MSRI

Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Elizabeth Gross Email/Phone: egross7@uic.edu

Speaker's Name: Greg Smith

Talk Title: Sums of squares & non-negative polynomials in multigraded rings

Date: 12 / 7 / 12 Time: 2 : 00 am / pm (circle one)

List 6-12 key words for the talk: sums of squares, non-negative polynomials, varieties of minimal ~~a~~ degrees, toric varieties, Ehrhart series, multigraded rings

Please summarize the lecture in 5 or fewer sentances:

Explores the question of when is ~~some~~ nonnegativity equivalent to being a sum of squares. Provides new examples in which every non-negative homogeneous polynomial is a sum of squares.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Sums of squares and nonnegative polynomials in multigraded rings

Gregory G. Smith

Queen's University

7 December 2012

Fundamental Problem

A polynomial $f \in S := \mathbb{R}[x_0, \ldots, x_n]$ is

- nonnegative if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$;
- a sum of squares if there exists $g_1, \ldots, g_\ell \in S$ such that $f = g_1^2 + \cdots + g_\ell^2$.

Sums of squares can provide efficient certificates for nonnegativity.

PROBLEM: When is nonnegativity the same as being a sum of squares?

The Solution?

Assume S has the standard $\mathbb{N}$-graded; in other words, S is the Cox ring of $\mathbb{P}^n$.

HILBERT (1888): When we have

- $n = 1$ (univariate nonhomogeneous),
- $2d = 2$ (quadratic forms), or
- $n = 2$, $2d = 4$ (ternary quartics),

every nonnegative $f \in S_{2d}$ is a sum of squares. In all other cases, there exists a $f \in S_{2d}$ that is not sums of squares.

Interpretation

TAGLINE: Except for a few coincidences, general nonnegative polynomials are not sums of squares.

MOTZKIN: $x_0^4 x_1^2 + x_0^2 x_1^4 + x_2^6 - 3 x_0^2 x_1^2 x_2^2$ is nonnegative, but not a sum of squares.

BLEKHERMAN (2003): Fix $d > 1$. As $n \to \infty$, there are significantly more nonnegative polynomials than sums of squares.

More Solutions!

CHOI-LAM-REZNICK (1980): Let S be the Cox ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}$; S is $\mathbb{N}^\ell$-graded. Every nonnegative $f \in S_{2d}$ is a sum of squares if and only if $\ell = 2$ and $2d = (2d_1, 2)$ or $(2, 2d_2)$.

QUESTIONS: Are there more examples? What explains the equality between nonnegativity and sums of squares?

General Setting

Let X be a projective variety over $\mathbb{R}$.

For line bundle $\mathscr{O}_X(D)$, $s \in H^0(X, \mathscr{O}_X(2D))$ is

- nonnegative if its evaluation at each point in $X(\mathbb{R})$ is nonnegative;
- a sum of squares if there are $t_1, \ldots, t_\ell \in V := H^0(X, \mathscr{O}_X(D))$ such that $s = \mu(t_1^2) + \cdots + \mu(t_\ell^2)$ where $\mu : \mathrm{Sym}^2(V) \to H^0(X, \mathscr{O}_X(2D))$

PROBLEM: Determine for which $(X, \mathscr{O}_X(D))$ nonnegativity equals a sum of squares.

Convex Cones

LEMMA: The collection of nonnegative sections (resp. sums of squares) forms a closed convex cone $P_{X,2D}$ (resp. $\Sigma_{X,2D}$).

Assume $\mathcal{O}_X(D)$ is globally generated. Let Y be the image of the induced map $\varphi : X \to \mathbb{P}^m$.

LEMMA: If φ surjects $X(\mathbb{R})$ onto $Y(\mathbb{R})$, then $P_{X,2D} = \Sigma_{X,2D}$ if and only if $P_{Y,2D} = \Sigma_{Y,2D}$.

LEMMA: If I_Y is not generated by quadrics then $P_{Y,2D} \neq \Sigma_{Y,2D}$.

The 'Real' Solution

Focus on $X \subseteq \mathbb{P}^m$ cut out by quadrics where $X(\mathbb{R})$ is Zariski dense and D is a (totally real) hyperplane section.

Count the non-Koszul first syzygies:
$b := h^0(X, \mathcal{O}(2D)) - (m+1)h^0(X, \mathcal{O}(D)) + \binom{m+1}{2}$.

THEOREM (Blekhermann-Smith-Velasco):
We have $b(D) = 0$ iff $P_{X,2D} = \Sigma_{X,2D}$.

IDEA FOR $\Longrightarrow$: Show that the extremal rays of $\Sigma^*_{X,2D}$ come from evaluation at a point.

Varieties of Minimal Degree

PROPOSITION: We have $b(D) = 0$ if and only if $\deg(X) = 1 + \operatorname{codim}(X)$.

DELPEZZO-BERTINI (1907): If $\deg(X) = 1 + \operatorname{codim}(X)$ then X is a cone over a smooth such variety. If X is smooth, then it is either

- a quadric hypersurface
- rational normal scroll, or
- the Veronese surface $\mathbb{P}^2 \subseteq \mathbb{P}^5$

Toric Varieties

Rational normal scrolls are toric varieties.
Let Δ be the polytope associated to $\mathscr{O}_X(D)$.
If the corresponding Ehrhart series is

$$\sum_{j\geq 0} |j\Delta \cap \mathbb{Z}^n|\, t^j = \frac{h_0^* + h_1^*\, t + \cdots + h_n^*\, t^n}{(1-t)^{n+1}},$$

then we have $b(D) = h_2^*$

BATYREV-NILL (2007) give a polyhedral description of the possible Δ.