Sparsity and non-Euclidean embeddings

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Embedding problems in Banach spaces and group theory
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Local theory of Banach spaces.

Banach Mazur distance :

$$d(X,Y) = \inf\left\{\|T\|\|T^{-1}\|, \quad T : X \to Y \text{ isomorphism}\right\}$$

Embedding of finite dimensional space E in a Banach space X : $E \overset{K}{\hookrightarrow} X$

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$$\exists T : E \to X \text{ with } \|T\|\|T^{-1}\| \leq K.$$

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The Euclidean space ℓ_2^n ?

Any other one ?

The Euclidean case.

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Let X be a Banach space of infinite dimension,

$\forall n \in \mathbb{N}, \forall \varepsilon > 0, \quad \ell_2^n \overset{1+\varepsilon}{\hookrightarrow} X.$

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$\forall n$, $\forall \varepsilon \in (0,1)$,

$$\ell_2^n \overset{1+\varepsilon}{\hookrightarrow} \ell_1^{c(\varepsilon)n}$$

where $c(\varepsilon) \simeq C \log(3/\varepsilon)/\varepsilon^2$.

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Proofs : random methods that can be described through the use of Gaussian operators,

$$G = (g_{ij}) : \ell_2^n \to \ell_1^N \text{ where } g_{ij} \sim \mathcal{N}(0,1).$$

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Main properties :

1) if $\theta, \theta_1, \dots, \theta_n$ are i.i.d. standard p-stable then for every $\alpha_1, \dots, \alpha_n$,

$$\sum \alpha_i \theta_i \sim \left(\sum |\alpha_i|^p \right)^{1/p} \theta$$

2) If θ is p-stable then $\theta \in L_r$ for all $r < p$.

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Consequence : for every $p > 1$, $\ell_p^n \overset{1}{\hookrightarrow} L_1$

Other embeddings.

X is of stable type p iff for some (every) $r < p$, there exists $C > 0$ such that for every finite collection of vectors $x_1, \ldots, x_n$

$$\left(\mathbb{E}\left\|\sum \theta_i x_i\right\|^r\right)^{1/r} \leq C \left(\sum \|x_i\|^p\right)^{1/p}.$$

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- Maurey-Pisier ['76] *Let* X *be a Banach space of infinite dimension,* $\forall n \in \mathbb{N}, \forall \varepsilon > 0$, $\ell_p^n \overset{1+\varepsilon}{\hookrightarrow} X$ *iff* X *is* *not* *of stable type* p*.*

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• Milman ['71] $\forall \varepsilon \in (0,1),\ \ell_2^n \overset{1+\varepsilon}{\hookrightarrow} E_N$ where $n = n(\varepsilon, N, M)$	• Pisier ['83] $\forall \varepsilon \in (0,1),\ \ell_p^n \overset{1+\varepsilon}{\hookrightarrow} E_N$ where $n = n(\varepsilon, N, ST_p(E_N))$

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- Naor-Zvavitch ['01] $\forall n$, $\forall \eta$, $\ell_p^n \overset{C(\log n, \eta)}{\hookrightarrow} \ell_1^{n(1+\eta)}$

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More generally,

$\forall n$, $\forall \eta$, $\ell_p^n \overset{c(\eta)}{\hookrightarrow} \ell_r^{n(1+\eta)}$ with $0 < r < p < 2$ and $r \leq 1$.

Definition of the random operator

Following Pisier ['83]

$Y = \pm e_i$ with probability $1/2N$, Y_{ij} independent copies of Y

$$\begin{aligned} T : \ell_p^n &\longrightarrow \ell_1^N \\ \alpha &\longmapsto \frac{\sigma_p}{N^{1/q}} \sum_{i=1}^{n} \sum_{j \geq 1} \alpha_i \, j^{-1/p} \, Y_{ij} \end{aligned}$$

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Key properties :

1) $\left| \mathbb{E} |T\alpha|_1 - |\alpha|_p \right| \leq D_p \left(\frac{n}{N}\right)^{1/q} |\alpha|_p \quad \rightarrow$ P['83]

2) Concentration of $|T\alpha|_1$ around its mean

$$\mathbb{P}\left\{ \Big| |T\alpha|_1 - \mathbb{E}|T\alpha|_1 \Big| \geq t \right\} \leq 2 \exp(-b_p N t^q)$$

where $1/p + 1/q = 1$. $\rightarrow$ J-S ['82]

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3) New ingredient : delicate small ball estimates.

Sparsity and compressed sensing

Reconstruction of a signal.

Let Φ be an $n \times N$ matrix. You receive $\Phi x := y$ where $x \in \mathbb{R}^N$ is the unknown signal.

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Restricted Isometry Property with constant $\delta \in (0, 1)$: an $n \times N$ matrix Φ such that for all m-sparse vectors $x \in \mathbb{R}^N$,

$$(1-\delta)|x|_2 \leq |\Phi x|_2 \leq (1+\delta)|x|_2.$$

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Candès, Romberg, Tao ['06]
For such matrices, if δ is small enough, any m-sparse vectors is uniquely defined by

$$\min\{|t|_1 \text{ subject to } \Phi t = y\}$$

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Random matrices are good !
$\Phi = (g_{ij})/\sqrt{N}$ satisfies RIP with $m \simeq c(\delta)\frac{n}{\log(eN/n)}$.
The same if $\Phi = (\pm 1/\sqrt{N})$, independent ψ_2 entries,
Mendelson, Pajor, Tomczak-Jaegermann
independent log-concave columns, rows ....
Adamczak, Litvak, Pajor, Tomczak-Jaegermann

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How to give an explicit construction of such a matrix ?

Sparsity and compressed sensing

- Donoho ['06] Connection with the study of Gelfand widths.

$$c_k(\mathrm{Id} : \ell_1^N \to \ell_2^N)$$

is the infimum over all subspaces S of codimension strictly less than k of the value of K such that

$$\forall x \in S, \quad |x|_2 \leq K|x|_1$$

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New proof of the result of Garnaev-Gluskin ['84]

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Let $\Phi : \mathbb{R}^N \to \mathbb{R}^n$, $\Phi = (g_{i,j})$ and take $S = \ker \Phi$, $k = n + 1$.

Sparsity and non Euclidean embeddings, the case $\ell_p^n \hookrightarrow \ell_1^{(1+\eta)n}$ with $1 < p < 2$

Property $\mathcal{P}_1(m, \alpha, \beta)$: $A : \ell_p^n \to \ell_1^{\eta n}$,

$$\forall x \in \mathrm{sparse}(m) \quad \alpha|x|_p \leq |Ax|_1 \leq \beta|x|_p.$$

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Foucart-Lei ['10] in the case $p = 2$

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Decomposition of $x \in \mathbb{R}^n$ according to the size m of sparsity : x_{I_1} the m first largest coordinates of x and so on...

$$x = \sum_{k=1}^{M} x_{I_k} \text{ with } M = [\frac{n}{m}].$$

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Property $\mathcal{P}_2(\kappa, m)$: $B : \ell_p^n \to \ell_1^n$

$$\forall x \in \mathbb{R}^n, \quad \sum_{k \geq 2} |x_{I_k}|_p \leq |Bx|_1 \leq (\kappa n)^{1/q} |x|_p$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Deterministic Theorem, $1 < p < 2$

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Theorem. [Friedland-Guédon '11]

Denote $U = \frac{1}{\beta} \left(\frac{m}{n}\right)^{1/q} A$ *and* $V = \frac{1}{(\kappa n)^{1/q}} B$.

Then for any $x \in \mathbb{R}^n$, $W = \begin{pmatrix} V \\ U \end{pmatrix} : \ell_p^n \to \ell_1^{(1+\eta)n}$ *satisfies*

$$\left(\frac{\alpha}{4\beta}\right) \left(\frac{\min(m, 1/\kappa)}{n}\right)^{1/q} |x|_p \le |Ux|_1 + |Vx|_1 \le 3|x|_p.$$

The two main properties

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Following Pisier ['83]

$Y = \pm e_i$ with probability $1/2\eta n$, Y_{ij} independent copies of Y

$$\begin{aligned} T : \ell_p^n &\longrightarrow \ell_1^{\eta n} \\ \alpha &\longmapsto \frac{\sigma_p}{(\eta n)^{1/q}} \sum_{i=1}^{n} \sum_{j \geq 1} \alpha_i \, j^{-1/p} \, Y_{ij} \end{aligned}$$

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Key properties :

1) $\forall \alpha \in \mathrm{sparse}(m) \cap S_p^{n-1}$, $|\mathbb{E}|T\alpha|_1 - 1| \leq D_p \left(\frac{m}{\eta n}\right)^{1/q}$

2) Concentration of $|T\alpha|_1$ around its mean

$$\mathbb{P}\left\{ \Big| |T\alpha|_1 - \mathbb{E}|T\alpha|_1 \Big| \geq t \right\} \leq 2\exp(-b_p \eta n t^q)$$

where $1/p + 1/q = 1$.

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Classical Union Bound gives

$$\mathbb{P}\left\{ \exists \alpha \in \mathrm{sparse}_p(m), \Big| |T\alpha|_1 - 1 \Big| \geq \frac{3}{8} \right\} \leq 2\binom{n}{m} \exp(-c_p \eta n)$$

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Take m of the order of

$$\frac{\eta}{\log\left(1 + \frac{1}{\eta}\right)} \, n.$$

Following Pisier ['83]

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Conclusion.

It satisfies Property $\mathcal{P}_1(m, \frac{5}{8}, \frac{11}{8})$ with m of the order of

$$c_p \frac{\eta}{\log\left(1 + \frac{1}{\eta}\right)} n :$$

The random operator $T : \ell_p^n \rightarrow \ell_1^{\eta n}$ satisfies with overwhelming probability

$$\forall x \in \text{sparse}(m) \quad \frac{5}{8}|x|_p \leq |Tx|_1 \leq \frac{11}{8}|x|_p.$$

The two main properties

Property $\mathcal{P}_1(m, \alpha, \beta)$: $A : \ell_p^n \to \ell_1^{\eta n}$,

$$\forall x \in \text{sparse}(m) \quad \alpha |x|_p \leq |Ax|_1 \leq \beta |x|_p.$$

Property $\mathcal{P}_2(\kappa, m)$: $B : \ell_p^n \to \ell_1^n$, $\frac{1}{p} + \frac{1}{q} = 1$

$$\forall x \in \mathbb{R}^n, \quad \sum_{k \geq 2} |x_{I_k}|_p \leq |Bx|_1 \leq (\kappa n)^{1/q} |x|_p$$

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The operator $\frac{1}{m^{1/q}} \mathrm{Id}_n : \ell_p^n \to \ell_1^n$ satisfies property $\mathcal{P}_2(\frac{1}{m}, m)$, that is for any $x \in \mathbb{R}^n$,

$$\sum_{k=2}^{M} |x_{I_k}|_p \leq \frac{1}{m^{1/q}} |x|_1 \leq \left(\frac{n}{m}\right)^{1/q} |x|_p$$

Conclusion : tight isomorphic embedding

The random operator $T : \ell_p^n \to \ell_1^{\eta n}$ satisfies Property $\mathcal{P}_1(m, \frac{5}{8}, \frac{11}{8})$ with $m = c_p \frac{\eta}{\log\left(1+\frac{1}{\eta}\right)} n$

The deterministic operator $\frac{1}{m^{1/q}}\mathrm{Id}_n : \ell_p^n \to \ell_1^n$ satisfies Property $\mathcal{P}_2(\frac{1}{m}, m)$

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Theorem. [Friedland-Guédon '11]

Denote $U = \frac{1}{\beta} \left(\frac{m}{n}\right)^{1/q} A$ *and* $V = \frac{1}{(\kappa n)^{1/q}} B$*.*

Then for any $x \in \mathbb{R}^n$*,* $W = \begin{pmatrix} V \\ U \end{pmatrix} : \ell_p^n \to \ell_1^{(1+\eta)n}$ *satisfies*

$$\left(\frac{\alpha}{4\beta}\right) \left(\frac{\min(m, 1/\kappa)}{n}\right)^{1/q} |x|_p \le |Ux|_1 + |Vx|_1 \le 3|x|_p.$$

Conclusion : tight isomorphic embedding

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Theorem. [Friedland-Guédon '11]

Denote $U = \left(c_p \frac{\eta}{\log\left(1+\frac{1}{\eta}\right)}\right)^{1/q} T$ *and* $V = \left(c_p \frac{\eta}{\log\left(1+\frac{1}{\eta}\right)}\right)^{1/q} \frac{\mathrm{Id}_n}{m^{1/q}}$.

Then for any $x \in \mathbb{R}^n$, $W = \begin{pmatrix} V \\ U \end{pmatrix} : \ell_p^n \to \ell_1^{(1+\eta)n}$ *satisfies*

$$\left(c_p \frac{\eta}{\log\left(1+\frac{1}{\eta}\right)}\right)^{1/q} |x|_p \le |Wx|_1 = |Ux|_1 + |Vx|_1 \le 3|x|_p.$$

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• Tightness. Take $\eta = 1/n \to$ Banach Mazur distance between ℓ_p^n and ℓ_1^{n+1}.

It is tight when $\eta \geq \frac{\log n}{n} \to$ connection with the study of Gelfand width.

Conclusion : tight isomorphic embedding

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- Mixture of random and deterministic method - Old story.

Conclusion : tight isomorphic embedding

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• Valid in a more general setting $\rightarrow$ arrival space is of stable type p, like in Pisier ['83],
and also r-Banach spaces like in Bastero, Bernués ['93]

Conclusion : tight isomorphic embedding

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• Valid for several new operators.

The random operator defined by Johnson and Schechtman satisfies the same property $\mathcal{P}_1$.

Property $\mathcal{P}_2$ is just an algebraic property and is satisfied for example when $B : \ell_p^n \to \ell_1^n$ is such that

$$|x|_1 \le |Bx|_1 \le (Cn)^{1/q} |x|_p$$

Conclusion : tight isomorphic embedding

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- Optimality of $\mathcal{P}_1(m, \alpha, \beta) \to$ Application to Gelfand width

$$c_k(\mathrm{Id} : \ell_1^N \to \ell_p^N)$$

which gives optimal results.

Other consequences of the properties - I

The random operator $T : \ell_p^n \to \ell_1^{\eta n}$ satisfies Property $\mathcal{P}_1(m, \frac{5}{8}, \frac{11}{8})$ with $m = c_p \frac{\eta}{\log\left(1+\frac{1}{\eta}\right)} n$ that is,

$$\forall x \in \mathrm{sparse}(m) \quad \frac{5}{8}|x|_p \leq |Tx|_1 \leq \frac{11}{8}|x|_p.$$

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Gelfand width of $\mathrm{Id} : \ell_1^n \to \ell_p^n$.

Find a subspace S of codimension less than k such that you control the diameter (in the ℓ_p^n-norm) of the section of the octahedron by S i.e.

$$\forall x \in S, \quad |x|_p \le D|x|_1$$

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We take $k = \eta n$ and $S = \ker T$

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We take $k = \eta n$ and $S = \ker T$
Let $h \in \ker T$ with $h \neq 0$ then

$$|h|_p^p = |h - h_{I_1}|_p^p + |h_{I_1}|_p^p$$

Other consequences of the properties - I

The random operator $T : \ell_p^n \to \ell_1^{\eta n}$ satisfies Property $\mathcal{P}_1(m, \frac{5}{8}, \frac{11}{8})$ with $m = c_p \frac{\eta}{\log\left(1+\frac{1}{\eta}\right)} n$ that is,

$$\forall x \in \text{sparse}(m) \quad \frac{5}{8}|x|_p \leq |Tx|_1 \leq \frac{11}{8}|x|_p.$$

- By property $\mathcal{P}_1$

$$|h_{I_1}|_p \leq \frac{8}{5}|Th_{I_1}|_1 = \frac{8}{5}\left|T\left(\sum_{k=2}^{M} h_{I_k}\right)\right|_1 \leq \frac{8}{5}\sum_{k=2}^{M}|Th_{I_k}|_1$$

Using again property $\mathcal{P}_1$, we get

$$|h_{I_1}|_p \leq \frac{11}{5}\sum_{k=2}^{M}|h_{I_k}|_p$$

And by the simple algebraic property $\mathcal{P}_2$,

$$|h_{I_1}|_p \leq \frac{11}{5}\frac{1}{m^{1/q}}|h|_1.$$

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The random operator $T : \ell_p^n \to \ell_1^{\eta n}$ satisfies Property $\mathcal{P}_1(m, \frac{5}{8}, \frac{11}{8})$ with $m = c_p \frac{\eta}{\log\left(1+\frac{1}{\eta}\right)} n$ that is,

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- For the other part

$$|h - h_{I_1}|_p^p = \left|\sum_{k=2}^{M} h_{I_k}\right|_p^p = \left(\sum_{k=2}^{M} |h_{I_k}|_p^p\right)^{1/p} \leq \left(\sum_{k=2}^{M} |h_{I_k}|_p\right)^p$$

and by the simple algebraic property $\mathcal{P}_2$, we get that

$$|h - h_{I_1}|_p^p \leq \frac{1}{m^{p/q}} |x|_1^p$$

Other consequences of the properties - I

In conlusion, for any $h \in \ker T$,

$$|h|_p \leq C_p \frac{1}{m^{1/q}} |h|_1 = C'_p \left(\frac{\log\left(1 + \frac{n}{k}\right)}{k} \right)^{1/q} |h|_1$$

since

$$m = c_p \, \frac{\eta}{\log\left(1 + \frac{1}{\eta}\right)} \, n \text{ and } k = \eta n$$

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since

$$m = c_p \frac{\eta}{\log\left(1 + \frac{1}{\eta}\right)} n \text{ and } k = \eta n$$

This means that

$$c_k(\mathrm{Id} : \ell_1^n \to \ell_p^n) \leq C_p' \left(\frac{\log\left(1 + \frac{n}{k}\right)}{k} \right)^{1/q}$$

and it is known to be optimal for $k \geq \log n$.

Other consequences of the properties - II

Let $W : \ell_p^n \to \ell_1^N$ such that

$$\frac{1}{D} |x|_p \leq |Wx|_1 \leq |x|_p.$$

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Let $S = \operatorname{Im} W$, $S \subset \ell_1^N$ and $\operatorname{codim} S = N - \operatorname{rank} W = \eta n = k$.
Then for any $y \in S$, $y = Wx$ with $x \in \ell_p^n$ and

$$|y|_1 = |Wx|_1 \ge \frac{1}{D}\,|x|_p$$

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This proves that

$$\forall y \in S, \quad |y|_p \leq D\,|y|_1$$

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We have proved that

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which means that

$$c_k(\mathrm{Id} : \ell_1^N \to \ell_p^N) \leq D = \left(\frac{\log\left(1 + \frac{N}{k}\right)}{k}\right)^{1/q}$$

(here $n \simeq N$) and this is known to be optimal for $k \geq \log N$
i.e. $\eta \geq \frac{\log N}{n}$

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- Optimality of the Theorem [FG].

Other consequences of the properties - III

The random operator $T : \ell_p^n \to \ell_1^{\eta n}$ satisfies Property $\mathcal{P}_1(m, \frac{5}{8}, \frac{11}{8})$ with $m = c_p \, \frac{\eta}{\log\left(1+\frac{1}{\eta}\right)} \, n$ that is,

$$\forall x \in \mathrm{sparse}(m) \quad \frac{5}{8}|x|_p \leq |Tx|_1 \leq \frac{11}{8}|x|_p.$$

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Reconstruction via ℓ_1-minimization

$$\Delta(y) = \text{argmin } |z|_1, \text{ subject to } Tz = Ty$$

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Reconstruction via ℓ_1-minimization

$$\Delta(y) = \text{argmin } |z|_1, \text{ subject to } Tz = Ty$$

If $s > 0$ satisfies

$$s \leq c_p \frac{k}{\log(1 + \frac{n}{k})}$$

then with probability greater than $1 - \exp(-b_p k)$, every s-sparse vectors y, is exactly reconstructed : $y = \Delta(y)$.

$$\forall y, |y - \Delta(y)|_1 \leq 4 \inf_{|I| \leq s} |y - y_I|_1$$

Sparsity and non-Euclidean embeddings

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