Probabilistic tools in nonlinear Dvoretsky theory (Assaf Naor)

<u>Dvoretsky's thm</u>

$\forall k \in \mathbb{N}$, $\forall D > 1$, $\exists n = n(k, D)$ s.t. $\forall$ n-dim normed space X $\exists$ linear subspace Y s.t.

1) $\dim Y = k$
2) $C_2(Y) \leq D$

where $C_2(M)$ = smallest D s.t. $\exists f: M \to \ell_2$ with
(M, d) metric space
$d(x,y) \leq \|f(x) - f(y)\| \leq D\, d(x,y)$ $\forall x, y \in M$

(Later, we will also discuss $C_1(M)$, the analogous thing for ℓ_1).

<u>Bourgain-Figiel-Milman</u> (1986)

$\forall n \in \mathbb{N}$, $\forall D > 1$, what is the largest $m = m(n, D) \in \mathbb{N}$ such that any n-point metric space (X, d) $\exists Y \subset X$ s.t.

1) $|Y| \geq m$
2) $C_2(Y) \leq D$?

<u>Tao 2006</u>

$\forall \alpha > 0$ $\forall D > 1$, what is the largest $\beta = \beta(\alpha, D)$ such that in any compact metric space X, $\dim_H(X) \geq \alpha$ $\exists$ closed $Y \subseteq X$ s.t.

1) $\dim_H Y \geq \beta$
2) $C_2(Y) \leq D$?

<u>Mendel, N 2011</u>: Answer to Tao's problem

2012?: Some more extensions.

2006: Answer to BFM: $\forall \varepsilon \in (0,1)$ $\forall$ n-point (X, d) $\exists Y \subseteq X$ s.t. 1) $|Y| \geq n^{1-\varepsilon}$
2) $C_2(Y) \leq \frac{100}{\varepsilon}$.

Bartal, Linial, Mendel, N (2002):

The estimate is sharp: $\forall \varepsilon, \forall n\ \exists$ n-point (X_n, d) s.t. if $Y \subseteq X_n$, $|Y| \geq n^{1-\varepsilon}$ then $C_2(Y) \geq c/\varepsilon$.

N, Tao (2010): $D > 2$. Let θ be the solution to $2/D = (1-\theta)\,\theta^{\theta/1-\theta}$. Then any n-point metric space X has $Y \subseteq X$ s.t.

1) $|Y| \geq n^{\theta}$
2) $C_2(Y) \leq D$.

Note: $D \to \infty \Rightarrow \theta \sim 1 - \frac{2e}{D}$

$D = 2+\varepsilon \Rightarrow \theta \sim \varepsilon/\log 1/\varepsilon$

This θ is the best for the method.

Majorizing measures thm is a cor of (2012?)

N-Tao 2010: application to Hardy - Littlewood inequality.

How to prove lower bounds of the BLMN type?

Consider $(\{0,1\}^n, \|\cdot\|)$.

Enflo 69: $C_2(\{0,1\}^n) = \sqrt{n}$.

Fourier-analytic gives the same bound.

What about $S \subseteq \{0,1\}^n$, $|S|$ big?

BLMN: $C_2(S) \geq \sqrt{n/1+\log 2^n/|S|}$.

(So $|S| = 2^{n(1-\varepsilon)} \Rightarrow C_2(S) \geq \min\{\frac{1}{\sqrt{\varepsilon}}, \sqrt{n}\}$.)

$\exists\ S \subseteq \{0,1\}^n$, $|S| \geq 2^{n(1-\varepsilon)}$ and $C_2(S) \leq \sqrt{\frac{\log 1/\varepsilon}{\varepsilon}}$

We will do a simpler case that contains all of the ideas: let G be an n-vertex k-regular graph with girth g. (ie. balls of radius $\le g/4$ are isometric to a tree).

Bourgain: If T_n is the complete binary tree of depth n then $c_2(T_n) \approx \sqrt{\log n}$

$$\Rightarrow c_2(G) \gtrsim \sqrt{\log g}.$$

Linial, Magen, N (2001): $c_2(G) \gtrsim c\sqrt{g}$ using Markov type, because random walks diverge at linear speed.

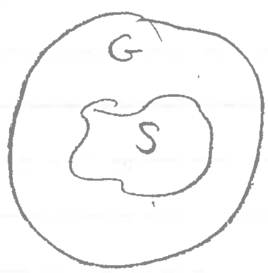
⊛ Thm: $S \subseteq G \Rightarrow c_2(S) \ge \sqrt{\frac{g}{1+\log(n/|S|)}}$.

Open: are there graphs with girth $\to \infty$ s.t. $c_2(G) = O(\sqrt{g})$

Ostrovskii (2011): $\exists\, G_n$ with girth $\to\infty$ s.t. $c_1(G_n) = O(1)$.

These are related to the question of whether graphs with large girth contain expanders.

Proof of ⊛: If G has average degree k & girth g then $c_2(G) \gtrsim (1-2/k)\sqrt{g}$.



Want to turn S into something with many edges.

General fact: If H is a graph on $\{1,\dots,n\}$.
$d = \lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n < 0$
then $\forall\, S \subseteq H$, $2E(S) \ge d|S|^2/n + \lambda_n |S|$

large girth tells us somethi[ng] about this.

Let $G^{(m)}$ be the distance-m graph, $m < g/2$.

$A_{G^{(m)}}$ = adjacency mat.

$$A_{G^{(m)}} = P_m^k (A_G)$$

Geronimous polynomials

$$p_0^k(x) = 1$$

$$p_1^k(x) = x$$

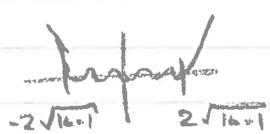
$$p_2^k(x) = x^2 - k$$

$$p_m^k(x) = x\, p_{m-1}^k(x) - (k-1)\, p_{m-2}^k(x)$$

$$\lambda_n(G^{(m)}) \geq \min_{x \in \mathbb{R}} P_m^k(x)$$

very crude!

$$P_m^k(2\sqrt{k-1}\cos\theta) = (k-1)^{m/2-1} \frac{(k-1)\sin((m+1)\theta)\ \sin((m-1)\theta)}{\sin\theta}$$



Now do a random walk on $G^{(m)} \cap S$ of length t, with $mt < g/2$, and optimize the parameters. $\square$.