideal Y. We can therefore determine the simplicial complexes associated to $\mathrm{init}(Y)$.

Example:



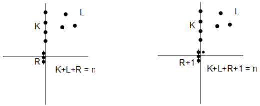
Thus, $\mathrm{init}(I(Y))$ is $\langle a_2 \rangle \cap \langle b_1 \rangle \cap \langle a_1 \rangle$ and the associated simplicial complex is



To each compatibly split subvariety of $U_{\langle x,y^n \rangle}$, the step by step degenerations allow us to associate "words" in the following "letters":

$$(1)\ \ a\uparrow,\quad (2)\ \ \hat{a}\uparrow,\quad (3)\ \ aa\uparrow,\quad (4)\ \ aa,\quad (5)\ \ \hat{a}$$

Let Y be a compatibly split subvariety of $U_{\langle x,y^n \rangle} \subset \mathrm{Hilb}^n(\mathbb{A}^2_k)$. Suppose that a general element of Y has L points in $\mathbb{A}^2_k \setminus \{xy = 0\}$, K points on the punctured y-axis, and R points "vertically stacked" at the origin. For example:



Proposition: The facets of the simplicial complex associated to $\mathrm{init}(Y)$ are in one-to-one correspondence with words of the following form:

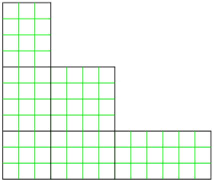
$$(\text{word in } (1),\ (2),\ (3))\ \mid\ (\text{word in } (4),\ (5))\ \mid\ (a \text{ iff "}R+1\text{" at origin})$$

such that

$$\#(1) + \#(3) + \#(4) = L,\quad \#(2) + \#(3) = K,\quad \#(4) + \#(5) = R.$$

What about other patches U_λ which are isomorphic to $\mathbb{A}^{2n}$?

Conjecture: There are specific coordinates $a_1, \dots, a_n, b_1, \dots, b_n$, chosen in an analogous manner to the $\lambda = \langle x, y^n \rangle$ case, such that $\{f_\lambda = 0\}$ is a residual normal crossings divisor. (This has been checked in Macaulay 2 for $n \leq 8$.) The term order such that $\mathrm{init} f_\lambda = a_1 b_1 \cdots a_n b_n$ depends on the shape of the partition associated to λ and is a generalization (in a precise way) of the term order in the $U_{\langle x, y^n \rangle}$ case.



Question: What about on patches that are not isomorphic to affine space? Can you do this in a formal neighborhood of a torus fixed point?

Finally, we consider a connection to Poisson geometry.

Consider $\mathbb{A}^2$ with the Poisson tensor $xy\frac{d}{dx} \wedge \frac{d}{dy}$. This induces a Poisson tensor on the Hilbert scheme. (See a paper of Bottacin for the general situation.)
On the open patch $U_{\langle x,y^n \rangle}$ the Poisson bracket is given by*:

$$\{b_i, b_j\} = 0, \qquad \{a_i, a_j\} = -\sum_{k=1}^{j-1} a_k a_{j-(k-i)}, \ i < j$$

$$\{a_i, b_j\} = \sum_{k=j}^{n} a_{k-(j-i)} b_k, \ i \leq j, \quad \{a_i, b_j\} = a_{i-j} - \sum_{k=1}^{j-1} a_{i-j+k} b_k, \ i > j$$

The Pfaffian of the $(2n \times 2n)$-matrix gives the anticanonical divisor that determines the induced splitting of $U_{\langle x,y^n \rangle}$. With respect to the same weighting of the variables which gives $\mathrm{init}(f_n) = a_1 b_1 \ldots a_n b_n$, we can degenerate the Poisson tensor to a *log canonical* tensor. Furthermore, all compatibly split subvarieties of $U_{\langle x,y^n \rangle}$ are Poisson subvarieties.

* Conjectural for large n.

Thank You.