Random Matrices, Exact Operator spaces, .. (G. Pisier) 9/20

Outline:

1. Operator spaces
2. Exactness
3. Random matrices
4. Grothendieck's Thm.

$E \subset B(\mathcal{H})$

closed subspace; bounded ops; Hilbert space

We will consider a norm stronger than the operator norm

$x_j \in E$, $\alpha_j \in \mathbb{C}$, $a_j \in B(H)$

We will take the norm of $\sum a_j \otimes x_j$ as an operator

$H \otimes_2 \mathcal{H} \to H \otimes_2 \mathcal{H}$.

Replace $B(E,F)$ by "$CB(E,F)$" "completely bdd"

$$\|u\|_{cb} \overset{def}{=} \sup_{\substack{a_j \in B(H) \\ x_j \in E}} \frac{\|\sum a_j \otimes u(x_j)\|}{\|\sum a_j \otimes x_j\|}$$

$$= \sup_{N \geq 1} \underbrace{\sup_{\substack{x_j \in E \\ a_j \in B(\ell_2^N)}} \text{" \qquad "}}_{\text{call this } \|\cdot\|_{cb_N}}$$

(whereas the usual operator norm is

$$\|u\|_{cb_1} = \|u\| = \sup_{\substack{x_j \in E \\ \alpha_j \in \mathbb{C}}} \frac{\|\sum \alpha_j u(x_j)\|}{\|\sum \alpha_j x_j\|}$$)

2. Exactness

$E \subset B(\mathcal{H})$ is called <u>exact</u> (Kirchberg ~ 90's)
if $\exists\, c = c_E$ s.t. $\forall\, F \subset E$, $\dim F < \infty$
$\exists\, k.\ \exists\, \tilde{F} \subset M_k$ $\exists\, u : F \to \tilde{F}$: $\|u\|_{cb}\, \|u^{-1}\|_{cb} \leq c$
$\left(\overset{\shortparallel}{k_F}\right)$ $k \times k$ matrices

(example : $K(H)$ = compact operators)

3. Random matrices (Haagerup Thorbjørnsen)

$Y^{(N)}, Y_1^{(N)}, \ldots,$ iid Gaussian $N \times N$ matrices.

$Y^{(N)}(i,j) = \frac{1}{\sqrt{N}} g_{ij}$ ← stand. gaussian.

<u>Thm (HT)</u>: $\forall\, E$ exact, $\forall\, x_1, \ldots, x_n \in E$

$$(*) \quad \lim_{N\to\infty} \|\sum Y_j^{(N)} \otimes x_j\| \overset{a.s.}{\leq} c_E \left(\|x\|_R + \|x\|_C\right)$$

$$\text{"} \quad \leq c_E \cdot 2\max\left(\|x\|_R, \|x\|_C\right)$$

where $\|x\|_R = \|(\sum x_j x_j^*)^{1/2}\|$, $\|x\|_C = \|(\sum x_j^* x_j)^{1/2}\|$.
(note: from free probability, we get $\|\sum c_j \otimes x_j\|$ in the limit, where c_j is standard free Gaussian).

Examples: $R = \overline{\text{span}}\,[e_{1j},\ j \geq 1]$, $C = \overline{\text{span}}\,[e_{i1},\ i \geq 1]$
$R \subset B(\ell_2)$ $C \subset B(\ell_2)$

If $x_j \in B(H)$ then
$\|\sum x_j \otimes e_{1j}\| = \|x\|_R$, $\|\sum x_j \otimes e_{j1}\| = \|x\|_C$

Easy to see that R and C are exact (they embed into $K(H)$).

"Operator Hilbert space"

Example:

$OH = \overline{\text{span}}\,[e_j] \subset B(\mathcal{H})$ ($\mathcal{H}$ can be ℓ_2)

with $\|\sum a_j \otimes e_j\| = \|\sum a_j \otimes \bar{a}_j\|^{1/2}$

This is not exact, but the finite-dimensional version OH_n has $c_{OH_n} \sim n^{1/4}$.

Example:

$$\|\sum a_j \otimes e_j\| = \sup_{\|u_j\| \le 1} \|\sum a_j \otimes u_j\|$$

This is the operator space analogue of ℓ_1. Finite dim version has $c_E \sim \sqrt{n}$. (This is the worst possible).

Proof of ⊛:

By routine concentration arguments, it suffices to prove that

$$(**) \quad \lim_{N\to\infty} \mathbb{E}\,\|\sum Y_j^{(N)} \otimes x_j\| \le c_E \cdot 2\max(\|x\|_R, \|x\|_C)$$

Since E is exact, can assume $x_1, \dots, x_n \in M_k$, $k < \infty$.

Claim: $\forall\, x_j \in M_k$, $\forall\, \varepsilon > 0$,

$$\mathbb{E}\,\|\sum Y_j^{(N)} \otimes x_j\| \le (1+\varepsilon)\left(2 + \varepsilon(N) + c_\varepsilon \sqrt{\frac{\log(kN)}{N}}\right) \cdot \max(\|x\|_R, \|x\|_C)$$

where $\varepsilon(N) \to 0$.

The Claim implies $(**)$ by taking $N \to \infty$.

To prove the claim, take $p \gg 2$ an even integer.

$$\left(\mathbb{E} \left((\tau_N \otimes \tau_k) \left| \sum Y_j^{(N)} \otimes x_j \right| \right)^p \right)^{1/p}$$

normalized trace

$$\leq \underbrace{\left(\mathbb{E}\, \tau_N \, |Y^{(N)}|^p \right)^{1/p}} \max\left(\|x\|_R, \|x\|_C \right)$$

$$= \left(\mathbb{E} \, \|Y\|^p_{L^p(\tau_N)} \right)^{1/p} \leftarrow \text{ordinary, well-kn Gaussian stuff.}$$

$$=: B_p^{(N)} \leq 2 + \varepsilon(N) + \sqrt{\frac{p}{N}}$$

Thus,

$$\mathbb{E} \left\| \sum Y_j^{(N)} \otimes x_j \right\| \leq (Nk)^{1/p} \cdot (I)_p .$$

If $p = \frac{1}{\varepsilon} \log(Nk)$ then $(Nk)^{1/p} \sim 1 + \varepsilon$, so

$$(Nk)^{1/p} B_p^{(N)} \leq (1+\varepsilon) \left[2 + \varepsilon(N) + \sqrt{\frac{\log(kN)}{N}} \right].$$

<u>Def</u>: E is called <u>subexponential</u> if $\exists\, c' = c'_E$ s.t

$\forall\, F$, $\dim F < \infty$, $\forall\, N$ $\exists\, k = k_F(N)$ $\exists\, \tilde{F} \subset M_k$
$\exists\, u_N : F \to \tilde{F}$ such that

$$\|u_N\|_{cb_N} \cdot \|u_N^{-1}\|_{cb_N} \leq c'.$$

and $\lim_{N \to \infty} \left(\log k_F(N) / N \right) = 0$.

Example: exact $\Rightarrow$ subexponential because $k_F(N) \leq k_F$.
(only known)

<u>Thm</u>: $(*)$ holds for subexponential E also.

Non-examples: OH and o.s. ℓ_1 are not subexponential.

Connections to Grothendieck's Thm: see slides.