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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Elizabeth Gross Email/Phone: egross7@uic.edu

Speaker's Name: Ezra Miller

Talk Title: Alexander duality & total positivity: a cluster/commutative algebra connection

Date: 12/7/12 Time: 3:30 am / pm (circle one)

List 6-12 key words for the talk: subword complex, determinantal ideals, initial ideals, Poincaré duality, cluster algebra, totally nonnegative unipotent matrices

Please summarize the lecture in 5 or fewer sentences:

Introduces subword simplicial complexes & their connection to the topology of Lusztig's parametrization of the totally ~~[illegible]~~ nonnegative unipotent matrices.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Poincaré duality & total positivity

Summary

determinantal ideals $I_w \subseteq k[n \times n \text{ matrices}]$

$\Rightarrow \mathrm{in}_<(I_w) = k[\Delta]$ $\quad w \in S_n$

$<$ antidiagonal

Δ = subword complex

[w/ Knutson ~ 2001]

$N = \begin{bmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{bmatrix}$ $\quad$ TNN = { matrices/$\mathbb{R}$ with all minors ≥ 0 }

$\mathbb{R}^d_{\geq 0} \xrightarrow{\text{word}} \mathrm{TNN} \cap N$ $\quad$ [Lusztig 1994]

fibers naturally stratified

poset of strata = face poset $\quad \nabla$

Δ - subword complex

[w / Davis + Hersh]

Connection to cluster algebras

cluster variable $\rightsquigarrow$ module X over preprojective algebra

$\rightsquigarrow \phi_X : N \to \mathbb{C}$ $\quad$ defined on words

$$x_{i_1}(t_1) \cdots x_{i_d}(t_d) \mapsto \sum_{a \in \mathbb{N}^d} \chi(T_{x,i,a}) \frac{t^a}{a!}$$

variety of composition series

Outline

I. subword complexes

II. I_w

III. TNN $\cap$ N

I. $S_n = \langle \Sigma \rangle \quad \Sigma = \{s_1, \dots, s_{n-1}\} \quad s_i = (i \;\; i+1)$

Def A word is $Q = (\sigma_1, \dots, \sigma_d)$ with $\sigma_j \in \Sigma$. A subword $P \subseteq Q$ is a subsequence.

$w \in S_n$ has length $\ell(w) = \min \{ |Q| \mid w = \prod Q \}$

$\prod Q = \sigma_1 \dots \sigma_d \in S_n$

Q is reduced if $|Q| = \ell(w)$)

e.g. $Q = (s_2, s_3, s_2, s_3)$ not reduced

$P \quad s_2 \; s_3 \; s_2$ reduced $\prod P =$
$(2\;3)(3\;4)(2\;3$
$= 1432$

$P' \quad s_3 \, s_2 \, s_3$ reduced
$\prod P' = 1432$

Def [KM]

word Q and $w \in S_n$ $\Rightarrow$ subword complex

$\Delta(Q, w)$ have

- vertex set $\sigma_1, \ldots, \sigma_d$
- facets $\{ Q \setminus P \mid P \subseteq Q \text{ \& } \prod P = w \text{ reduced} \}$

$$\Delta((s_2, s_3, s_2, s_3), 1432)$$

$$= \quad \underset{a}{\bullet} \qquad \underset{d}{\bullet}$$

$$\Delta(Q, \underset{s_2 s_3}{\underset{\parallel}{1342}})$$

Thm [KM] $\Delta(Q, w) \cong$ ball or sphere

Q reduced $\Rightarrow$ ball

Pf Exercise.

works for all Coxeter groups.

II $\quad I_w \qquad w \in S_n \longleftrightarrow$ permutation matrix

e.g.
$$\begin{array}{c} 2 \\ 4 \\ 3 \\ 1 \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

rank(w) =

0	1	1	1
0	1	1	2
0	1	2	3
1	2	3	4

~~IIII~~

variety of I_w is

$\{n \times n$ matrices whose upper left ~~submatrix~~ submatrix has rank $\le r_{ij}\ \forall i,j\}$

$$I_{2341} = \langle z_{11}, z_{21}, z_{31}, z_{13}z_{22} - z_{12}z_{23} \rangle$$

<u>Thm</u> [KM] $\text{in}_< I_w = I(\Delta(Q_n, w))$ for any antidiagonal term order $<$.

$$\begin{array}{|c|c|c|c|l}
\hline
1 & 2 & 3 & 4 & \leftarrow 1 \\ \hline
2 & 3 & 4 & 5 & \leftarrow 2 \\ \hline
3 & 4 & 5 & 6 & \leftarrow 3 \\ \hline
4 & 5 & 6 & 7 & \leftarrow 4 \\ \hline
\end{array}$$

$$4321543265437654 = Q_4$$

$$\text{in}_< I_w = \langle z_{11}, z_{21}, z_{31}, z_{13} \rangle \qquad \begin{array}{|c|c|c|c|}
\hline
+ & & + & \\ \hline
+ & & & \\ \hline
+ & & & \\ \hline
 & & & \\ \hline
\end{array}$$

$$\cap \langle z_{11}, z_{21}, z_{31}, z_{22} \rangle \qquad \begin{array}{|c|c|c|c|}
\hline
+ & & & \\ \hline
+ & + & & \\ \hline
+ & & & \\ \hline
 & & & \\ \hline
\end{array}$$

<u>III</u> TNN $\cap$ N

$$\begin{array}{cc}
 & \scriptstyle i+1 \\
i & \begin{bmatrix} 1 & & & \\ & 1 & t & \\ & & \ddots & \\ & & & 1 \quad 1 \end{bmatrix}
\end{array} = x_i(t) \text{ is TNN for } t \geq 0$$

$x_{i_1}(t_1)\cdots x_{i_d}(t_d) \in TNN \cap N \quad \forall$ words $Q = (s_{i_1}, \ldots, s_{i_d})$

$\quad \| \qquad\qquad t_0 \in \mathbb{R}^d_{\geq 0}$

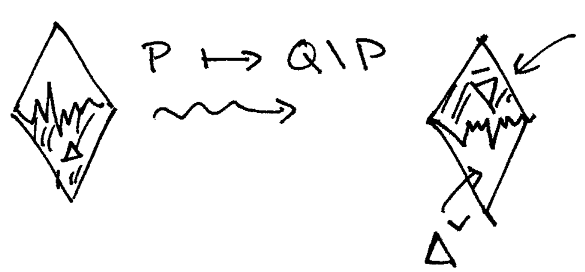
$x_Q(t)$

fiber F of $x_Q \Rightarrow F_P = F \cap \mathbb{R}^P_{>0}$ for $P \subseteq Q$

w_F = row eschelon form of $x_Q(F)$

<u>Thm</u> [DHM] $F_P \cong \mathbb{R}^{|P| - \ell(w_F)}$ if non-empty & $\{P \mid F_P \neq \emptyset\} = \nabla(Q, w_F)$ which is a contractible regular CW complex.

$P \mapsto Q \setminus P$



$\nabla = \{ Q \setminus P \mid$ interior face of $\Delta \}$

$\Delta \cong$ ball $\Rightarrow$ this makes sense