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Uniform Isoperimetric Constants via Random Forests

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Abstract

A finitely generated group is non-amenable if any of its Cayley graphs has a positive isoperimetric constant. That constant depends on the generating set chosen. We show that groups whose first ℓ^2-Betti number is positive have a positive lower bound to their isoperimetric constant over all generating sets. We do this by considering the free uniform spanning forests in the Cayley graphs. This is joint work with Mikaël Pichot and Stéphane Vassout.

1 Background

Let G be the Cayley graph of the (countably infinite) group Γ with respect to the generating set S. E.g. the Cayley graph of $\Gamma = \mathbb{Z}^n$ with the set of generators $\{\pm e_i\}_{i=1}^n$ (where e_i is the unit vector with a 1 in the i^{th} coordinate) is the infinite grid in $\mathbb{R}^n$.

Let B_m be the ball of radius m about the identity. We know that $B_m \cdot B_n = B_{m+n}$, so we have the supermultiplicative property

$$|B_m|\,|B_n| \geq |B_{m+n}|\,.$$

Thus the limit

$$\lim_{n\to\infty} |B_n|^{1/n} =: \operatorname{gr}(G) \equiv \operatorname{gr}(\Gamma, S)$$

exists, and is called the exponential growth rate of G. If $\operatorname{gr}(G) > 1$, then we say that G has exponential growth. Note that the exponential growth rate does depend on the set of generators S, but whether or not G has exponential growth does not depend on S—it is a property of the group Γ.

In 1981, Gromov asked the following question. If $\operatorname{gr}(G) > 1$, is $\inf_S \operatorname{gr}(\Gamma, S) > 1$? If $\inf_S \operatorname{gr}(\Gamma, S) > 1$ then we say the group Γ satisfies the uniform exponential growth property. Examples of groups satisfying this property include free

groups, word hyperbolic groups, solvable groups, linear groups, etc. However, the answer in general to Gromov's question is no: J.S. Wilson provided a counterexample in 2004. This counterexample Γ actually contains F_2 (the free group on 2 generators, which has uniform exponential growth) as a subgroup, and all of the generating sets S have $|S| = 2$.

For $K \subset \Gamma$, define the following two notions of boundary.

$$\partial_S^{\text{int}} K := \{x \in K : xS \nsubseteq K\}$$
$$\partial_S^{\text{ext}} K := \{x \notin K : xS \cap K \neq \emptyset\}.$$

Define also the isoperimetric constant

$$\Phi(\Gamma, S) := \inf\left\{\frac{|\partial_S K|}{|K|} : \emptyset \neq K \subseteq \Gamma, K \text{ finite}\right\}.$$

The two notions of boundary are related via

$$\frac{\left|\partial^{\text{int}} K\right|}{|K|} = \frac{\left|\partial^{\text{ext}} K^0\right|}{|K^0| + |\partial^{\text{ext}} K^0|},$$

and so consequently

$$\Phi^{\text{int}}(\Gamma, S) = \frac{\Phi^{\text{ext}}(\Gamma, S)}{1 + \Phi^{\text{ext}}(\Gamma, S)},$$

i.e. it does not matter which notion of boundary we use in the definition of the isoperimetric constant when we care about positivity / uniformity questions.

Whether $\Phi(\Gamma, S)$ is positive or not does not depend on the generating set S—it is a property of the group Γ. When $\Phi(\Gamma, S) > 0$, we say that Γ is non-amenable. For example, the d-dimensional Euclidean lattice is amenable, since if we take a box with sidelength n, then this will have volume n^d, but the volume of its boundary is of order n^{d-1}, and the ratio of the two goes to 0 as n goes to infinity.

Note that

$$\text{gr}(\Gamma, S) \geq 1 + \Phi^{\text{ext}}(\Gamma, S)$$

since

$$\frac{|B_{n+1}|}{|B_n|} \geq 1 + \Phi^{\text{ext}}(\Gamma, S).$$

So non-amenable groups have exponential growth. The converse is not true: there exist amenable groups with exponential growth, e.g. solvable groups and lamplighter groups.

Another question (analogous to Gromov's question) is the following. If $\Phi(\Gamma, S) > 0$, is $\inf_S \Phi(\Gamma, S) > 0$ (i.e. is there a uniform bound on the expansion constant)? The answer was given by Arzhantseva, Burillo, Lustig, Reeves, Short, and Ventura in 2005. The statement is true for many families of groups, e.g. free groups, hyperbolic groups, Burnside groups, etc., but not true in general. A counterexample is

$$\text{BS}(m, n) := \left\langle a, t \middle| t^{-1} a^m t = a^n \right\rangle,$$

where $(m, n) = 1$, $m, n > 1$. Actually, Wilson's example above is a counterexample too; it is non-amenable, since it contains F_2 as a subgroup.

2 Main result

The main result is the following.

Theorem 2.1 (Lyons, Pichot, Vassout, 2008)**.**

$$\Phi^{ext}(\Gamma, S) \geq 2\beta_1(\Gamma),$$

where $\beta_1(\Gamma)$ *is the first* ℓ^2*-Betti number.*

A few properties of the first ℓ^2-Betti number:

- $\beta_1(\Gamma) = 0$ if Γ is finite or amenable.
- $\beta_1(\Gamma_1 \times \Gamma_2) = \beta_1(\Gamma_1) + \beta_2(\Gamma_2) + 1 - \frac{1}{|\Gamma_1|} - \frac{1}{|\Gamma_2|}$.
- $\beta_1(\Gamma) = 2g - 2$ if Γ is the fundamental group of an orientable surface of genus g.
- $\beta_1(\Gamma_1 \times \Gamma_2) = 0$ if Γ_1, Γ_2 are infinite.

Mann asked the following question this year: are there free products whose uniform growth rate lies in the interval $\left(\sqrt{2}, \frac{1+\sqrt{5}}{2}\right)$? The answer to this question is a corollary of the main theorem. The only possible examples are $\mathbb{Z}_2 \times \mathbb{Z}_5$ and $\mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Where do the interval edges come from? The uniform growth rate of $\mathbb{Z}_2 \times \mathbb{Z}_3$ is $\sqrt{2}$, while the uniform growth rate of $\mathbb{Z}_2 \times \mathbb{Z}_4$ is $\frac{1+\sqrt{5}}{2}$. Why is the answer true? All other free products (except $\mathbb{Z}_2 * \mathbb{Z}_2$) have $1 + 2\beta_1 \geq \frac{5}{3}$.

The proof of the main theorem goes via a connection to free uniform spanning forests in Cayley graphs. For a finite connected graph, its spanning tree is the subset of edges that is maximal without cycles. If we choose one uniformly at random, we call it a Uniform Spanning Tree (UST). If we have a sequence of subgraphs of a graph G, the n^{th} subgraph giving the uniform spanning tree UST_n, then UST_n converges weakly to what is called the Free Uniform Spanning Forest (FUSF). The FUSF is invariant with respect to all automorphisms of G. In particular, in a Cayley graph $G = (\Gamma, S)$, the expected degree of a vertex in the FUSF is the same for every vertex, so it makes sense to talk about the expected degree of the FUSF on (Γ, S). Lyons showed in 2003 that this is exactly $2\beta_1(\Gamma) + 2$.

Now let us use this fact to prove the theorem. Here is the idea of the talk. Let us look at the FUSF $\mathcal{F}$. Let $\mathcal{F}'$ be the part of $\mathcal{F}$ that touches K (i.e. the edges with at least one vertex in K). Let $L := V(\mathcal{F}') \setminus K$. Since $\mathcal{F}$ is a forest, we have

$$\begin{aligned}\sum_{x \in K} \deg_{\mathcal{F}}(x) \leq \sum_{x \in K \cup L} \deg_{\mathcal{F}'}(x) - |L| &= 2|E(\mathcal{F}')| - |L| \\ &< 2|V(\mathcal{F}')| - |L| = 2|K| + |L| \leq 2|K| + \left|\partial_S^{\text{ext}} K\right|.\end{aligned}$$

Now let us take expectation. The right hand side is not random, while the value of the left hand side is given by Lyons (2003), so we get

$$|K|\,(2\beta_1 + 2) \leq 2\,|K| + \left|\partial_S^{\text{ext}} K\right|$$

and dividing by $|K|$ we get the theorem.

Finally, let us show that β_1 does not depend on the generating set S. We begin with an old observation about uniform spanning trees (although of course it was not originally formulated in this language).

Theorem 2.2 (Kirchoff, 1847)**.** $\mathbb{P}(e \in UST) = \langle P_\diamond^\perp \chi^e, \chi^e \rangle$*, where* χ^e *is the indicator function of the edge* e *and* $\diamond$ *is the space spanned by indicators of cycles.*

One can take this theorem to a limit, and prove the exact same thing for a FUSF:

Theorem 2.3 (BLPS, 2001)**.** $\mathbb{P}(e \in FUSF) = \langle P_\diamond^\perp \chi^e, \chi^e \rangle$*, where* χ^e *is the indicator function of the edge* e *and* $\diamond$ *is the space spanned by indicators of cycles.*

Thus, the expected degree of the identity in an FUSF (which is the same, of course, as the expected degree of any edge) is

$$\sum_{s \in S \cup S^{-1}} \langle P_\diamond^\perp \xi^{(id,s)}, \xi^{(id,s)} \rangle = 2 \dim_\Gamma \diamond^\perp,$$

where $\dim_\Gamma$ denotes the von Neumann dimension. But this last quantity is independent of S!