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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Elizabeth Gross **Email/Phone:** egross7@uic.edu

Speaker's Name: Adam Van Tuyl

Talk Title: Persistence of square-free monomial ideals

Date: 12 / 7 / 12 **Time:** 11 : 30 am / pm (circle one)

List 6-12 key words for the talk: monomial ideals, persistence, associated primes, edge ideals, cover ideals, chromatic number

Please summarize the lecture in 5 or fewer sentances:

Surveys & introduces ~~[illegible]~~ new families of squarefree monomial ideals that ~~[illegible]~~ satisfy the persistence property

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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Do square-free monomial ideals have the persistence property?

Adam Van Tuyl

Lakehead University

December 2012

Associated Primes

Let $R = k[x_1, \ldots, x_n]$, k a field.

Definition

For any ideal $I \subseteq R$ a prime ideal P is an *associated prime* of I if there is an element $M \in R$ such that $I : \langle M \rangle = P$. The set of associated primes of I is denoted $\mathrm{Ass}(I)$.

A problem about associated primes first(?) raised by Ratliff (1976):

Problem

Describe the sets $\mathrm{Ass}(I^s)$ *as* s *varies?*

Eventual behaviour

Theorem (Brodmann (1979))

Let $I \subseteq R = k[x_1, \ldots, x_n]$. *Then there exists an integer* s_0 *such that*

$$\mathrm{Ass}(I^{s_0}) = \mathrm{Ass}(I^s) \text{ for all } s \geq s_0.$$

Definition (Index of stability)

$$\mathrm{astab}(I) = \min\{s_0 \mid \mathrm{Ass}(I^{s_0}) = \mathrm{Ass}(I^s) \text{ for all } s \geq s_0\}.$$

Three Problems

1. Bound $\operatorname{astab}(I)$ in terms of invariants of I and R.
2. Describe the elements of $\operatorname{Ass}(I^{\operatorname{astab}(I)})$.
3. What is the initial behaviour of $\operatorname{Ass}(I^s)$ for $s \leq \operatorname{astab}(I)$.

This talk will focus on 3.

Definition

$I \subseteq R$ has the **persistence property** if

$$\operatorname{Ass}(I^s) \subseteq \operatorname{Ass}(I^{s+1}) \text{ for all } s \geq 1.$$

"Wild" initial behaviour

Brodmann showed the existence of an ideal without the persistence property. Here is a monomial example:

Example (Herzog, Hibi)

Consider

$$I = (a^6, a^5b, ab^5, b^6, a^4b^4c^4, b^4d, a^4e^2f^3, b^4e^3f^2)$$

in the ring $R = k[a,b,c,d,e,f]$. Then

$$\begin{array}{rcl} (a,b,c,d,e,f) & \in & \mathrm{Ass}(I) \\ (a,b,c,d,e,f) & \notin & \mathrm{Ass}(I^2) \\ (a,b,c,d,e,f) & \in & \mathrm{Ass}(I^3) \\ (a,b,c,d,e,f) & \notin & \mathrm{Ass}(I^4) \end{array}$$

Original context for this example: studying the depth function $f(s) = \mathrm{depth}(R/I^s)$.

The previous example can be generalized:

Theorem (Bandari, Herzog, Hibi)

For any given integer $m \geq 1$, there exists a monomial I ideal in $2m+4$ variables such that

$$\mathfrak{m} \in \mathrm{Ass}(I^s) \quad \textit{for all } s \textit{ odd and } s \leq 2m+1$$

and

$$\mathfrak{m} \notin \mathrm{Ass}(I^s) \quad \textit{for all } s \textit{ even and } s \leq 2m.$$

Here $\mathfrak{m} = \langle x_1, \cdots, x_{2m+4} \rangle \subseteq k[x_1, \ldots, x_{2m+4}]$.

Monomial ideals can very "un-persistent"

HOWEVER....

HOWEVER....

... there are no examples of square-free monomial ideals that do *not* have the persistence property.

Question

Do square-free monomial ideals have the persistence property?

Positive Evidence

Simplest examples of square-free monomial ideals:

1. monomial ideals generated in degree 1, i.e.,

$$I = \langle x_{i_1}, \ldots, x_{i_t} \rangle$$

2. unmixed height 1 monomial ideals, i.e.,

$$I = \langle x_{i_1} \rangle \cap \langle x_{i_2} \rangle \cap \cdots \cap \langle x_{i_t} \rangle = \langle x_{i_1} x_{i_2} \cdots x_{i_t} \rangle$$

Both ideals are complete intersections, so

$$\mathrm{Ass}(I^s) = \mathrm{Ass}(I) \text{ for all } s \geq 1$$

Note: Two classes are related via Alexander Duality.

Positive Evidence II

Next simplest classes of square-free monomial ideals:

1. monomial ideals generated in degree 2.
2. unmixed height 2 monomial ideals.

Again, two classes are related via Alexander Duality.

More common names:

1. *Edge ideals* of finite simple graphs
2. *Cover ideals* of finite simple graphs

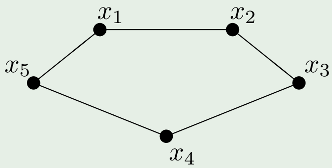
Edge Ideals

$G = (V_G, E_G)$ will denote a finite simple (no loops or multiple edges) graph with vertex set $V_G = \{x_1, \ldots, x_n\}$ and edge set E_G.

Definition (Edge Ideal)

$$I(G) = \langle x_i x_j \mid \{x_i, x_j\} \in E_G \rangle \subseteq R$$

Example



$$I(G) = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1 \rangle$$

Persistence of Edge Ideals

Theorem (Martínez-Bernal, Morey, Villarreal)

For any finite simple graph G, $I(G)$ has the persistence property.

Completes the program of

- Chen, Morey, Sung (2002)
- Morey, Villarreal (2010)

Cover ideal

Definition (Cover Ideal)

$$J = J(G) = \bigcap_{\{x_i,x_j\} \in E_G} \langle x_i, x_j \rangle \subseteq R$$

A subset $W \subseteq V_G$ is a *vertex cover* if for every edge $e \in E_G$, $W \cap e \neq \emptyset$.

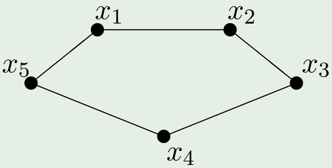
Lemma

Let G be a finite simple graph. Then

$$J = \langle x_{i_1} \cdots x_{i_r} \mid W = \{x_{i_1}, \ldots, x_{i_r}\} \text{ is a vertex cover} \rangle.$$

Example

For the graph G below:



$$J(G) = \langle x_1x_2x_4, x_2x_3x_5, x_3x_4x_1, x_4x_5x_2, x_5x_1x_3 \rangle$$

Perfect Graphs

- Let G be a simple graph and let $P \subseteq V_G$. The *induced graph* on P, denoted G_P, is the graph with

$$V_{G_P} = P \text{ and } E_{G_P} = \{\{x_i, x_j\} \in E_G \mid \{x_i, x_j\} \subseteq P\}.$$

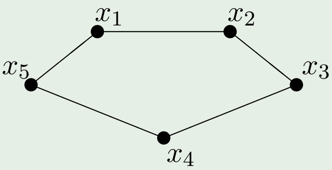
- The *complete graph of order* n, denoted $\mathcal{K}_n$, is the graph on n vertices and edge set $E_G = \{\{x_i, x_j\} \mid 1 \leq i < j \leq n\}$.
- An induced subgraph G_P is a *clique* if $G_P = \mathcal{K}_{|P|}$.
- The *chromatic number* of G, denoted $\chi(G)$, is the least number of colours in a vertex-colouring of G

Definition

A graph G is *perfect* if for every induced graph G_P, $\chi(G_P)$ equals the size of largest clique in G_P.

Example

The graph G below is not perfect:



We have $\chi(G) = 3$, but the size of the largest induced clique is 2.

Persistence of Cover Ideals

Theorem (Villarreal; Francisco, Hà, VT)

For any finite simple graph G that is perfect, $J(G)$ has the persistence property. Moreover,

$$\chi(G) - 1 = \operatorname{astab}(J(G))$$

• There are also families of non-perfect graphs for which we know that $J(G)$ has the persistence property. For example if G is an odd cycle or odd anti-cycle (the complement of a cycle).

• To come: a colouring conjecture that implies that for all $J(G)$ has the persistence property for all G

1. monomial ideals generated in degree $d \geq 2$.
2. unmixed height $d \geq 2$ monomial ideals.

For the first class, we have the following results:

Theorem

1. *[Herzog, Rauf, Vladoiu] Let I be a polymatroidal ideal (so I is a square-free monomial ideal generated in degree d). Then I has the persistence property.*
2. *[Herzog, Qureshi] Let I be square-free monomial ideal with the property that I^s has a linear resolution for all $s \geq 1$. Then I has the persistence property.*

Generalizing Cover Ideals

Definition

Fix an integer $t \geq 1$. The **partial** t**-cover ideal** of G is the monomial ideal

$$J_t(G) = \bigcap_{x \in V_G} \left(\bigcap_{\{x_{i_1},\ldots,x_{i_t}\} \subseteq N(x)} \langle x, x_{i_1}, \ldots, x_{i_t} \rangle \right).$$

- When $t = 1$, $J_1(G) = J(G)$, the cover ideal
- $J_t(G)$ is a unmixed height $(t+1)$ square-free monomial ideal

Further Positive Evidence

Theorem (Bhat-Biermann-VT)

Fix a $t \geq 1$. If G is a tree, then $J_t(G)$ has the persistence property. Moreover

$$\operatorname{astab}(J_t(G)) = \begin{cases} 1 & \text{if } t = 1 \\ \min\{s \mid s(t-1) \geq \Delta(G) - 1\} & \text{if } t > 1 \end{cases}$$

where $\Delta(G)$ is largest degree of a vertex in G.

Strategies for proving persistence

Because I is square-free, all associated primes are monomial, i.e., $P = \langle x_{i_1}, \ldots, x_{i_t} \rangle$. Localization gives:

Theorem

$P = \langle x_{i_1}, \ldots, x_{i_t} \rangle$ is an associated prime of I^s in $k[x_1, \ldots, x_n]$ if and only if $PR_P = \langle x_{i_1}, \ldots, x_{i_t} \rangle$ is an associated prime of $(I_P R_P)^s$ in $R_P = k[x_{i_1}, \ldots, x_{i_t}]$

To prove persistence, enough to show:

Question

Let I be a square-free monomial ideal. If $\mathfrak{m} \in \operatorname{Ass}(I^s)$, then is $\mathfrak{m} \in \operatorname{Ass}(I^{s+1})$?

New annihilators from old

Lemma

Suppose that $I^s : \langle T \rangle = \langle x_1, \ldots, x_n \rangle$. *If there exists an element* $M \in I$ *such that* $MT \notin I^{s+1}$, *then* $I^{s+1} : \langle MT \rangle = \langle x_1, \ldots, x_n \rangle$.

Proof.

Since $MT \notin I^{s+1}$,

$$I^{s+1} : \langle MT \rangle \subseteq \langle x_1, \ldots, x_n \rangle.$$

For any i, $Tx_i \in I^s$, so $MTx_i \in I^{s+1}$. Thus

$$\langle x_1, \ldots, x_n \rangle \subseteq I^{s+1} : \langle MT \rangle.$$

□

When does such an element M exist?

Theorem

Let I be a square-free monomial ideal such that $I^{s+1} : I = I^s$ for all $s \geq 1$. Then I has the persistence property.

Under the hypothesis, if there exists a T such that $I^s : \langle T \rangle = \mathfrak{m}$, then there must exist a generator $M \in I$ such that $MT \notin I^{s+1}$.

Theorem (Martínez-Bernal, Morey, Villarreal)

If $I = I(G)$, then $I^{s+1} : I = I^s$ for all s.

To prove this result, a graph theory result of Berge on matchings in a graph was required.

Ratliff Ideals

Definition (Herzog-Qureshi)

An ideal I is *Ratliff* if $I^{s+1} : I = I^s$ for all $s \geq 1$.

- For any ideal in R, Ratliff showed that $I^{s+1} : I = I^s$ for $s \gg 0$.
- Edge ideals
- Polymatroidal ideals
- If I is normal, i.e., $I^k = \overline{I^k}$ for all k, where $\overline{J}$ is the the integral closure of J.

  Villarreal proved that $J(G)$, the cover ideal of G, is normal when G is perfect.

Not all square-free monomial ideals are Ratliff.

Example (Martínez-Bernal, Morey, Villarreal)

The ideal

$$I = \langle x_1x_2x_5, x_1x_3x_4, x_1x_2x_6, x_1x_3x_6, x_1x_4x_5, x_2x_3x_4, x_2x_3x_5, x_2x_4x_6, x_3x_5x_6, x_4x_5x_6\rangle$$

has $I^3 : I \neq I^2$. (It does satisfy the persistence property!)

A combinatorial interpretation

Return to the cover ideal of $J(G)$.

Definition

A graph G is **critically s-chromatic** if $\chi(G) = s$, but $\chi(G \setminus x) = s - 1$ for every $x \in V_G$. If G is critically s-chromatic for some s, G is called a **critical graph**.

Example

Any odd cycle is a critically 3-chromatic graph.

Expansion

Definition

The **expansion** of G at a vertex x_i is the graph $G' = G[\{x_i\}]$ whose vertex set is given by $V_{G'} = (V_G \setminus \{x_i\}) \cup (x_{i,1}, x_{i,2})$ and with edge set

$$E_{G'} = \{\{u,v\} \in E_G \mid u \neq x_i \text{ and } v \neq x_i\} \cup$$

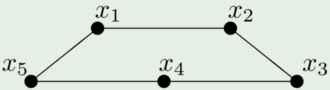
$$\{\{u, x_{i,1}\}, \{u, x_{i,2}\} \mid \{u, x_i\} \in E_G\} \cup \{\{x_{i,1}, x_{i,2}\}\}$$

For any $W \subseteq V_G$, the **expansion** of G at W, denoted $G[W]$, is formed by successively expanding all the vertices of W in any order.

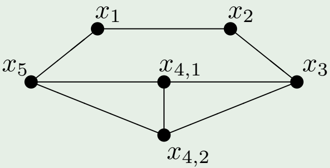
Example of Expansion

Example

Consider the 5-cycle on the vertex set $\{x_1, \ldots, x_5\}$.



Then the expansion of G at the vertex x_4 is $G[\{x_4\}]$:



The s-th expansion

Definition

Let G be a simple graph with vertex set V_G. The **second expansion** of G, denoted G^2, is the graph $G^2 = G[V_G]$, i.e., expand all the vertices of G.

The s-**th expansion** of G, denoted G^s, is the graph $G^s = G^{s-1}[V_G]$ for $s \geq 2$. (G^{s-1} contains a copy of G, expand those vertices inside G^{s-1}.)

Theorem (Francisco, Hà, VT)

Let G be a finite simple graph with cover ideal $J = J(G)$. Then

$$\langle x_{i_1}, \ldots, x_{i_r} \rangle \in \mathrm{Ass}(J^s)$$

if and only if there exists some set T with

$$\{x_{i_1,1}, \ldots, x_{i_r,1}\} \subseteq T \subseteq \{x_{i_1,1}, \ldots, x_{i_1,s}, \ldots, x_{i_r,1}, \ldots, x_{i_r,s}\}$$

such that G_T^s is critically $(s+1)$-chromatic.

Holds for *all* square-free monomial ideals. Identify the square-free monomial ideal with a *hypergraph*. The associated primes are then related to critically chromatic hypergraphs.

A conjecture

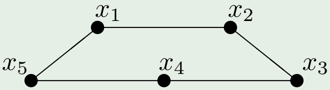
Conjecture

If G is a critically s-chromatic graph, then there exists a subset $W \subseteq V_G$ such that $G[W]$ is a critically $(s+1)$-chromatic graph.

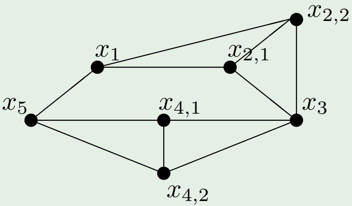
- If the conjecture is true, we can construct critically $(s+d)$-chromatic graphs for any $d \geq 1$ by repeated applying the result.
- Conjecture true for cliques $\mathcal{K}_s$ (expand a clique at any vertex and the new graph is the clique $\mathcal{K}_{s+1}$).

Example

The 5-cycle is a critically 3-chromatic graph:



If we expand G at $W = \{x_2, x_4\}$, we get a critically 4-chromatic graph $G[W]$:



The persistence property and the conjecture

Theorem

Suppose the Conjecture is true. Then the cover ideal J has the persistence property.

Proof.

(Sketch) Suppose $\langle x_{i_1}, \ldots, x_{i_r} \rangle \in \mathrm{Ass}(J^s)$. There exists a set T with

$$\{x_{i_1,1}, \ldots, x_{i_r,1}\} \subseteq T \subseteq \{x_{i_1,1}, \ldots, x_{i_1,s}, \ldots, x_{i_r,1}, \ldots, x_{i_r,s}\}$$

such that G_T^s is critically $(s+1)$-chromatic.

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such that G_T^s is critically $(s+1)$-chromatic.

By Conjecture, there exists $W \subseteq V_{G_T^s}$ such that $G_T^s[W]$ is a critically $(s+2)$-chromatic graph.

The persistence property and the conjecture

Theorem

Suppose the Conjecture is true. Then the cover ideal J has the persistence property.

Proof.

(Sketch) Suppose $\langle x_{i_1}, \ldots, x_{i_r} \rangle \in \mathrm{Ass}(J^s)$. There exists a set T with

$$\{x_{i_1,1}, \ldots, x_{i_r,1}\} \subseteq T \subseteq \{x_{i_1,1}, \ldots, x_{i_1,s}, \ldots, x_{i_r,1}, \ldots, x_{i_r,s}\}$$

such that G^s_T is critically $(s+1)$-chromatic.

By Conjecture, there exists $W \subseteq V_{G^s_T}$ such that $G^s_T[W]$ is a critically $(s+2)$-chromatic graph.

The graph $G^s_T[W]$ is isomorphic to a subgraph of G^{s+1}. So, exists T':

$$\{x_{i_1,1}, \ldots, x_{i_r,1}\} \subseteq T' \subseteq \{x_{i_1,1}, \ldots, x_{i_1,s+1}, \ldots, x_{i_r,1}, \ldots, x_{i_r,s+1}\}$$

such that $G^{s+1}_{T'}$ is critically $(s+2)$-chromatic. So $\langle x_{i_1}, \ldots, x_{i_r} \rangle \in \mathrm{Ass}(J^{s+1})$. □

Evidence for the Conjecture

Theorem

The Conjecture holds if we also assume

$$\chi(G) - 1 < \chi_f(G) \leq \chi(G)$$

Here, $\chi_f(G)$ is the fractional chromatic number.

(Our proof is algebraic.)

Corollary

The Conjecture holds for the following critical graphs: cliques, odd holes, and odd antiholes.

Future directions?

- Look at families of square-free monomial ideals which are not generated in the same degree nor height unmixed.
- What square-free monomial ideals are Ratliff?
- Does the colouring conjecture hold?