New applications of rearrangement inequalities. (Perla So

<u>Theorem:</u> (Peres – S)
Let (A_s) be open sets in $\mathbb{R}^d$.
Let $(\xi(s))$ be BM in $\mathbb{R}^d$. Then $\forall t$,

$$\mathbb{E}\,\mathrm{vol}\bigcup_{s\le t}(\xi(s)+A_s) \ge \mathbb{E}\,\mathrm{vol}\bigcup_{s\le t}(\xi(s)+B(0$$

where $B(0,r_s)$ is the Euclidean ball with the same volume as A_s.

<u>Motivation:</u> $\Pi = \{X_i\}$ is a PPP in $\mathbb{R}^d$ w/intensity λ
Move the points according to BM: at time t, the position of the particles is $\{X_i + \xi_i(t)\}$

$$T_{det} = \inf\{t\ge 0 : \exists i,\ X_i + \xi_i(t) \in B(0,r)\}$$

"detection time"

$$T_{det}^f = \inf\{t\ge 0 : \exists i,\ X_i + \xi_i(t) \in B(f(t),r)\}.$$

How is this related to the Brownian sausage?

<u>Classical result:</u> $\mathbb{P}(T_{det}^f > t) = \exp\big(-\lambda\,\mathbb{E}\,\mathrm{vol}\bigcup_{s\le t}(\xi(s)+B(f(s)$

<u>Proof:</u> $\Phi = \{X_i \in \Pi : \exists s\le t,\ X_i + \xi_i(s) \in B(f(s),r)\}$
$=$ thinned PPP w/intensity
$\Lambda(x) = \lambda\cdot\mathbb{P}(x \in \bigcup_{s\le t} B(\xi(s)+f(s), r$

$$\mathbb{P}(T_{det}^f > t) = \mathbb{P}(\Phi(\mathbb{R}^d)=0) = \exp\int -\Lambda(x)\,dx.$$

Theorem (Peres–Sinclair–Stauffer–S.)

$d=1$, f cts.

$$\forall t \quad \mathbb{E}\,\mathrm{vol} \bigcup_{s \le t} B(\xi(s)+f(s), r)$$

$$\ge \mathbb{E}\,\mathrm{vol} \bigcup_{s \le t} B(\xi(s)+r).$$

(i.e. best strategy to avoid detection is to stay still).

In higher dimensions, the bound was worse:

$d=2$ gives LHS $\ge (1-o(1)) \cdot$ RHS

$d \ge 3$ gives LHS $\ge c(d) \cdot$ RHS.

<u>Analogy</u>:

Analysis of escape times via discretization:

$$\mathbb{P}(\tau_D > t) = \lim_{n\to\infty} \mathbb{P}\left(\xi\left(\tfrac{it}{n}\right) \in D \ \forall\, i=0,\dots,n\right)$$

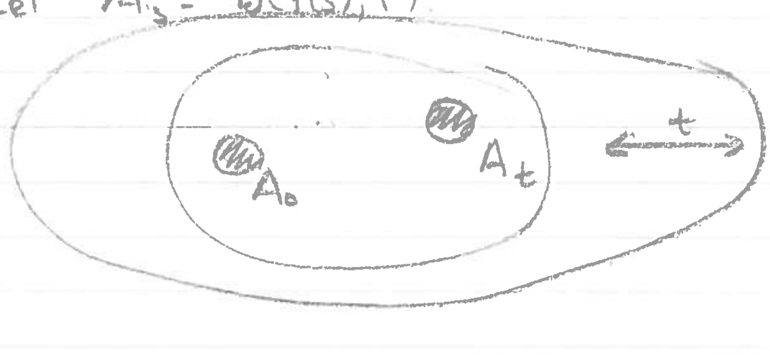
$$= \lim_{n\to\infty} \int \cdots \int \prod p(x_1,\dots,x_n)\ \mathbb{1}(x_i \in D)\ d\underline{x}$$

<u>Same idea for detection problem:</u>

$$\mathbb{P}(T^f_{det} > t) = \mathbb{P}(\forall s > 0 \dots t,\ \nexists\, i \text{ s.t. } X_i + \xi_i(s) \in B(f(s), r))$$

$$= (*)$$

Let $A_s = B(f(s), r)$



Points outside the big ball will not be detected by time t.

$$(*) = \mathbb{E}\left(\mathbb{P}\left(\forall s = 0,\dots,t,\ \xi(s) \in A_s\right)^N\right)$$

$$= \int\cdots\int \prod p(x_1,\dots,x_n)\ \mathbb{1}(x_i \in A_i^c)\, d\underline{x}$$

Problem: on $\mathbb{R}^d$, the complement of a ball is not a ball.

Solution: lift to the sphere.

Then apply the previous (ie. last lecture) re-arrangement inequalities to conclude that the maximizing sets are balls. Project down to Euclidean space by throwing away one coordinate, and then take the radius of the sphere to ∞.

Area of BM with drift: see slides.