Quantitative differentiation

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Let $\mathcal L_n$ denote Lebesgue measure on $\bf R^n$. There is a natural measure, $\mathcal C=r^{-1}dr\times \mathcal L_n$ on the collection of balls, $B_r(x)\subset \be R^n$, for which the subcollection of balls $B_r(x)\subset B_1(0)$, has infinite measure. Let $f: B_r(x)\to \bf R^n$ have bounded differential $|f'|\leq 1$. For any $B_r(x)\subset B_1(0)$, there is a natural scale invariant quantity which encodes the deviation of $f\, |\, B_r(x)$ from being an affine linear function. The most basic case of quantitative differentiation (due to Peter Jones) asserts that for all $\epsilon>0$, the measure of the collection of balls on which the deviation from linearity is $\geq \epsilon$ is finite and controlled by ϵ, independent of the particular function f. We will explain the sense in which this model case is actually a particular instance of a general phenomenon which is present in many different geometric/analytic contests. In each case which fits the framework, to prove the relevant quantitative diffentiation theorem, one must verify a single estimate which we term {\it coercivity of relative defects}. We indicate a number of recent applications.