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Alexander Volberg: Random Geometric Constructions for the Sharp Estimates of Singular Operators

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- no prob. in statement
- but only probabilistic sol'ns known.

Motivation (Convex Analysis)

1898 Painlevé

$E \subset \mathbb{C}$ compact

$\exists f \in Hol(\mathbb{C} \setminus E) \cap L^\infty(\mathbb{C} \setminus E)$

$f(\infty) = 0$, $f \not\equiv 0$. ?

When does such a fn. exist?

Thm $H^1(E) = 0 \Rightarrow$ only 0 fn

$\overset{?}{\Leftarrow}$

~~[illegible]~~ $\gamma(E) = 0$

Reformulated in abstract nonsense way:

What are E, s.t. $\exists S$ distr. on E s.t. $\langle S, \frac{1}{\xi - z} \rangle$ is bdd?

Harmonic analysis

I. RG Construction

II. Bellman fn. technique (Stoch. Opt. Control applied to HA)

$\hookrightarrow$ not in this talk

GMT

? E? $\mu > 0$

$\int \frac{d\mu(\xi)}{\xi - z}$ bdd.

Easy to see that then $\mu(B(x,r)) \le Cr$.

Consequence:

if $\dim E < 1 \Rightarrow \mu \equiv 0$.

$\dim E = s > 1$, $\mu(B(x,r)) \le r^{s-\varepsilon}$.

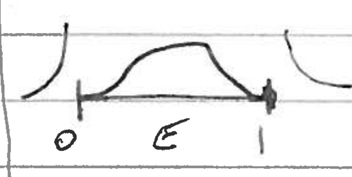
$\dim E = 1$

$\rightsquigarrow$ now H-measure kicks in.

$0 = H^1(E) \rightsquigarrow$ Painlevé knew

$0 < H^1(E) < \infty$.

$E = [0,1]$ works



$\int \frac{\varphi(x)\,dx}{x - z}$

E? $\exists\, \mu > 0$ on E s.t.

$$\left\langle \mu, \frac{1}{\xi - z} \right\rangle = \int_E \frac{d\mu(\xi)}{\xi - z}$$

is bdd?

E? μ real?

μ complex?

$\gamma(E) = 0 \implies H^1(E) = 0$

Denjoy 1919

YES.

Vitushkin 1956

NO

$\gamma(E_V) = 0, \quad 0 < H^1(E_V) < \infty$

wild, not nice

$H^1(\Gamma \cap E_V)$ $\forall$ rectifiable Γ

Vitushkin asked: can you find example not like above?

? E on Γ-rectif. s.t.

$0 < H^1(E) < \infty$

$\gamma(E) = 0$.

Thm 1

If $0 < H^1(E) < \infty$

Then $\gamma(E) = 0 \iff$

E is completely irregular in the sense of Besicovitch

Two proofs:

(G. David, P. Mattila)

(Nazarov, Treil, V.)

$\exists\, f$ nontriv bdd

$\gamma(E)$

$\exists\, f = \int \frac{d\nu}{\cdots}$ ν

$\gamma_c(E) = \sup\{ |\nu(E)| : \ldots \}$

$\gamma_+(E) = \sup \mu(E);\ \mu > 0, \int \ldots$

$\gamma_+ \le \gamma_c \le \gamma$.

Tolsa

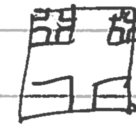
(Nazarov, Treil, V.)

Thm 2 If E is s.t. it supports complex measure $\nu \neq 0$ s.t.

$\int \frac{d\nu}{\xi - z}$ is bdd

$\implies \exists\, \mu$ on E, $\mu > 0$

$\int \frac{d\mu}{\xi - z}$ bdd.



$0 < H^1(C_{1/4}) < \infty$

$\implies$ No. $\nexists$ such μ

Conj $0 < H^1(E) < \infty$

$E = \operatorname{supp} \mu$ s.t.

$\int \frac{d\mu}{\xi - z}$ bdd.

$\implies H^1(\Gamma \cap E) > 0$.

$\implies$ True!

$\implies$ Thm 3.

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$$\mu_n(E) \geq C_1 |\nu_n(E)| \left(\frac{|\nu_n(E)|}{|\nu_n|(E)}\right)^{256}.$$

$\mathbb{R}^d$, $K(x,y)$ Calderon-Zygmund kernel

$$|K(x,y)| \leq \frac{C}{d(x,y)^s}$$

$$|K(x,y) - K(x',y)| \leq \frac{d(x,x')^{\varepsilon}}{d(x,y)^{s+\varepsilon}}$$

Operator w/ CZ kernel. introduced by CZ in '50s

$$f \xmapsto{T} \int K(x,y) f(y) dy$$

$s=d$ $L^p \to L^p$ $1<p<\infty$

$L^1 \to L^{1,\infty}$

$L^2 \to L^2$

1985 David-Journé. — T1 theory.

Q: ball or cube

$\|T\mathbb{1}_Q\|_{L^2} \leq C\|\mathbb{1}_Q\|_{L^2}$ $\forall Q$

$\|T'\mathbb{1}_Q\|_{L^2} \leq C\|\mathbb{1}_Q\|_{L^2}$ $\forall Q$

T' same as T but x and y switched in kernel.

$$f \longmapsto \int K(x,y) f(y)\, d\mu(y)$$

M. Christ 1991.

(X,μ) Yes. T1 thm.

$\mathbb{R}^2$, $s=1$, $K(z,\xi) = \frac{1}{z-\xi}$

$$T_\mu f = \int K(x,y)\, f(y)\, d\mu(y)$$

$$(T_\mu f, g) = \sum_{I,J \in \mathcal{D}} (T h_I, h_J)\,(f, h_I)\,(g, h_J).$$

L^1	$L^\infty L^\infty$	$\mu(I)^{1/2}\mu(J)^{1/2}$
$L^\infty L^1$	$L^1 L^\infty$	$\frac{\mu(I)^{1/2}}{\mu(J)^{1/2}}$
L^∞	$L^1 L^1$	$\frac{1}{\mu(I)^{1/2}\mu(J)^{1/2}}$

... breaks down.

Can replace L^2 in DJ thm by $L^2(\mu)$.

Random Geometric Construction

I is bad if $\exists\, J$ $\quad \ell(J) \geq 2^r \ell(I)$

$$\frac{\operatorname{dist}(I,J)}{\ell(J)} \leq \left(\frac{\ell(I)}{\ell(J)}\right)^{\text{small power } \delta}$$

$\rightsquigarrow \Omega = \{\mathcal{D}_w\}$

prob. space of diadic lattices.

$$\mathbb{P}(I \text{ bad}) = 2^{-ar\gamma}.$$

"Something totally outrageous and false".

$$\sum_{I,J \in \mathcal{D}_w} (Th_I, h_J)\,(f, h_I)(g, h_J) =$$

$$= \sum_{\substack{I,J \\ I,J \text{ good}}} + \sum_{\text{bad, good}} + \sum_{--} + \sum_{--}$$