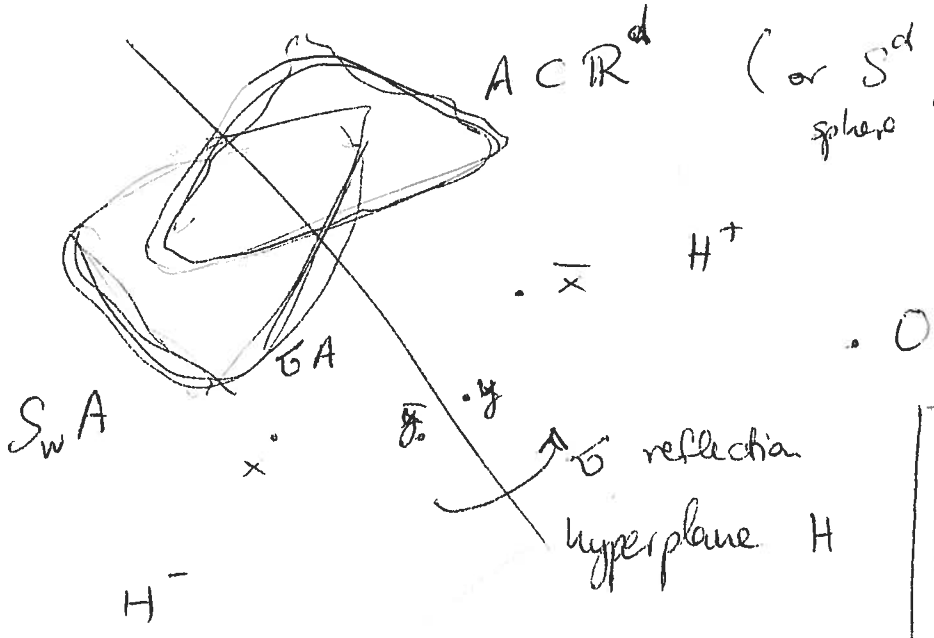
Almut: Random Sequences of Simple [redacted] rearrangements

2011.09.20

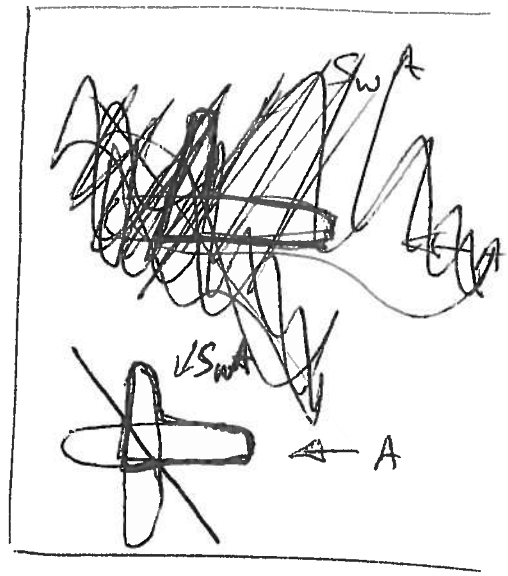
(1)

Polarization (two-point symmetrization)

$A \subset \mathbb{R}^d$ (or S^d sphere, H^d hyperbolic space, $\{0,1\}^d$ a space w/ many reflections)



$w = (r,u)$ (polar coords.)



$f : \mathbb{R}^d \to \mathbb{R}$.

$$Sf(x) = \begin{cases} \max\{f(x), f(\bar{x})\}, & x \in H^+ \\ \min\{f(x), f(\bar{x})\}, & x \in H^- . \end{cases}$$

Hope: do this two-point symmetrization many many times
$\rightarrow$ arrive at a ball.

Properties

- Preserves size
  $|A| = |SA|$.
- Increases overlaps
  $|A \cap B| \le |SA \cap SB|$

Hardy - Littlewood

$$\int_{\mathbb{R}^d} fg \, dx \le \int (Sf)(Sg) \, dx .$$

- decreases perimeter (Brunn–Minkowski)

$$|A_\delta| \geq |(SA)_\delta|$$

$$\mathrm{Per}(A) \geq \mathrm{Per}(SA)$$

$\circledast$ (Riesz)

$$\iint f(x) g(y) K(|x-y|)\,dx\,dy \leq$$

$$\leq \iint Sf(x)\, Sf(y)\, K(|x-y|)$$

if K is non-increasing.

Good point about doing only two-point symm.: you don't just have an ineq., you can actually compute the difference.

Let's compute $\displaystyle\int_{\mathbb{R}^d} Sf(x)\, Sg(x)\,dx - \int_{\mathbb{R}^d} f(x) g(x)\,dx$.

$$I = \int_{\mathbb{R}^d} f(x) g(x)\,dx = \int_{H^+} \big(f(x) g(x) + f(\bar{x}) g(\bar{x}) \big)\,dx$$

$$II = \int_{\mathbb{R}^d} Sf(x)\, Sg(x)\,dx = \int_{H^+} \max\{f(x), f(\bar{x})\} \max\{g(x), g(\bar{x})\} + \min\{\_\_\} \min\{\_\_\}$$

$$II - I = \ldots = \int_{H^-} \big[(f(x) - f(\bar{x}))(g(x) - g(\bar{x})) \big]^- \,dx$$

[not sure if all the $\pm$'s are correct]

More elaborate version 1996$^+$ (2000) B, Schmuckenschläger (3)

$$\int \int$$

Interesting case $K_{i,i+1} = \left(4\pi \frac{t}{n}\right)^{-d/2} e^{-\frac{|x-y|^2}{4t/n}}$ take limit $n \to \infty$.

get: consider $u_A(x,t)$ sol'n of

$$u_t = \Delta u \quad \text{on } A$$
$$u(x,0) = 1$$
$$u(x,t) = 0 \quad \text{on } \partial A$$
$$u_A(x,t) = P^x(T_A > t).$$

$$\int u_{SA}(x,t)\,dx - \int u_A(x,t)\,dx =$$

$$= \int_{SA} \int_{SA} W^t_{x,y}(E^t)\,dx\,dy.$$

What do we need for polarization to work?

- Reflections $S^2 = I$.
- $d(x, sy) \geq d(x,y)$ if $x,y \in H^-$.
- LOTS OF THESE.

Typical application

- isoperimetric ineq. on S^d.

Similar ineqs for path integrals

Key point: need sequences $S_{w_n} \dots S_{w_1} A \longrightarrow$ ball.

Not so easy to write down.

Parametrize by $\Omega = \{(r,u) \mid r \geq 0, u \in S^{d-1}\}$

It's neither necessary nor sufficient for $\{w_i\}$ to be dense in Ω. (4)

Point: writing down random seq.s of polarizations that converges is easy.

μ prob. measure on Ω, positive density.

$W = (R, U)$ r.v.s $\qquad \mu(R=0)=0$.

Thm (vS 2005, B, F, 2011$^+$)

$\{w_i\}_{i\geq 1}$ iid seq. on Ω according to μ.

then: $\mathbb{P}\left(\forall A \text{ compact}: \lim_{n\to\infty} d_H(S_{w_n}\cdots S_{w_1}A, A^*)=0\right)=1$.

centered ball of same volume

Note: positive density assumption not needed.

Claim: it's an easy thm

2 ingredients to proof

(1) compactness to get candidates for limits

(2) Identify

Lemma $\quad A = A^* \iff S_w A = A$ for all $w \in \Omega$.

$A = A^*$ up to translation $\iff \forall w\in\Omega$: Either $S_wA = A$ or $S_wA = \Sigma_w A$

1'') Compactness

Look at $f \in C_c^+(\mathbb{R}^d)$ ← or S^d.

Look at measures ν on C_c^+.

$$S\#\nu(\mathcal{A}) := \int_{C_c^+} P(S_w g \in \mathcal{A})\, d\nu(g).$$

Borel set in C_c^+

Then we get $\nu_0 = \delta_f$,

get easily $S^n \# \nu \xrightarrow[\text{subseq.}]{} \tilde{\nu}$

Look at $I(f) = \int f(x)\, d(x,0)$ decreases under S (strictly unless $f = f^*$)

Ultimate conclusion

$\tilde{\nu}$ is supported on functions g s.t.

$$g(\tau_u(x)) \geq g(x) \quad \forall x, \forall u \in G$$

If I can produce dense orbits $(x \in S^{d-1})$

$\{\tau_{u_k} \cdots \tau_{u_1} x\}_{k \geq 1}$ dense in S^{d-1}

then I get full radial symmetry.

$G = \{u \in S^{d-1} \mid (0,u) \in \text{supp}\,\mu\}$

$$\tau_u(x) = \begin{cases} x & u \cdot x \geq 0 \\ x - 2xuu & ux < 0 \end{cases}$$

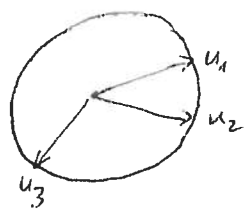
Compression

Q) Under what conditions on G can I produce, for every $x \in S^{d-1}$, a dense orbit $\{\tau_{u_k} \cdots \tau_{u_1} x\}_{k \geq 1}$ in S^{d-1}.

Sufficient

Problem

$d=2$, S^1



obvious necessary conds

1. u_1, u_2, u_3 span $\mathbb{R}^2$
2. half-circles $\{x \cdot u \geq 0\}$ cover S^1
3. u_1, u_2 enclose an irrational angle.

Do $\tau_{u_1}, \tau_{u_2}, \tau_{u_3}$ generate dense orbit?