Dimensionality in the Stability of the Brunn-Minkowski Inequality: A blessing or a curse?

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Berkeley, September 23rd 2011

The Brunn-Minkowski Inequality

- Let $K, T \subset \mathbb{R}^n$ be convex bodies. The dimension n is generally assumed to be larger than some constant.
- The **Minkowski sum** of K and T is defined by,

$$K + T = \{x + y \ \ ; x \in K, y \in T\}$$

- The Brunn Minkowski inequality (mainly due to Brunn and Minkowski) states that

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- If both bodies are convex and closed, equality is attained if and only if K and T are homothetic to each other (up to measure zero).
- What if there is *almost* an equality in the above? Does that mean that K and T are, in some sense, *almost* homothetic?

Stability results

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- Let us try to understand what these inequalities look like.
- For simplicity, we assume from now on that $|K| = |T| = 1$. Define,

$$\epsilon = \left| \frac{K + T}{2} \right| - 1$$

  A stability result is therefore of the following form:

$$d(K, T) < c(n, \epsilon)$$

  where d is a certain distance between the two bodies and $c(n, \epsilon)$ should be small when ϵ is small.

Stability results - continued

Some examples of possible metrics are:

- The Hausdorff metric, defined by

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- The Wasserstein distance between the uniform measures on K and T.
- A certain distance between the support functions or the norms induced by K and T (Diskant, Schneider).
- The quantity $\sup_\rho \left|\frac{\int_K \rho(x)dx}{\int_T \rho(x)dx} - 1\right|$ where ρ belongs to a certain class of functions.

The F-M-P result

- As an example, let us review the (relatively recent) result by Figalli-Maggi-Pratelli. Their result reads,

$$|(K + x_0)\Delta T|^2 \leq n^7\left(\left|\frac{K + T}{2}\right| - 1\right)$$

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- Unfortunately, the above inequality is essentially applicable only when $\left|\frac{K+T}{2}\right| - 1 = O(n^{-7})$.
- The goal of this lecture will be to demonstrate some results which are already applicable when, for example, $\left|\frac{K+T}{2}\right| < 10$.

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- Clearly, $L := \left\{ \frac{x+F(x)}{2} \;\middle|\; x \in K \right\} \subseteq \frac{K+T}{2}$.
  Denote by $\lambda_1(x), \ldots, \lambda_n(x)$ the eigenvalues of $\nabla F(x) = Hess\varphi$. One has,

$$|L| = \int_K \det\left(\frac{\nabla F(x) + Id}{2}\right) dx = \int_K \prod_j \frac{\lambda_j(x) + 1}{2} dx < 1 + \epsilon$$

while $\prod_j \lambda_j(x) = 1$.

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- The best known spectral gap known for general convex bodies is probably far from the optimal one (KLS conjectured spectral gap which is $n^{1/4}$ times better than what is currently known, due to Bobkov).

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This proof has some obstructions from giving a tight result:

- The best known spectral gap known for general convex bodies is probably far from the optimal one (KLS conjectured spectral gap which is $n^{1/4}$ times better than what is currently known, due to Bobkov).
- This method does not give a good bound for large values of λ, so even if we knew the KLS spectral gap conjecture to be true, we would still lose a power of n in our best bound.

Dimensionality

- Recall that this method gives the bound,

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- Considering the example of products of low dimensional bodies shows that the behavior of $|K\Delta T|$ necessarily can't become **better** as the dimension grows, since in any dimension one can construct example which are essentially two-dimensional.
- What about other distance functions? Can dimensionality, in fact, be a blessing?

Stability of the Covariance Matrix

- Let K be a convex body with barycenter at the origin, and $X = X_K$ be a random vector uniformly distributed on K. There exists a matrix $M(K)$ which satisfies,

$$Var[\langle \theta, X \rangle] = \langle \theta, M(K)\theta \rangle, \quad \forall \theta \in S^{n-1}$$

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- We would like to say something about the matrix $M(K)M^{-1}(T)$ under the assumption $\left|\frac{K+T}{2}\right| < 100$.
- In case both $M(K)$ and $M(T)$ are multiples of the identity, this is reduced to bounding the quantity $\left|\frac{\int_K \rho(x)dx}{\int_T \rho(x)dx} - 1\right|$ for $\rho(x) = |x|^2$.

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- A convex body K is said to be isotropic if $|K| = 1$ and $M(K) = \alpha^2 Id$. In that case, α is called the **isotropic constant** of K. The hyperplane conjecture states that α is smaller than some universal constant, independent of the dimension.

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- The **thin shell conjecture** states that for any isotropic convex body K, one has $Var[|\frac{X_k}{\alpha}|] < C$ for some universal constant C.
- Let us normalize X such that $\alpha = 1$. The latter becomes equivalent to $Var[|X|^2] < Cn$.

$$Var[|X|^2] = \sum_i \mathbb{E}[X_i^2 |X|^2] - \mathbb{E}[X_i^2]\mathbb{E}[|X|^2]$$

This shows that it is enough to show that for all $1 \leq i \leq n$,

$$COV(X_i^2, |X|^2) < C\sqrt{n}$$

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- Recall that the marginals of convex bodies are log-concave measures. A useful fact about the space of isotropic one-dimensional log-concave measures is the fact it's **compact**. Moreover, log-concave measures have a sub-exponential tail.
- It follows that the density of X is bounded and has an exponentially-decreasing tail.

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- It follows that the density of X is bounded and has an exponentially-decreasing tail.
- Denote $K_t = K \cap \{x_1 = t\}$. For most t_1, t_2,

$$\left| \frac{K_{t_1} + K_{t_2}}{2} \right| < 10$$

(we're cheating a bit because the volumes are not equal, but this can be easily overcome via rescaling).

Stability of the second moment implies the thin shell conjecture

Theorem: Let K, T be unit volume convex bodies such that $\left|\frac{K+T}{2}\right| < A$. One has

$$\left| \frac{\int_K |x|^2 dx}{\int_T |x|^2 dx} - 1 \right| < \frac{c(A)}{n^\alpha}$$

where $c(A)$ depends only on A and does not depend on the dimension.

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- We prove the theorem with $\alpha = \frac{1}{2}$ in the **unconditional case**. An unconditional convex body is a body for which $(x_1, \ldots, x_n) \in K \Leftrightarrow (\pm x_1, \ldots, \pm x_n) \in K$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

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- In the general case our methods only yield $\alpha = \frac{1}{10}$.

Formulation of the results - the general case

Theorem 1: There exists a constant $C > 0$ such that the following holds. Let $T, K \subset \mathbb{R}^n$ be two unit volume convex bodies. Denote $A = \left| \frac{K+T}{2} \right|$ and denote,

$$M = M(K)M^{-1}(T)$$

and let $\lambda_1, ... \lambda_n$ be the eigenvalues of M in increasing order. Then,

$$\#\{i; \quad |\lambda_i - 1| > \delta\} < C \left(\frac{A}{\delta} \right)^{10}$$

In particular, if $M = \alpha Id$, then $|\alpha - 1| < C \frac{A^{10}}{n^{1/10}}$

Formulation of the results - the unconditional case

Theorem 2: Let $K, T \subset \mathbb{R}^n$ be unconditional convex bodies of volume one. Denote,

$$A = Vol_n\left(\frac{K+T}{2}\right) \geq 1.$$

Then,

$$\|M(K)^{-1}M(T) - Id\|_{HS} \leq CA^5$$

where $C > 0$ is a universal constant. In particular, when $M(K)$ is proportional to the identity,

$$\left|\frac{I(K)}{I(T)} - 1\right| \leq \frac{\tilde{C}A^5}{\sqrt{n}},$$

where $\tilde{C} > 0$ is a universal constant.

The Knothe Map

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- Define $F(x_1, ..., x_n) = (F_1(x_1, ..., x_n), ..., F_n(x_1, ..., x_n))$. For all $1 \le j \le n$, F_j depends only on the variables $x_1, .., x_j$.

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- F_j is increasing in x_j (when keeping the other coordinates fixed).

The unconditional case

- Define $\lambda_j = (\nabla F)_{j,j}$. Note that $\det(\nabla F) = \prod \lambda_j = 1$.

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- Fix $1 \leq j \leq n$. We would like to show that $Var[\pi_j(K_1)] \approx Var[\pi_j(K_2)]$. To this end, let μ, ν be the projection of K, T respectively onto the subspace $E = sp\{x_1, ..., x_j\}$. By the definition of the knothe map, the restriction of F onto E is well defined.

The unconditional case - continued

- Consider a single "fiber" ℓ by fixing the coordinates $x_1, ..., x_{j-1}$. The restriction of ℓ onto such a fiber is a measure-preserving transformation, satisfying $F'(x) = \lambda_j(x)$. The poincare inequality yields,

$$\int_{\ell} |F(x) - x|^2 d\mu(x) \leq C \int |F'(x) - 1|^2 d\mu(x)$$

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- Using Fubini's theorem (by transporting each fiber separately) gives,

$$W_2(\pi_j(K), \pi_j(T)) < \int_K (1 - \lambda_j(x))^2 dx$$

The unconditional case - continued

- A small W_2 distance implies that the variances are approximately the same. Namely,

$$Var(\mu_{|F}) - Var(\nu_{|F}) < C \int_F (\lambda_j(x) - 1)^2 d\mu(x)$$

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$$Var(\mu_{|F}) - Var(\nu_{|F}) < C \int_F (\lambda_j(x) - 1)^2 d\mu(x)$$

- Recall that,

$$\int_{K_1} \exp(c \sum_{j=1}^{n} \min\{(\lambda_j(x) - 1)^2, 1\}) \leq A$$

from this point it is reasonable that with some extra work we'll be able to bound the expression,

$$\sum_j |Var[\pi_j(K)] - Var[\pi_j(T)]|^2$$

which is exactly the Hilbert-Schmidt distance between $M(K)$ and $M(T)$.

The general case

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- The main component here will be the **central limit theorem for convex sets**, initially proven by B. Klartag, which states roughly that a typical low dimensional marginal of the uniform distribution over a convex body is approximately gaussian.
- We will use a pointwise version formulated as follows: Theorem (E., Klartag 2008): Let X be an isotropic random vector in $\mathbb{R}^n$ with a log-concave density. Let $1 \leq \ell \leq n^{c_1}$ be an integer. Then there exists a subset $\mathcal{E}$ of the ℓ-dimensional Grassmanian with measure $1 - C\exp(-n^{c_2})$ such that for any $E \in \mathcal{E}$, the following holds: Denote by f_E the density of the random vector $Proj_E(X)$. Then,

$$\left| \frac{f_E(x)}{\gamma(x)} - 1 \right| \leq \frac{C}{n^{c_3}} \tag{1}$$

for all $x \in E$ with $|x| \leq n^{c_4}$. Here, $\gamma(x) = (2\pi)^{-\ell/2} \exp(-|x|^2/2)$ is the standard gaussian density in E, and $C, c_1, c_2, c_3, c_4 > 0$ are universal constants.

The general case - continued

- Consider the body $L \subset \mathbb{R}^{n+1}$ which is defined as the minimal convex body satisfying,

$$\{(x_1, ..., x_n) \mid (x_1, ..., x_n, -1) \in L\} = K$$

and,

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- Let us try to understand the projection, μ, of the uniform measure on L onto a random subspace of dimension roughly $n^{0.1}$ which contains the direction x_{n+1}.

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- Let us try to understand the projection, μ, of the uniform measure on L onto a random subspace of dimension roughly $n^{0.1}$ which contains the direction x_{n+1}.
- If we suppose K and T are isotropic, then the restriction of μ to $x_{n+1} \in \{-1, 1\}$ should be approximately gaussian. Denote the densities of these restrictions by $f, g : \mathbb{R}^n \to \mathbb{R}$.

The general case - continued

- Consider the body $L \subset \mathbb{R}^{n+1}$ which is defined as the minimal convex body satisfying,
$$\{(x_1, ..., x_n) \mid (x_1, ..., x_n, -1) \in L\} = K$$
and,
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- If we suppose K and T are isotropic, then the restriction of μ to $x_{n+1} \in \{-1, 1\}$ should be approximately gaussian. Denote the densities of these restrictions by $f, g : \mathbb{R}^n \to \mathbb{R}$.
- What about $h = \frac{d\mu}{dx}_{|\{x_{n+1}=0\}}$?

The general case - continued

- According to the Prekopa-Leindler theorem, μ is log-concave, which means that,

$$h(x) \geq \sup_{y \in \mathbb{R}^n} \sqrt{f(x+y)g(x-y)}$$

on the other hand,

$$\int h(x)dx < \left| \frac{K+T}{2} \right|$$

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- A calculation shows that the supremum convolution two gaussian densities is close to 1 only if their variances are roughly the same. Namely,

$$\left(\int h(x) \right)^C > \left| \frac{Var(K)}{Var(T)} - 1 \right| n^\kappa$$

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- In order to finish the proof, we still have to overcome the fact that K and T are not necessarily isotropic.

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- This implies that $\delta < \frac{A^{1/C}}{n^\kappa}$

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- The proof of the general case may be generalized to the class of lipschitz functions defined on the Radon transforms of K, T.
- What is the best dependence of funcitonal $|K \Delta T|$ on dimension? And of the Wasserstein distance?

Thank you. It's been a lovely conference.