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Assaf Naor: Probabilistic tools in nonlinear Dvoretzky theory

Goal: indicate a certain way of thinking.

- No history and motivation now.
  (but there are many applications)

Dvoretzky's thm

$\forall k \in \mathbb{N}\ \forall D > 1\ \exists n(k,D)$ s.t.

$\forall$ n-dim. normed space X $\exists$ linear subspace Y s.t.

1) $\dim Y \geq k$
2) $C_2(Y) \leq D$.

For (M,d) metric space — **Euclidean distortion**

$C_2(M)$ = smallest D s.t. $\exists f: M \to \ell_2$ with

$$d(x,y) \leq \|f(x) - f(y)\|_2 \leq D\, d(x,y) \quad \forall x,y \in M.$$

$C_1(M)$ = ... with ℓ_1.

Bourgain – Figiel – Milman problem (1986)

$\forall n \in \mathbb{N}$, $\forall D > 1$, what is the largest $m = m(n,D) \in \mathbb{N}$ s.t. any n-point metric space (X,d) $\exists Y \subseteq X$ s.t.

1) $|Y| \geq m$
2) $C_2(Y) \leq D$?

Legitimate criticism: why pass to finite spaces?

A: lots of applications.

closer to Dvoretzky

Perhaps a more natural Q:

Tao (2006)

$\forall \alpha > 0\ \forall D > 1$ what is the largest $\beta = \beta(\alpha, D)$ s.t. in any compact metric space $\underbrace{X}_{\text{s.t. } \dim_H X \geq \alpha}$, $\exists$ closed $Y \subseteq X$ s.t.

1) $\dim_H Y \geq \beta$
2) $C_2(Y) \leq D$?

Our current understanding: second problem is much more interesting; first problem is very special case of second prob.

2011 Mendel, N.: answer Tao's prob.

2012? "nonlinear Dvoretzky thm for measures".

2006 (Mendel, N.)

$\forall \varepsilon \in (0,1)$ $\forall$ n-point metric space (X, d) $\exists\, Y \subseteq X$ s.t.

1) $|Y| \geq n^{1-\varepsilon}$
2) $C_2(Y) \leq \frac{100}{\varepsilon}$.

Bartal, Linial, Mendel, N. (2002)

The above thm is tight.

$\forall \varepsilon \in (0,1)$ $\forall n$ $\exists$ n-pt. metric space (X_n, d) s.t. if $Y \subseteq X_n$, $|Y| \geq n^{1-\varepsilon}$ then $C_2(Y) \geq \frac{c}{\varepsilon}$.

N., Tao (2010)

$D > 2$. Let θ be the sol'n to $\frac{2}{D} = (1-\theta)\theta^{\frac{\theta}{1-\theta}}$, then any n-pt metric space X has $Y \subseteq X$ s.t.

1) $|Y| \geq n^{\theta}$
2) $C_2(Y) \leq D$.

$D \to \infty$ $\quad \Theta = 1 - \frac{2e}{D}$

$D = 2+\varepsilon$ $\quad \Theta \approx \frac{\varepsilon}{\log(1/\varepsilon)}$

this Θ is the best for the method.

Proof of these: probabilistic method.

2012? paper has majorizing measures as a corollary.

2010. N., Tao $\Longrightarrow$ different approach

so area is very related to maximal ineq.s.

Now: how to prove thms of the BLMN-type?
fun application of Markov-types.

$(\{0,1\}^n, \|\cdot\|_1)$ Enflo '69 : $C_2(\{0,1\}^n) = \sqrt{n}$.

proof: tensorization argument
$\exists$ Fourier analytic argument (uses group structure)

to get an impossibility result: $S \subseteq \{0,1\}^n$, $|S|$ big,
need to show that can't ~~[illegible]~~ embed in Eucl. space well.

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<u>BLMN</u> $C_2(S) \gtrsim \sqrt{\frac{n}{1+\log\left(\frac{2^n}{|S|}\right)}}$

$|S| = 2^{n(1-\varepsilon)}$

$C_2(S) \gtrsim \min\left\{\frac{1}{\sqrt{\varepsilon}}, \sqrt{n}\right\}$

$\exists\ S \subseteq \{0,1\}^n$

$|S| \geq 2^{n(1-\varepsilon)}$

and $C_2(S) \leq \sqrt{\frac{\log(1/\varepsilon)}{\varepsilon}}$

G n-vertex k-regular graph,

girth$(G) = g$.

$\rightsquigarrow$ looks like a tree locally, up to ball of radius $\frac{g}{4}$.

<u>Bourgain</u> If T_n is the complete binary tree of depth n, then $C_2(T_n) \approx \sqrt{\log n}$

$C_2(G) \geq \sqrt{\log g}$

Actually:

<u>Linial, Magen, N. (2001)</u> $C_2(G) \geq c\sqrt{g}$.

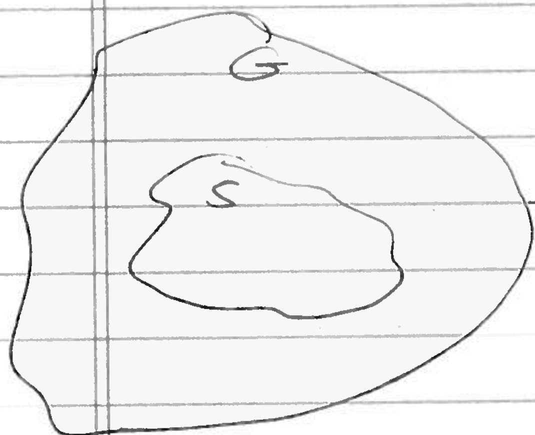
<u>Thm</u> $S \subseteq G$: $C_2(S) \geq \sqrt{\frac{g}{1+\log_k\left(\frac{n}{|S|}\right)}}$

<u>Open</u> : Are there graphs with girth $\to \infty$ with $C_2(G) = O(\sqrt{g})$

<u>Ostrovskii</u> (2011) $\exists G_n$ with girth $\to \infty$ s.t. $C_1(G_n) = O(1)$.

$\rightsquigarrow$ very far from being an expander, but has large girth (going to ∞)

⇝ Connection betw girth & expanders leads to statements as above. Still a weak understanding as of yet.



if G has avg degree k, and girth g then

$$C_2(G) \gtrsim \left(1-\frac{2}{k}\right)\sqrt{g}.$$

General fact

$H = (\{1,\dots,n\}, E_H)$. d-reg.

A adj. mx., $\lambda_1 \geq \dots \geq \lambda_n$, $\lambda_1 = d$

$\forall S \subseteq H$:
$$2E(S) \geq \frac{d|S|^2}{n} + \lambda_n |S|.$$

$G^{(m)}$: $u \sim v \iff d_G(u,v) = m$

$A_{G^{(m)}}$ adj. mx.

$m < \frac{g}{2}$.

degree.

$$A_{G^{(m)}} = P_m^k(A_G)$$

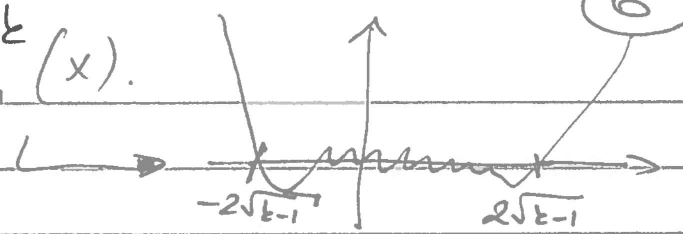
Geronimus polynomials

$P_0^k(x) = 1$
$P_1^k(x) = x$
$P_2^k(x) = x^2 - k.$

Recursion (from local tree structure):
$$P_m^k(x) = x P_{m-1}^k(x) - (k-1) P_{m-2}^k(x).$$

m even

$$\lambda_n(G^{(m)}) \geq \min_{x\in\mathbb{R}} P_m^k(x).$$



$$\{P_m^k(\lambda_i(G))\}_{i=1}^{n}.$$

$$P_m^k(2\sqrt{k-1}\cos\theta) = (k-1)^{\frac{m}{2}-1} \cdots$$

(a simple formula)

$$(k-1)\frac{\sin((n+1)\theta)-\sin((n-1)\theta)}{\sin\theta}.$$

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