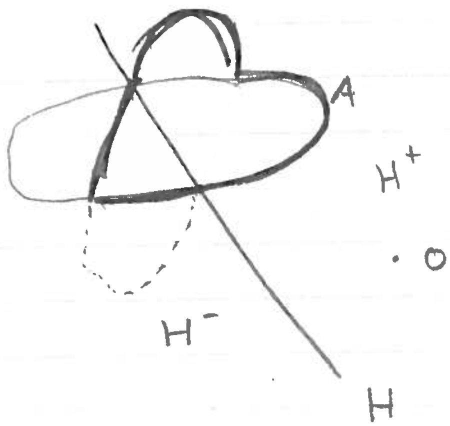
Random sequences of simple rearrangements (Almut Burchard)

Polarization: take a hyperplane not passing through the origin. (aka 2-point symmetrization)



On one side, take $A \cup \sigma A$, on the other, take $A \cap \sigma A$

Functional version:

$f: \mathbb{R}^d \to \mathbb{R}$

$$Sf(x) = \begin{cases} \max\{f(x), f(\bar{x})\} & x \in H^+ \\ \min\{f(x), f(\bar{x})\}, & x \in H^- \end{cases}$$

Properties

- Preserves size: $|A| = |SA|$, $\|f\|_p = \|Sf\|_p$
- Increases overlaps: $|A \cap B| \leq |SA \cap SB|$, $\int_{\mathbb{R}^d} fg\,dx \leq \int (Sf)(Sg)\,dx$ (Hardy-Littlewood).
  $\Rightarrow \|f-g\|_{L^p} \geq \|Sf - Sg\|_{L^p}$
- Decreases perimeter: $|A_\delta| \geq |(SA)_\delta|$ (Benyamini)
- $\iint f(x)g(y)K(|x-y|)\,dx\,dy \leq \iint Sf(x)\,Sg(y)\,K(|x-y|)\,dx\,dy$
  if K is non-increasing.

Turns out it's easy to write down the gap in the Hardy-Littlewood inequality:

$$\int_{\mathbb{R}^d} Sf(x)\, Sg(x)\, dx - \int_{\mathbb{R}^d} f(x) g(x) dx$$

$$(\mathrm{I}) \quad \int_{\mathbb{R}^d} f(x) g(x) = \int_{H^+} f(x) g(x) + f(\bar{x})\, g(\bar{x})\, dx$$

$$(\mathrm{II}) \quad \int_{\mathbb{R}^d} Sf(x)\, Sg(x) = \int_{H^+} \max(f(x), f(\bar{x}))\max(g(x) g(\bar{x})) + \min(\cdot)\min(\cdot)\, dx$$

$$\mathrm{II} - \mathrm{I} = \int_{H^+} (f(x) - f(\bar{x}))(g(x) - g(\bar{x}))^- \, dx.$$

<u>More elaborate version</u>: (B, Schmuckenschlager, 96)

$$\int \cdots \int \prod_{i=1}^{n} f_i(x_i) \prod_{i<j} K_{ij}(|x_i - x_j|)\, dx_1 \cdots dx_n \le \text{same for } Sf_i$$

non-increasing, positive

<u>Interesting case</u>: $K_{i,i+1} = \left(4\pi \frac{t}{n}\right)^{-d/2} e^{-|x-y|^2/4t/n}$

and take limit as $n \to \infty$:

consider $u_A(x,t)$ a solution of $\begin{cases} u_t = \Delta u \text{ on } A \\ u(x,0) = 1 \\ u(x,t) = 0 \text{ on } \partial A \end{cases}$

Then

$$\int u_{SA}(x,t) dx - \int u_A(x,t) dx$$

$$= \int_{SA} \int_{SA} w^t_{x,y}(E^t)\, dx dy.$$

What do we need for this to work?

- reflections that are measure-preserving involutions & isometries
- $d(x, \sigma y) \ge d(x,y)$ for $x, y \in H^+$
- <u>lots</u> of reflections.

Typical application
- isoperimetric inequality on S^d
- Similar inequalities for path integrals.

key point: need sequences $S_{\omega_k} \cdots S_{\omega_1} A \to$ ball.
Not so easy to write down!

Parametrize hyperplanes by $\Omega = \{(r,k) : r \geq 0,\ k \in S^{n-1}\}$.
It is neither necessary nor sufficient for $\{\omega_i\}$ to be dense in Ω (consider the case of a ball not around 0).

However, it's easy to write down a random sequence that works! Take μ a prob measure on Ω, with positive density (for now), $\mu(R=0)=0$.

Thm (vS 2005, BF 2011$^+$)
Take $\omega_i \overset{iid}{\sim} \mu$. Then

centered ball of same volume

$$P(\forall\, A \text{ compact},\ \lim_{n\to\infty} d_H(S_{\omega_n} \cdots S_1 A, A^*) = 0) = 1$$

Proof: two ingredients
(1) compactness to get candidates for limits
(2) identify

Lemma: $A = A^* \iff S_\omega A = A \ \forall\, \omega \in \Omega$.

$\left(\begin{array}{c} A = A^* \\ \text{up to translation} \end{array}\right) \iff \forall\, \omega \in \Omega$, either $S_\omega A = A$ or $S_\omega A = \sigma A$.

(1) compactness: look at $f \in C_c^+(\mathbb{R}^d)$ non-negative
look at measures on $C_c^+(\mathbb{R})$.

$$S\#\nu(\mathcal{A}) := \int_{C_c^+} P(S_\omega g \in \mathcal{A})\, d\nu(g)$$

Borel set in C_c^+

Then we get $\nu = \delta_f$, get easily $S^n \# \nu \longrightarrow \tilde{\nu}$ along a subsequence.

Look at $I(f) = \int f(x)\, d(x, 0)$, which decreases under S (strictly unless $f = f^*$).

Conclusion: $\tilde{\nu}$ is supported on functions g s.t.

$$g(\tau_u(x)) \geq g(x) \quad \forall x,\ \forall u \in G.$$

$$\boxed{\begin{aligned} &G = \{u \in S^{n-1} : (0, u) \in \operatorname{supp}(\mu)\} \\ &\tau_u(x) = \begin{cases} x, & ux \geq 0 \\ x - 2xu\,u, & ux < 0. \end{cases} \end{aligned}}$$

What condition on G will then imply that g is radially symmetric?

It suffices to produce dense orbits $\{\tau_{u_t} \cdots \tau_{u_1} x\}_{t \geq 1}$.

Q: under what conditions on G can we produce, for every $x \in S^{d-1}$, a dense orbit?

Sufficient condition: Let $G' = \{u \in G : -u \in G\}$. then it is enough for G' to contain a basis $u_1, \ldots, u_d$ that cannot be composed into mutually orthogonal parts, and with some irrational angles.

On S^1, there are obvious necessary conditions:

- u_1, u_2, u_3 span $\mathbb{R}^2$
- half-circles $\{x \cdot u \geq 0\}$ cover S^1
- u_1, u_2 enclose an irrational angle

Undergrad exercise: Is this sufficient?