Random Cayley graphs & Expanders for FSG 9/21

(Emmanuel Breuillard)

<u>Def</u> $\{\mathcal{G}_n\}$ a sequence of d-regular graphs is a family of expanders if $\exists\, \varepsilon > 0$ s.t.

$$|\partial A| \geq \varepsilon |A| \quad \forall A \subset \mathcal{G}_n,\ |A| \leq \tfrac{1}{2}|\mathcal{G}_n|$$

- $\mathrm{diam}(\mathcal{G}_n) = O_\varepsilon(\log|\mathcal{G}_n|)$.

History:

- Pinsker 73, Barzdin–Kolmogorov 69 ← random graphs
- explicit construction by Margulis 73, using Kazhdan's property (T) ← representation theory.

In 2005, Bourgain & Gamburd used random walks and combinatorics to give a completely different construction of expanders

Are Cayley graphs expanders?
Not all of them: $\mathrm{Cay}(S_n, \{(1\,2), (1\,2 \cdots n)\})$ has diameter $n^2 \sim (\log|\mathcal{G}_n|)^{2-\varepsilon}$, so it is not an expander.

Kassabov: In σ_n there are 7 generators s.t. the associated Cayley graphs are expanders.

Thm (Kassabov, Lubotzky, Nikolov):
$\forall^*$ finite simple groups G, there is a Cayley graph on 1000 generators s.t. $\{\mathcal{G}_G\}$ is a family of expanders.

$*$ = except for Suzuki groups.

Finite simple groups (FSG)

- FSGLT (Lie type): $PSL_n(\mathbb{F}_p)$, SO_n, Sp_n, 5 except. families "twisted" versions
- Sporadic groups
- A_n

Def: A family of FSG has bdd rank if the "n" is bounded.

Remark: Proof of KLN works by finding many copies of $SL_2(q)$ in a FSG. This is not possible for the Suzuki groups $Suz(q)$, $q = 2^{2n+1}$ because $|Suz(q)| \equiv 2 \ (3)$
$|SL_2(q)| \equiv 0 \ (3)$.

Thm (B, Green, Tao): The KLN thm is also true for $\{Suz(q)\}_q$. In fact, for $k \in \mathbb{N}$, $k \geq 2$, a random $2k$-regular Cayley graph is an expander: $\exists \ \varepsilon > 0$ s.t.

$$\mathbb{P}_{a_1, \dots, a_k}\left(\mathrm{Cay}(Suz(q), \{a_1, \dots, a_k\}) \text{ is connected and } \varepsilon\text{-expander}\right) \xrightarrow{q \to \infty} 1$$

The point is that $Suz(q)$ does not have many non-commutati[ve] subgroups. With some more effort, the proof can be extende[d]

Thm (B, Green, Guralnick, Tao): Given $r \geq 1$, a random Cayley graph of a finite simple group of Lie type and rank $\leq r$ (e.g. $PSL_r(q)$) is an expander.

Question: Does ε have to depend on r? Even for σ_n?

Thm: (Bourgain, Gamburd) Take $SL_2(\mathbb{Z}/p\mathbb{Z})$ and an arbitrary Cayley graph whose girth is at least $c \log |\mathcal{G}|$. Then $\mathcal{G}$ is an $\varepsilon(c)$-expander.

Q: What is the relation between girth and expansion? For Ramanujan graphs, the girth must tend to ∞, but how fast?

Thm (Gamburd, Hoory, Shahshahani, Shalev, Virag):
A random Cayley graph of a FSG with bdd rank has girth at least $c_r \log|G|$

Thus the girth is at least a constant fraction of the diameter.

Idea behind Bourgain - Gamburd:

Step 1: $\mu = \frac{1}{|S|} \sum_{s \in S} \delta_s$.

$\mu^n = \mu * \mu * \cdots * \mu$.

$$|G| \mu^{*2n}(e) = \sum_{i=1}^{|G|} \lambda_i^{2n} \leftarrow \text{eigenvalues of } \mu.$$

$$\geq 1 + \mathrm{mult}(\lambda_2) \cdot \lambda_2^{2n}$$

Fact: $\mathrm{mult}(\lambda_2) \geq |G|^{\alpha}$, $\alpha = \alpha(r)$

$$\Rightarrow \lambda_2 \leq \left(|G|^{1-\alpha} \mu^{2n}(e) \right)^{1/2n}$$

So if $\mu^{2n}(e) \leq \frac{1}{|G|^{1-\varepsilon}}$ for $n \simeq c \log|G|$ then

$$\lambda_2 \leq e^{-1/2c\,(\alpha - \varepsilon)}$$

Step 2: $\mathrm{girth}(\mathcal{G}) \geq c \log|G|$, so

$\mu^{2n}(e) \leq \rho^{2n}$, $\rho = \frac{\sqrt{2k-1}}{2k} < 1$

for $n \simeq c\log|G|$

$$\Rightarrow \mu^{2n}(e) \leq \frac{1}{|G|^{\varepsilon}}.$$

How to get from $c \log|G|$ to $C \log|G|$?

small big

Thm: if $\nu \in \text{proba}(G)$ s.t. $\sup_{H \lneq G} \nu(H) \leq \|\nu\|_2^{\delta}$

then $\|\nu * \nu * \nu\|_2 \leq \|\nu\|_2^{1+\delta}$

Important ingredient:

Product thm: If $A \subseteq G$, $\langle A \rangle = G$ then

$$|AAA| \geq |A|^{1+\varepsilon}, \quad \varepsilon = \varepsilon(n) > 0.$$

$\uparrow$

Helfgott for SL_2

B-Green-Tao } independently in general.
Pyber-Szabo }

To check the condition $\sup_{H \lneq G} \nu(H) \leq \|\nu\|_2^{\delta}$ for a large girth graph on SL_2, you can use the fact that subgroups of SL_2 are solvable and solvable groups don't intersect much with trees. For more general groups, it is more complicated.