Brownian motion with variable drift

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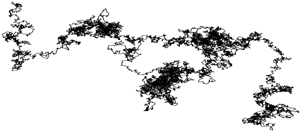
Planar Brownian motion



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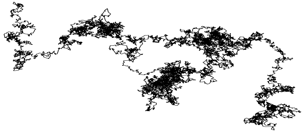

Theorem (Lévy 1940)

Let B be a planar Brownian motion. Then

$$\mathcal{L}(B[0,1]) = 0 \ a.s.$$

Area of planar Brownian motion with drift

Question

Let f be a continuous function. Does $(B+f)[0,1]$ still have 0 area?

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An a.s. property insensitive to the drift:

For any f continuous, $B+f$ is nowhere differentiable a.s.

Cameron–Martin Theorem

Denote by $D[0,1]$ the **Dirichlet space**

$$D[0,1] = \left\{ f \in C[0,1] : \exists g \in \mathbf{L}^2[0,1] \text{ s.t. } f(t) = \int_0^t g(s)ds, \forall t \in [0,1] \right\}.$$

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Theorem (Cameron–Martin 1944)

If $f \in D[0,1]$, then the law of B is mutually absolutely continuous w.r.t. the law of $B + f$.

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Theorem (Cameron–Martin 1944)

If $f \in D[0,1]$, then the law of B is mutually absolutely continuous w.r.t. the law of $B + f$.

Hence, if $f \in D[0,1]$, then $\mathcal{L}(B+f)[0,1] = 0$ a.s.

Graversen's result

Theorem (Graversen 1982)

For all $0 < \alpha < 1/2$*, there exists a Hölder(*α*) continuous function* $f : \mathbb{R}_+ \to \mathbb{R}^2$ *s.t.* $\mathbb{E}[\mathcal{L}(B + f)[0,1]] > 0$.

Le-Gall's result

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We will see: same transition from Hölder(α) for $\alpha < 1/2$ to $\alpha = 1/2$ applies to a large variety of properties of Brownian motion.

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Theorem (Antunović et al 2010)

For any $\alpha < 1/2$*, there exists a Hölder(*α*) function* $f : \mathbb{R}_+ \to \mathbb{R}^2$ *for which* $(B + f)[0, 1]$ *completely covers an open set a.s.*

A remaining question

In all these works it was not clear whether for any continuous f

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This was the impetus for our work.

0-1 Laws

Let $(B_t, 0 \leq t \leq 1)$ be a standard Brownian motion in $\mathbb{R}^d$ and let $f : [0,1] \to \mathbb{R}^d$ be a continuous function.

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- $\mathbb{P}(\mathcal{L}(B+f)[0,1] > 0) \in \{0,1\}$.
- $\mathbb{P}(\text{interior of } (B+f)[0,1] \neq \emptyset) \in \{0,1\}$.
- $\dim(B+f)[0,1] = c$ *a.s., where c is a positive constant and* dim *is the Hausdorff dimension.*

Beyond the Cameron–Martin theorem

Again the same setting, B is a standard Brownian motion and $D[0,1]$ is the Dirichlet space

$$D[0,1] = \left\{ f \in C[0,1] : \exists g \in \mathbf{L}^2[0,1] \text{ s.t. } f(t) = \int_0^t g(s)ds, \forall t \in [0,1] \right\}.$$

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Theorem (Cameron–Martin 1944)

If $f \notin D[0,1]$, then the law of B and the law of $B+f$ are singular.

As a consequence, when $f \notin D[0,1]$, there is some a.s. property of Brownian motion that fails for $B+f$.

Hitting probabilities

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Question

Does $B + f$ hit the same sets as B, if f is Hölder(1/2)?

Hitting probabilities

Theorem (Peres and S.)

Let A be a closed set of $\mathbb{R}^d$, for $d \geq 2$, and f a Hölder(1/2) continuous function. If $\mathbb{P}_x(B \text{ hits } A) > 0$, for all $x \in \mathbb{R}^d$, then $\mathbb{P}_x(B + f \text{ hits } A) > 0$, for all $x \in \mathbb{R}^d$.

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In 2 dimensions, if $\mathbb{P}_x(B \text{ hits } A) > 0$, then by neighborhood recurrence, $\mathbb{P}_x(B \text{ hits } A) = 1$. The same is true for $B + f$, if f is Hölder$(1/2)$.

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Concerning the existence of multiple points, $B + f$ behaves in the same way as B, if f is Hölder(1/2).

(This can fail if f is not Hölder(1/2), e.g. for f fractional Brownian motion.)

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For every $\alpha \geq 0$, the α-Hausdorff content of a metric space E is defined

$$\mathcal{H}^{\alpha}_{\infty}(E) = \inf\{\sum_{i=1}^{\infty}(\text{diam}(E_i))^{\alpha} : E_1, E_2, \ldots \text{ is a covering of } E\}.$$

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The **Hausdorff dimension** of E is defined to be

$$\dim E = \inf\{\alpha \geq 0 : \mathcal{H}_{\infty}^{\alpha}(E) = 0\}.$$

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Question

Can we provide bounds for $\dim(B+f)[0,1]$*?*

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Theorem (Peres and S.)

$$\dim(B+f)[0,1] \geq \max\{2 \wedge d, \dim f[0,1]\} \ a.s.$$

Back to the 0-1 law

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Theorem (0-1 law for $\mathcal{L}$)

$$\mathbb{P}(\mathcal{L}(B+f)[0,1] > 0) \in \{0,1\}.$$

Proof of the 0-1 law for $\mathcal{L}$

For an interval $I \subset [0,1]$, define $\Psi(I) = \mathcal{L}(B+f)(I)$.

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The limit of $\mathbb{E}[Z_n]$ exists and can be either infinite or finite.

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Letting $n \to \infty$ gives $\mathbb{P}(\Psi([0,1]) = 0) = 0$.

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Since $\Psi(\text{good points}) = 0 \Rightarrow \Psi([0,1]) = 0$ a.s.

More on 0-1 laws

Let $(B_t, 0 \le t \le 1)$ be a standard Brownian motion in $\mathbb{R}^d$, let $f : [0,1] \to \mathbb{R}^d$ be a continuous function and A a closed set in $[0,1]$.

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