Speed exponents for random walks on groups (Balint Virag)
(joint work w/ G. Amir)

Let X_n be a random walk on a group
"speed exponent β" means $|X_n| \sim n^{\beta+o(1)}$.

Trees have exponent $1/2$, $\mathbb{Z}$ has exponent 1.
What about other exponents?
A construction to get $\beta = 3/4$ is based on the lamplighter construction, with base graph $\mathbb{Z}$ and lamps $\mathbb{Z}$ (in other words, $\mathbb{Z} \text{ wr } \mathbb{Z}$).
wreath product

This construction can be iterated: if G has exponent β then $G \text{ wr } \mathbb{Z}$ has exponent $\frac{\beta+1}{2}$.
This gets exponents of the form $1-(\frac{1}{2})^n$.

__Thm__: All exponents in $[3/4, 1]$ are possible.

__Open__: Do there exist exponents in $(1/2, 3/4)$?

__Remark__: For amenable groups, $\beta = 1$

__Permutation wreath product__
Suppose G acts on S,
Take another group L, and we will define $L \text{ wr}_S G$.

Let $H = L^S$ and take $(h,g) \in (H,G)$.

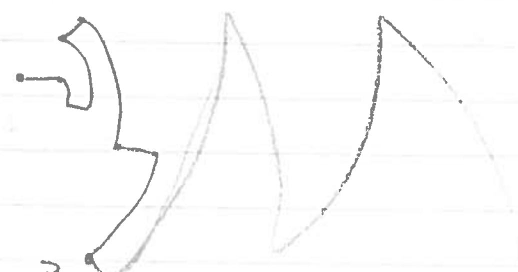
$$(h,g)\cdot(\overbrace{(id,\dots id, \underset{\underset{0}{\uparrow}}{\ell}, id, \dots id)}^{\tilde{\ell}},\ id) = (h\,\tilde{\ell}^{g^{-1}}, g)$$

light is switched at $0\cdot g^{-1}$!

So for a sequence like $G_1 L_1 G_2 L_2 \dots$,
we first switch the lamp at $O \cdot G_1^{-1}$,
then the lamp at $O \cdot G_2^{-1} \cdot G_1^{-1}$
... $O \cdot G_3^{-1} \cdot G_2^{-1} \cdot G_1^{-1}$

inverse orbit of O under G_1, $G_1 G_2$, $G_1 G_2 G_3$

Example: Consider a r.w. on the isometries of $\mathbb{R}^2$, given by rotating $5°$ to left or right and moving ahead 1 space



bigger strides as you go further!

Whereas the inverse orbit is much nicer (although non-Markovian).

Lemma:
$$\left.\begin{array}{l} O \\ O \cdot G_1^{-1} \\ O \cdot G_2^{-1} \cdot G_1^{-1} \\ O \cdot G_3^{-1} \cdot G_2^{-1} \cdot G_1^{-1} \\ \vdots \\ O \cdot G_n^{-1} \cdots G_1^{-1} \end{array}\right\} R_n = \text{range of inv. orbit.}$$

Under technical conditions, if $|R_n| \sim n^\rho$ then
$$|X_n| \sim n^{\frac{\rho+1}{2}}$$
where X_n is r.w. on $\mathbb{Z} \text{ wr}_S G$.

Proof: Suppose $|R_n| \gtrsim n^\rho$. Then each light sees about n/n^ρ switches.

The distance of the light from off is $\sim \sqrt{\frac{n}{n^\rho}}$ and so need $n^\rho \cdot \sqrt{n/n^\rho}$ steps to turn off the lights

$$\Rightarrow |X_n| \gtrsim n^{(\rho+1)/2}.$$

Other direction: suppose $|R_n| \lesssim n^\rho$.

$$\mathcal{E}nt(X_n) \leq \mathcal{E}(G_1 \dots G_n) + \mathcal{E}(R_n) + \underbrace{\log n}_{\substack{\text{entropy of} \\ \text{each light}}} \cdot \mathbb{E}|R_n|$$

Technical condition: the last term above is the dominant term.

$$\Rightarrow \mathcal{E}nt(X_n) \lesssim n^\rho$$

Now use Varopoulos - Carne bounds:

$\exists$ a set A of size $\exp(n^\rho(1+\varepsilon))$ s.t. $P(X_n \in A) \geq (1-\varepsilon)$.

and (V-C): $P(X_n = v) \leq c \exp(-d^2/2n)$ for $|v| = d$.

$$\Rightarrow P(|X_n| \leq d) \leq \exp(-d^2/2n) \cdot \exp(n^\rho) \sim 1$$

for $n^\rho \sim d^2/2n$

$\Rightarrow d = n^{(1+\rho)/2}$

Consider finite or infinite strings

$(w_\ell) \dots w_3 w_2 w_1$, $w_i \in \{0, \dots, m-1\}$,

and 2 operations:

1) randomize w_1
2) randomize w_τ from $\{1, \dots, m-1\}$
$w_{\tau+1}$ from $\{0, \dots, m-1\}$
where w_τ is the first nonzero letter.

This can be thought of as a random walk on the finitary boundary of an infinite tree.

example when $m=2$:

0000 $\longleftrightarrow$ 0001 $\longleftrightarrow$ 0011 $\longleftrightarrow$ 0010

this is a r.w. on $\mathbb{Z}$, represented by the Gray code. For $m>2$, the walk is more interesting!

Want to study $|R_n|$ for this walk.

Lemma (exercise): $\mathbb{E}|R_n| = \mathbb{E}(T ; T \leq n)$
where T = return time to 0 for the ordinary walk (not the inverse walk).

For $m=2$, $\mathbb{E}(T; T \leq n)$ is understood. If $m>2$, the walk doesn't depend on the letters' values, except that they are non-zero. Thus, the walk can be projected to the $m=2$ case, and we get a birth-death chain on $\mathbb{Z}$.

You can compute : $|R_n| \approx \dfrac{\log m}{2\log m - \log(m-1)}$.

This can be tweaked further to get any speed exponent, by taking $w_i \in \{0, \dots, m_i - 1\}$ (ie. alphabet depending on i).