Reconstruction in Trees

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MSRI Connections for Women: Discrete Lattice Models in Mathematics, Physics, and Computing

January 13, 2012

In This Talk

- Introduction
  - Reconstruction in Trees.
  - Gibbs measures.
- Applications and Questions
  - Reconstruction Threshold for the Hardcore model.
  - Random Constraint Satisfaction and Clustering.
  - Mixing of Glauber Dynamics.

Broadcast Process on a Tree

Model

- T^k infinite $k+1$-regular tree.

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- Gives rise to the *free measure* μ on configurations.
- Model for random spin configurations of T^k.
  E.g. proper colorings, Ising and Potts models, independent sets....

The Free measure

- Uniform distribution on proper colorings.

$$P(i,j) = \frac{\delta_{i \neq j}}{q-1}, \qquad \pi \text{ uniform on } [q]$$

- Hardcore measure with fugacity $\lambda = \omega(1+\omega)^k$. Choose $\sigma_\rho \in \{0,1\}$ according to (π_0, π_1).

$$M = \begin{pmatrix} P_{1,1} & P_{1,0} \\ P_{0,1} & P_{0,0} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\omega}{1+\omega} & \frac{1}{1+\omega} \end{pmatrix}.$$

Gibbs Measures, Uniqueness and Extremality

- Free measure μ is a *Gibbs measure* on spin configurations.
- As q, k and M vary, is μ unique? extremal?

Reconstruction

Inference problem:

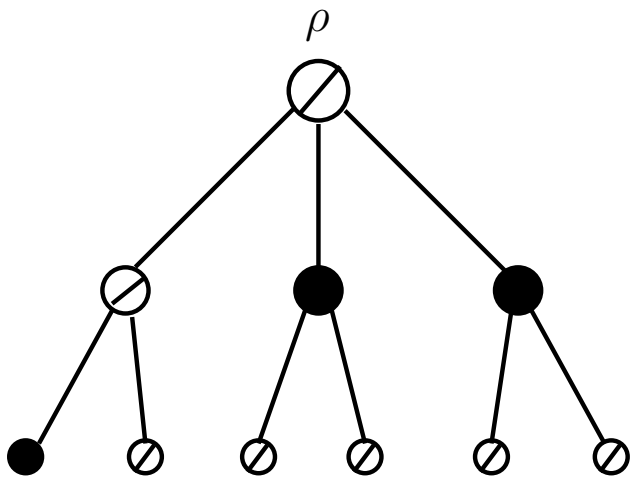
Given $L(n)$, the configuration at depth n, is it possible to reconstruct the value at the root? (with probability better than guessing randomly)

Reconstruction non-solvability:

$$\forall\ i \quad \lim_{n\to\infty} P_T[\sigma_r = i \mid L(n)] \to \pi(i)$$

[Georgii '88] Extremality of μ is equivalent to reconstruction non-solvability.

The Hardcore Measure on the Rooted $k+1$-regular Tree



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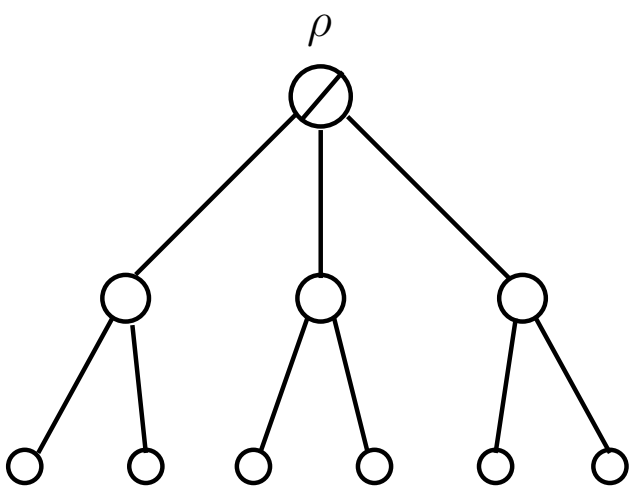

Let $\omega > 0$ and $\lambda = \omega(1+\omega)^k$.

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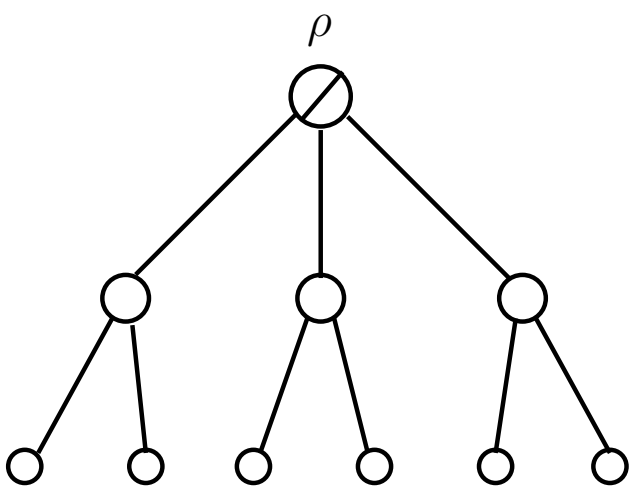

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Think of ω as a small number in $[0,1]$.

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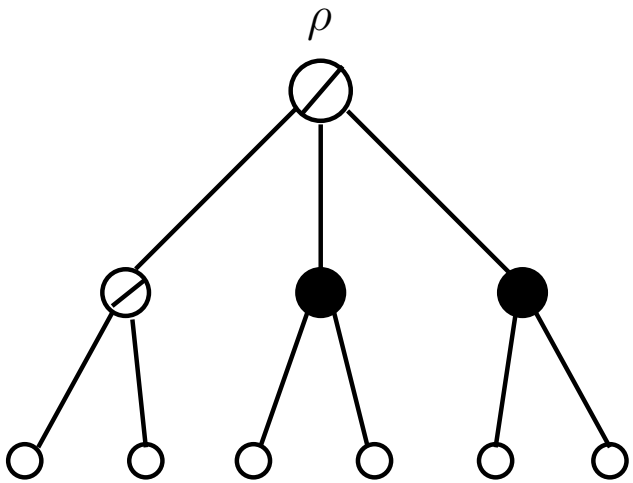
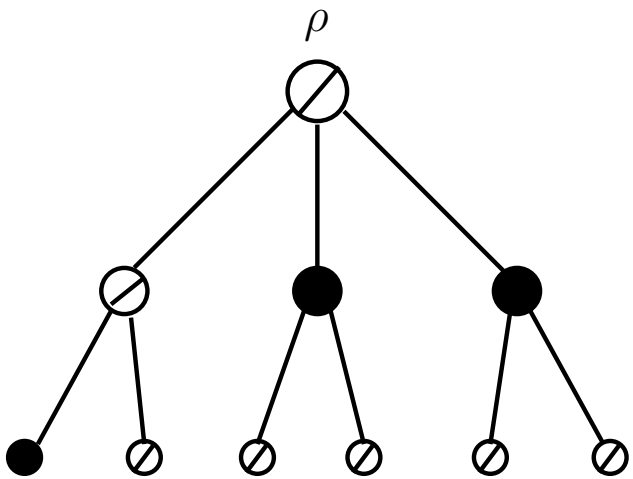


Generate vertex states recursively from parents' states according to

$$M = \begin{pmatrix} p_{11} & p_{10} \\ p_{01} & p_{00} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\omega}{1+\omega} & \frac{1}{1+\omega} \end{pmatrix}.$$

The Hardcore Measure on the Rooted k-regular Tree



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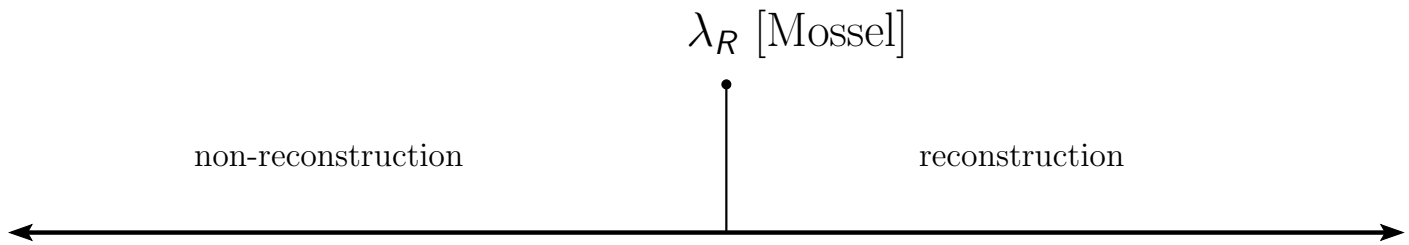
Phase Transitions

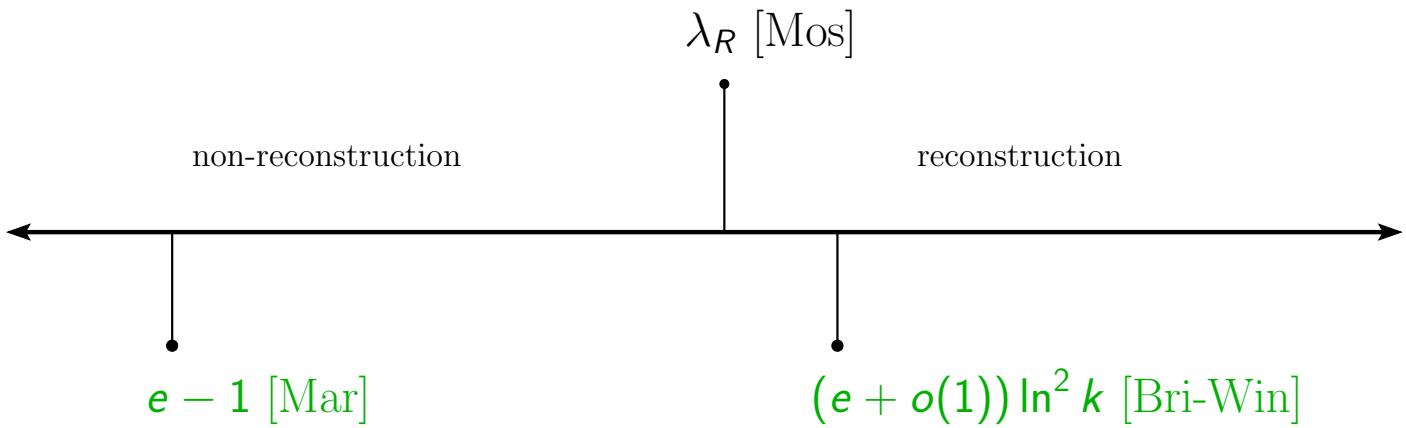
- [Kelly '85] Uniqueness threshold at $\lambda_U = k^k/(k-1)^{k+1}$.
- "Census", "second eigenvalue" or Kesten Stigum bound. Reconstruction for $(\Lambda_2(M))^2 k > 1$. Tight for Ising model.
- [Mossel '01, Brightwell-Winkler '04] Possible to reconstruct below the KS-bound for Potts models, binary asymmetric channels and hardcore model.
- [Mossel '01] Coupling to show there is a threshold for reconstruction.
- Finding the reconstruction threshold arises naturally in biology, information theory and statistical physics.

The Reconstruction Threshold on the k-regular Tree



- [Mossel '01] Non rec. for $\lambda < \lambda_R$, reconst. for $\lambda > \lambda_R$.

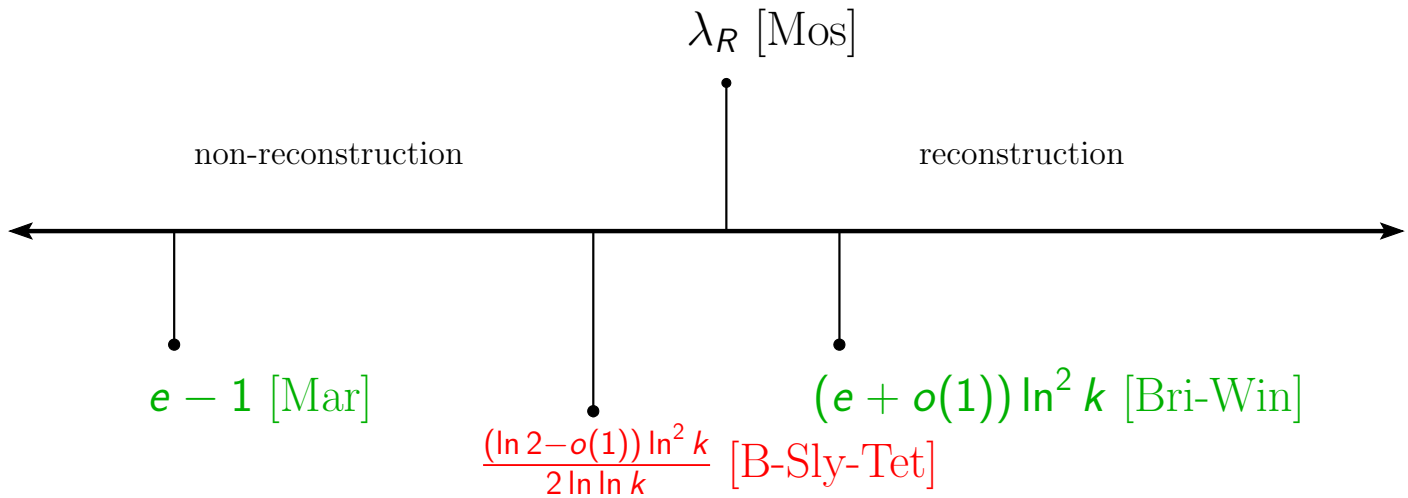
Bounds on the Reconstruction Threshold



- [Martin '03] Non-reconstruction for $\lambda < e-1$.
- [Brightwell-Winkler '04] Reconstruction when

$$\lambda \geq (e+o(1))(\ln k)^2, \quad \omega \geq \frac{\ln k + \ln\ln k + 1 + \varepsilon}{k}.$$

Improved Bound for Non-Reconstruction



- [B-Sly-Tetali '10] Non-reconstruction for

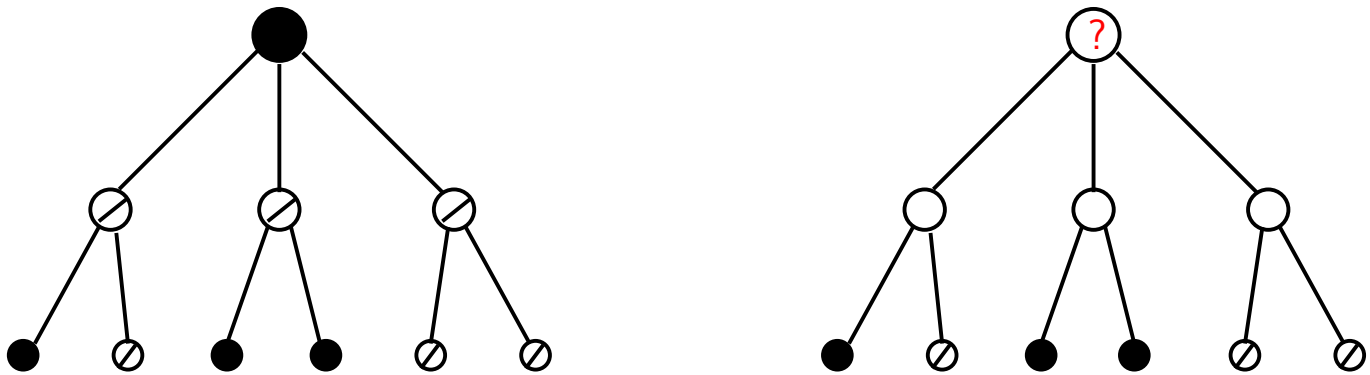
$$\lambda \leq \frac{(\ln 2 - o(1))\ln^2 k}{2\ln\ln k},$$

$$\omega \leq \frac{\ln k + \ln\ln k - \ln\ln\ln k - \ln 2 + \ln\ln 2 - o(1)}{k} := \omega^*.$$

An Equivalent Definition of Reconstruction Using Conditional Posterior

Magnetization:

$$X(n) = \pi_0^{-1}[\mathbb{P}(\sigma_\rho = 1 | \sigma(L(n)) = A) - \pi_1]$$



Theorem

Non-reconstruction is equivalent to

$$\lim_{n\to\infty} \mathbb{E}^1_{T_n}[X(n)] = 0,$$

where $\mathbb{E}^1_{T_n}[X(n)]$ *denotes the measure conditioned on* $\sigma_\rho = 1$.

Steps for Showing Non-Reconstruction

1. Tree recursions for posterior probabilities

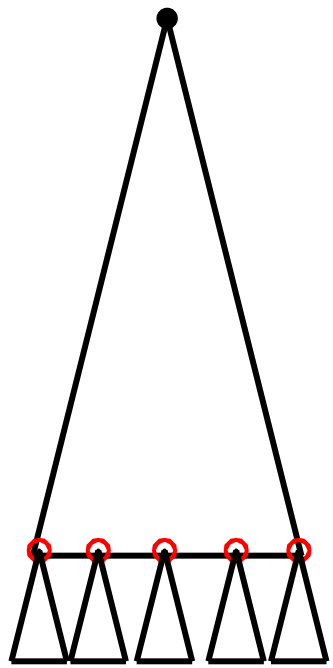
$$\mathbb{P}(\sigma_\rho = 1 | \sigma(L) = A) = \frac{\lambda \prod_{i=1}^{k} \mathbb{P}(\sigma_{\rho_i} = 0 | \sigma(L) = A_i)}{1 + \lambda \prod_{i=1}^{k} \mathbb{P}(\sigma_{\rho_i} = 0 | \sigma(L) = A_i)}$$

Steps for Showing Non-Reconstruction

2. Using methods from [Borgs-Chayes-Mossel-Roch '06]

   Tree recursion for X $\Rightarrow$ if $\mathbb{E}^1_{T_\ell}[X] < \frac{1}{2}$, then

$$\mathbb{E}^1_{T_{\ell+1}}[X(\ell+1)] \leq (\omega^2 e^{\frac{1}{2}\omega k} k)\mathbb{E}^1_{T_\ell}[X(\ell)].$$



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For $\omega \leq \omega^*$, $\omega^2 e^{\frac{1}{2}\omega k} k <\sim \frac{(\ln k)^{3/2}}{k^{1/2}} < 1$ for large k.

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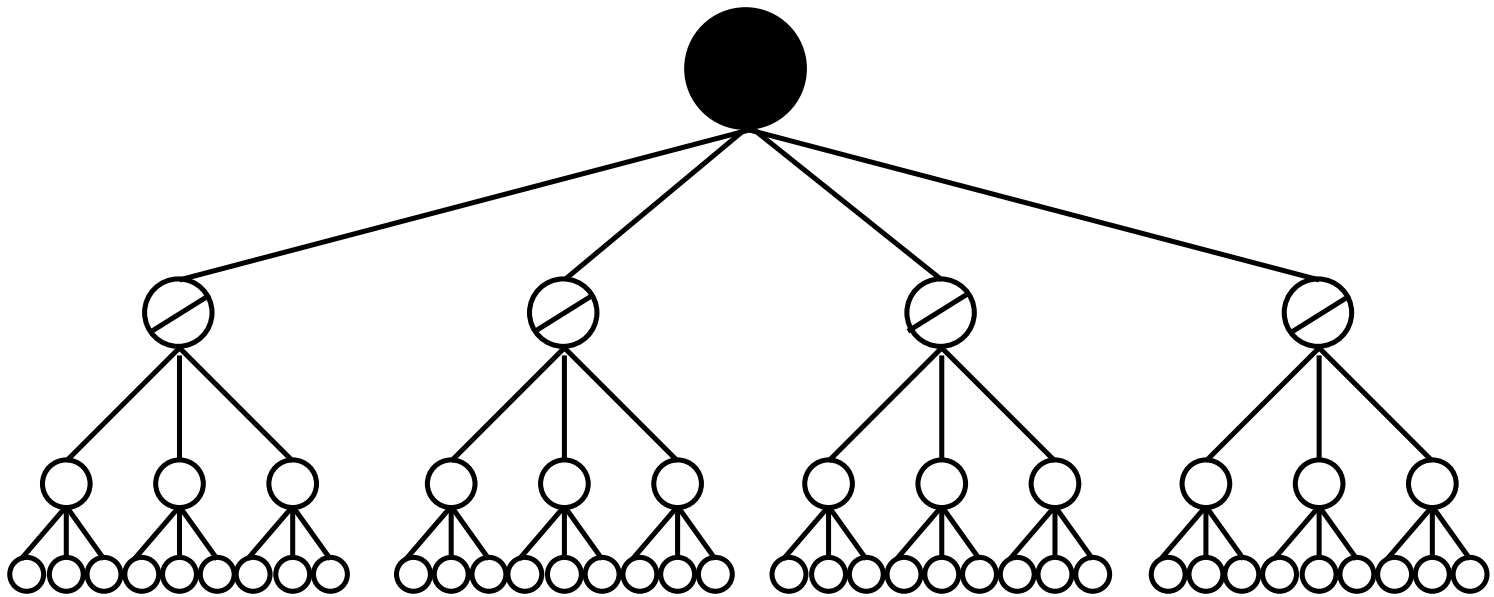
3. For a depth 3 tree, typical conditional posterior probability $\mathbb{E}^1_{T_3}[X] < 1/2$.

   - We show $\mathbb{E}^1_{T_3}[\mathbb{P}[\sigma_{root} = 1 \mid \sigma(L)]] < \frac{1}{2}$.
   - This implies $\mathbb{E}^1_{T_3}[X] < 1/2$.

What is the Typical Conditional Posterior in T_3?



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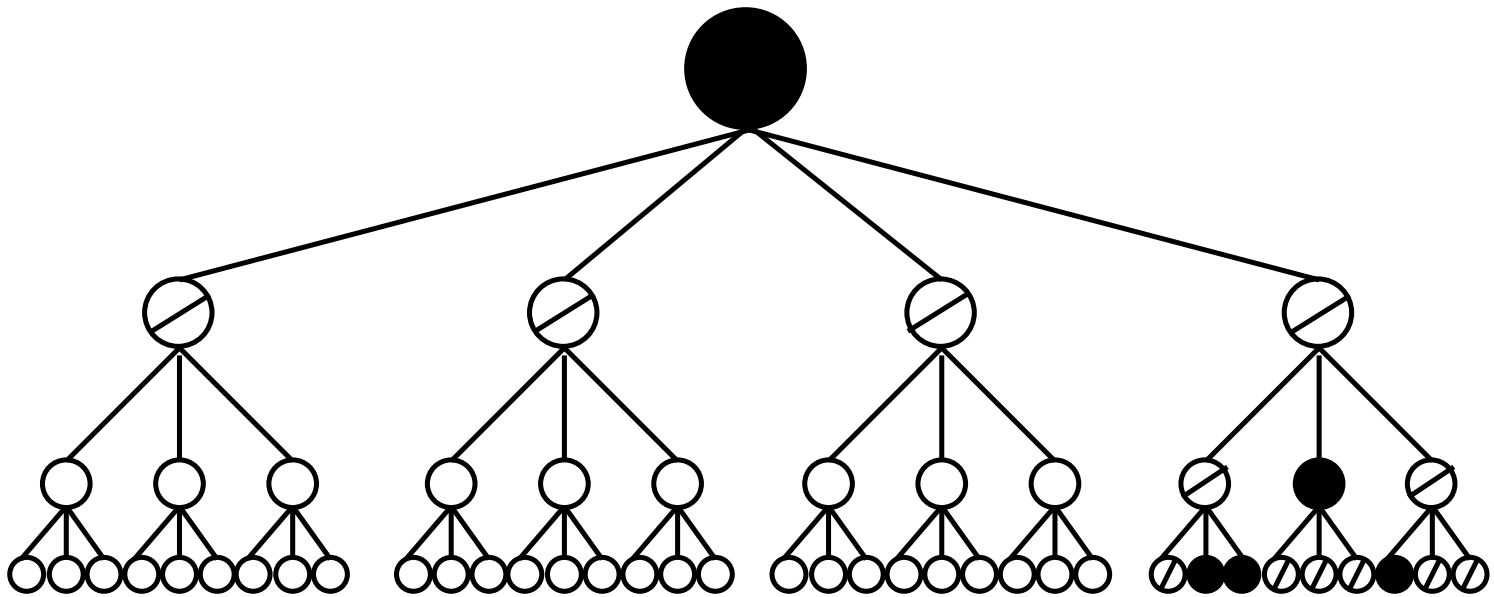

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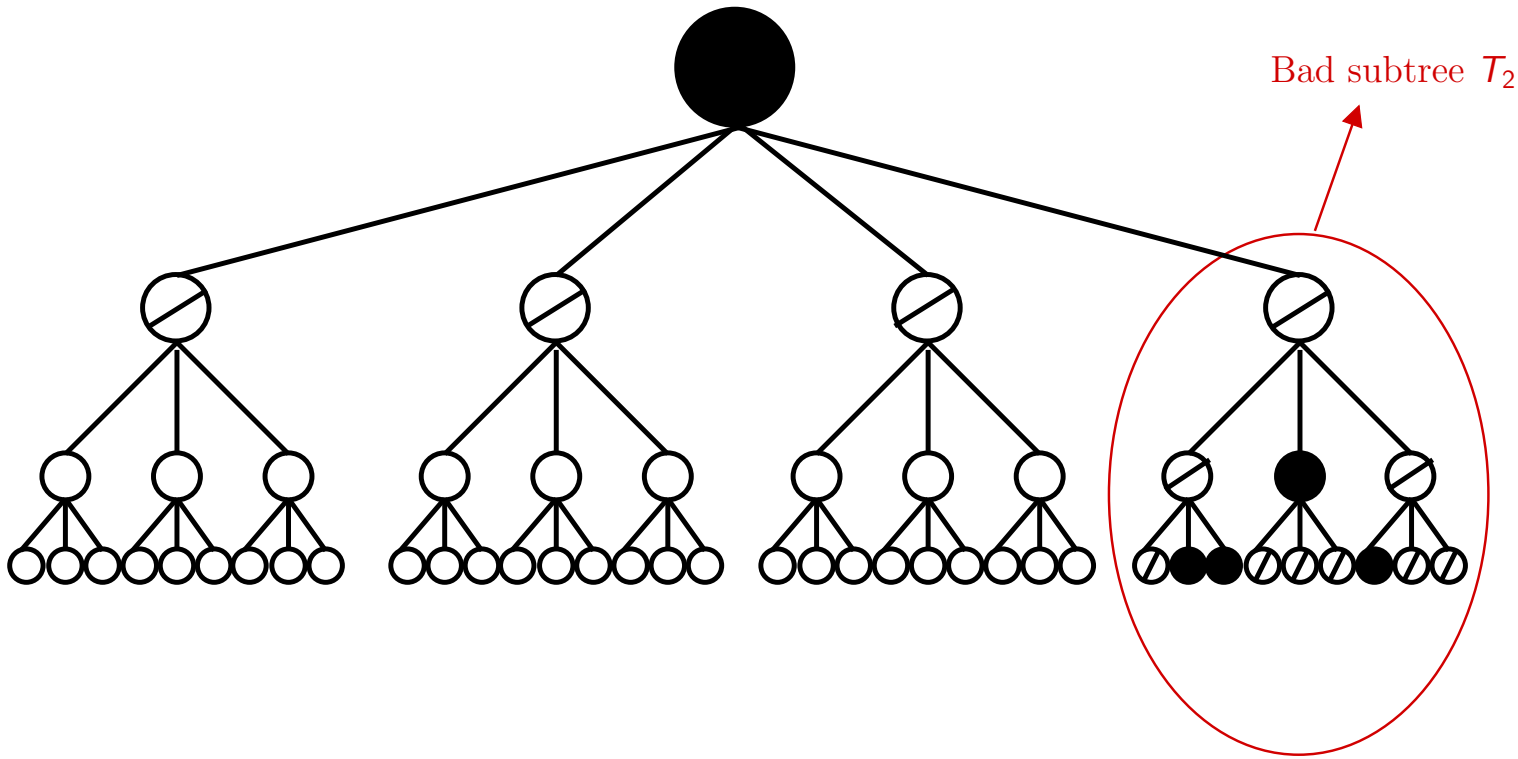

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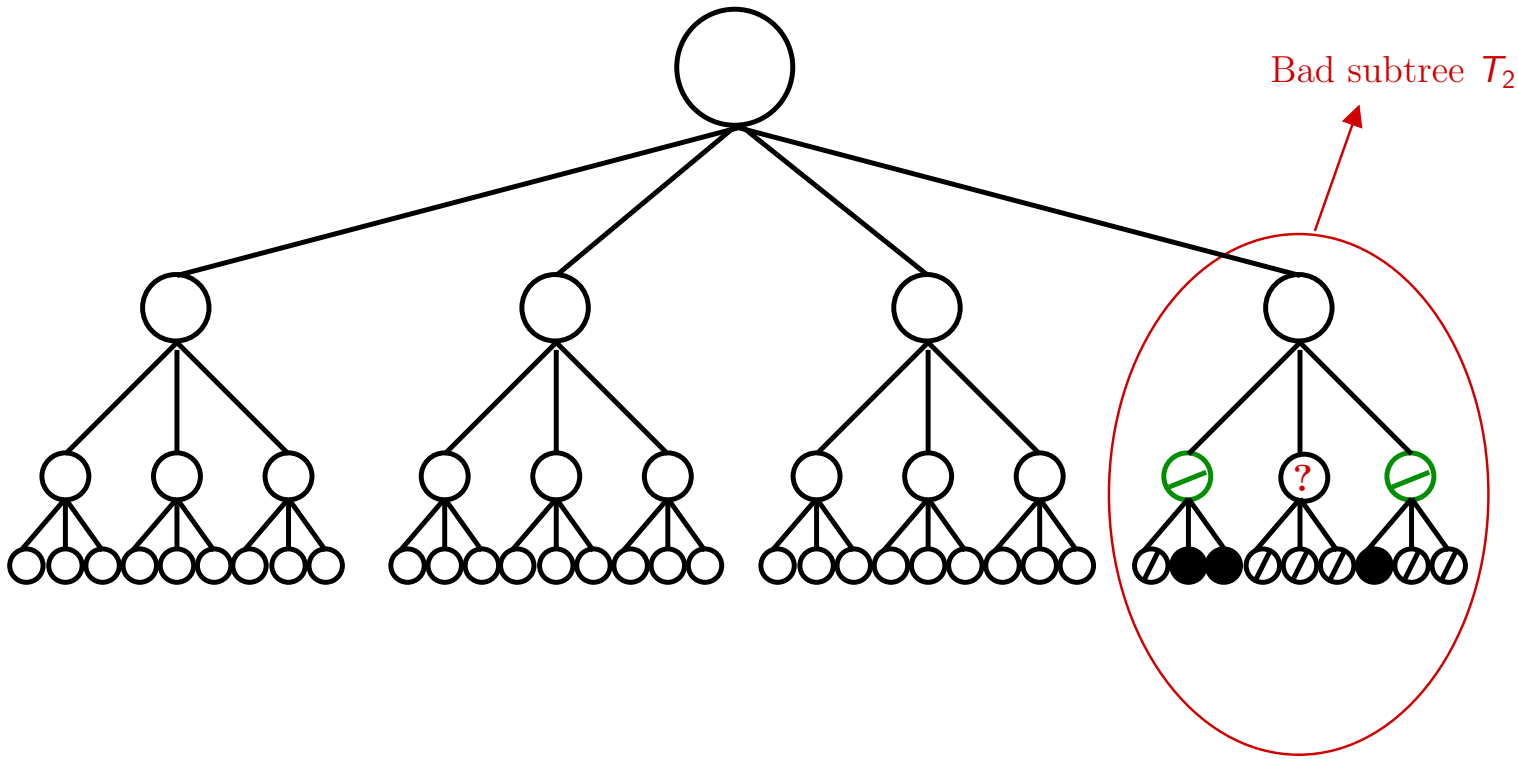

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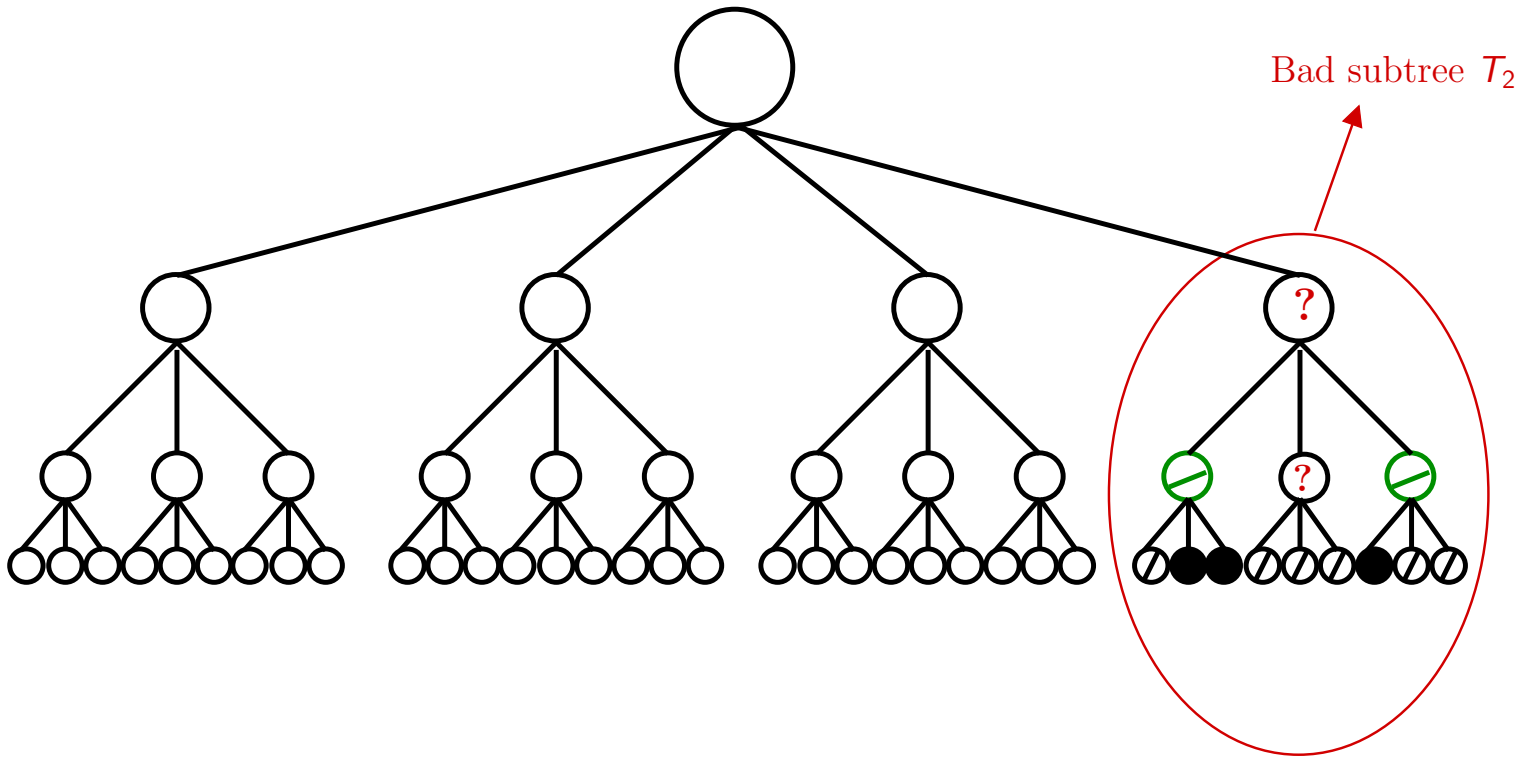

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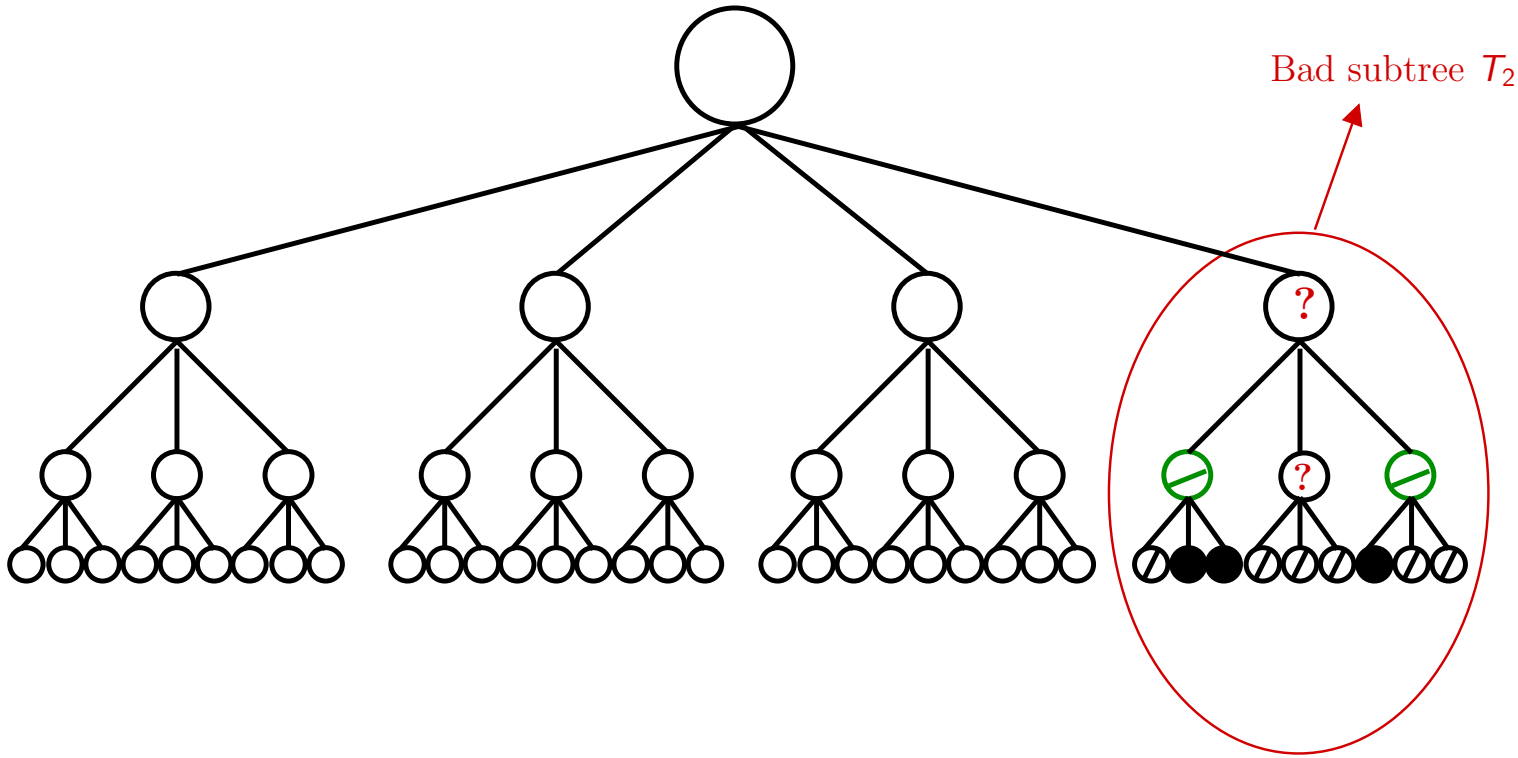

In a *Bad subtree* T_2

$$\mathbb{P}^0_{T_2}[\sigma_{root} = 0 \mid \sigma(L(2))] = \frac{1}{2}\left(1 + \frac{1}{1+2\lambda}\right) \sim \frac{1}{2}$$

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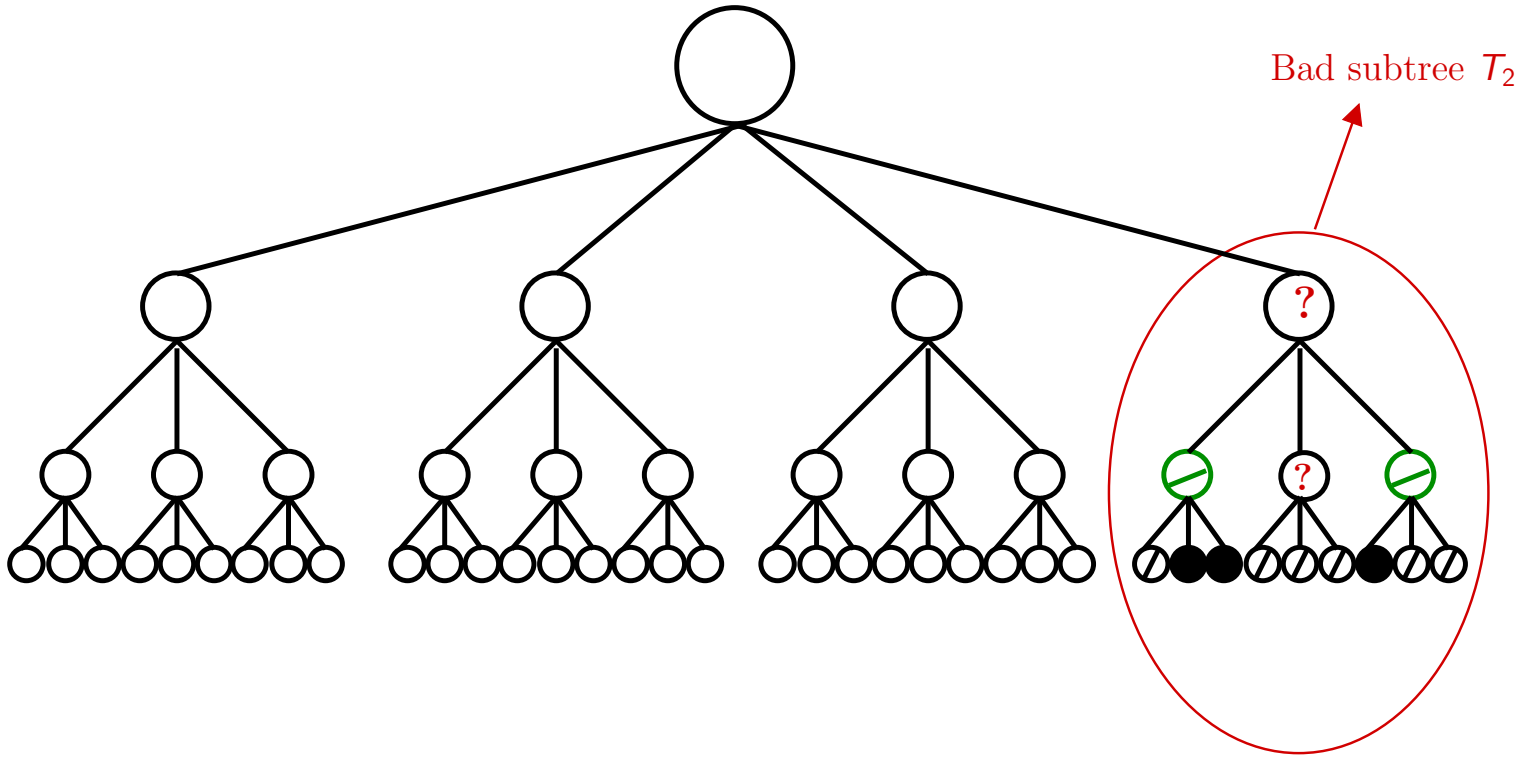

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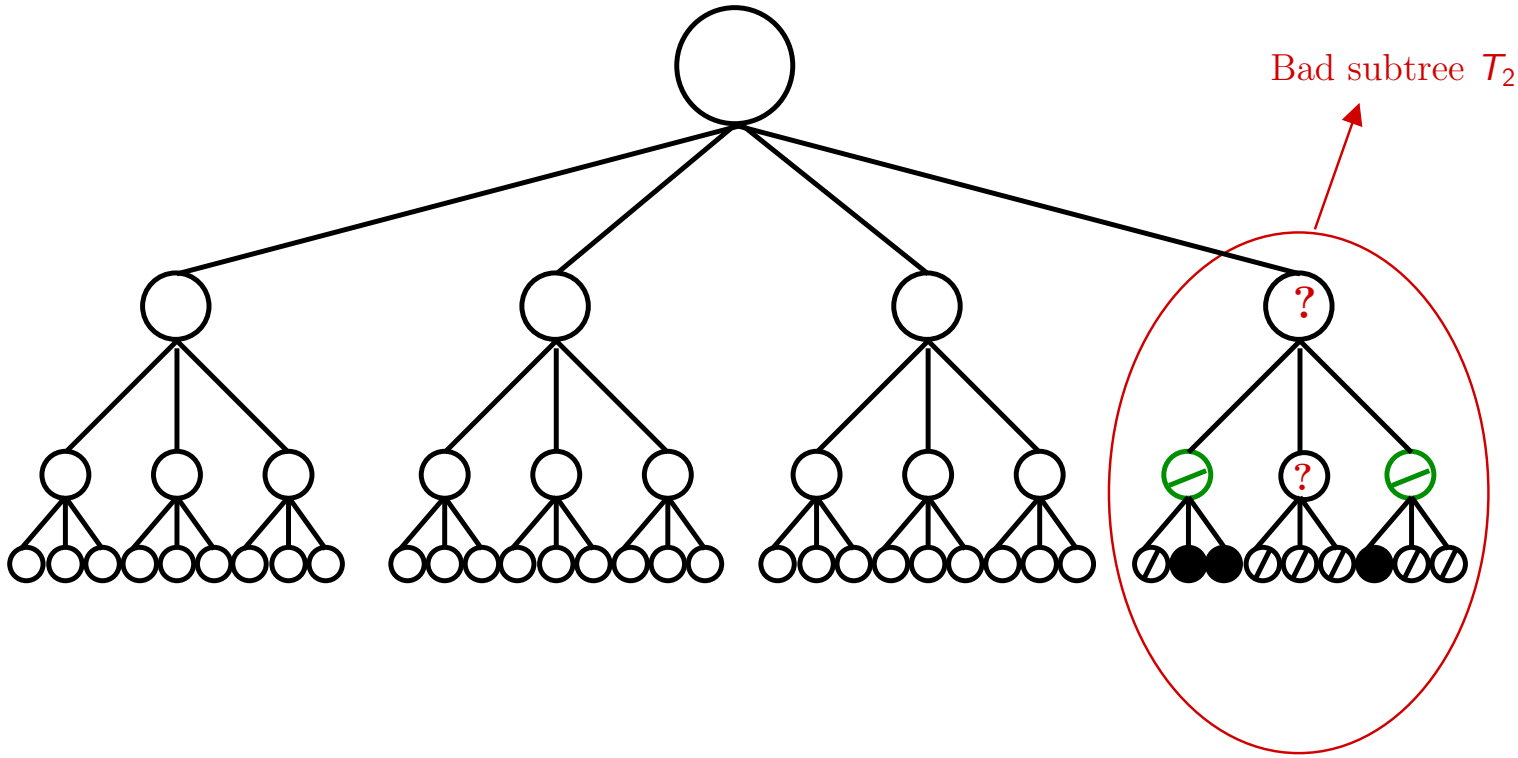

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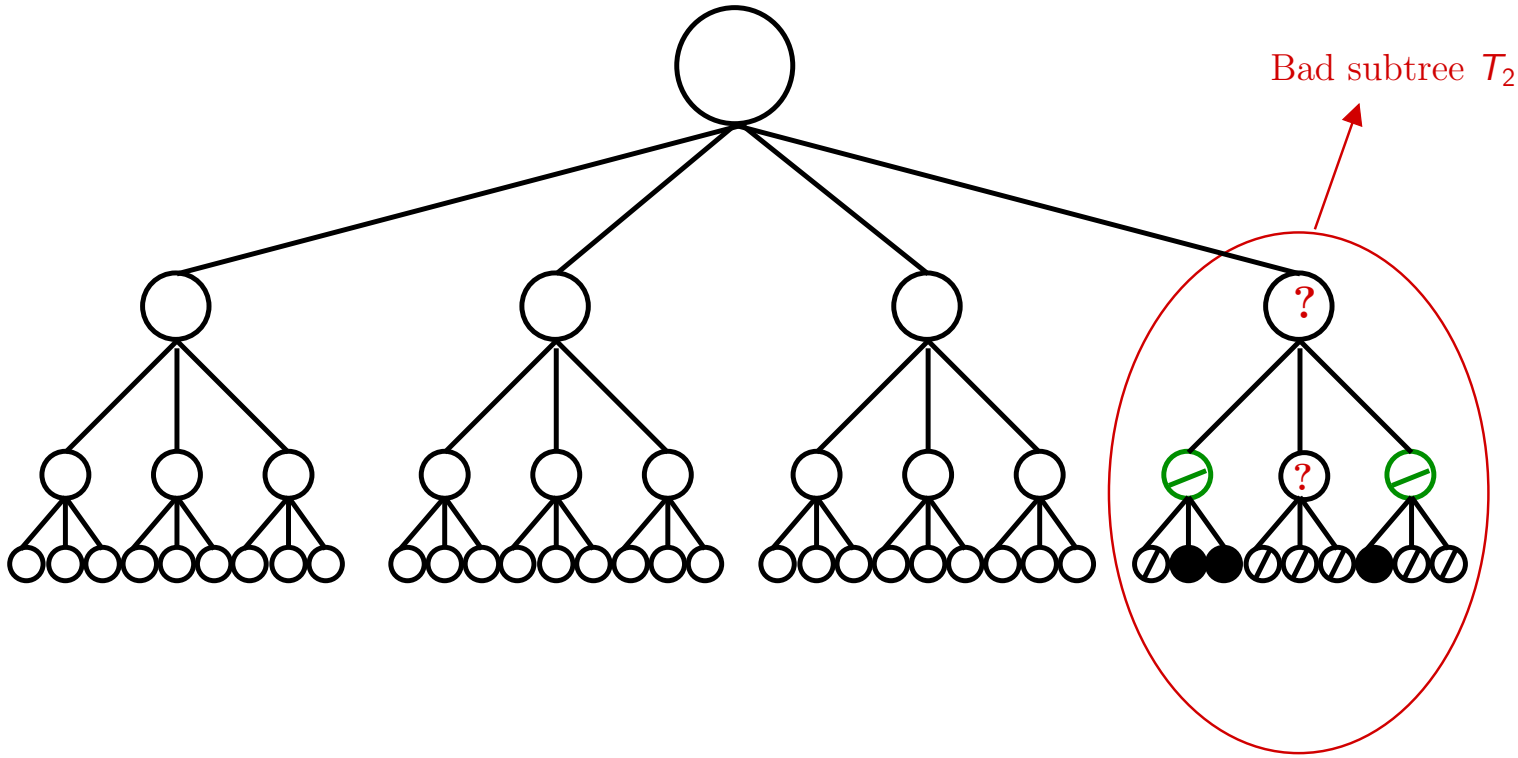

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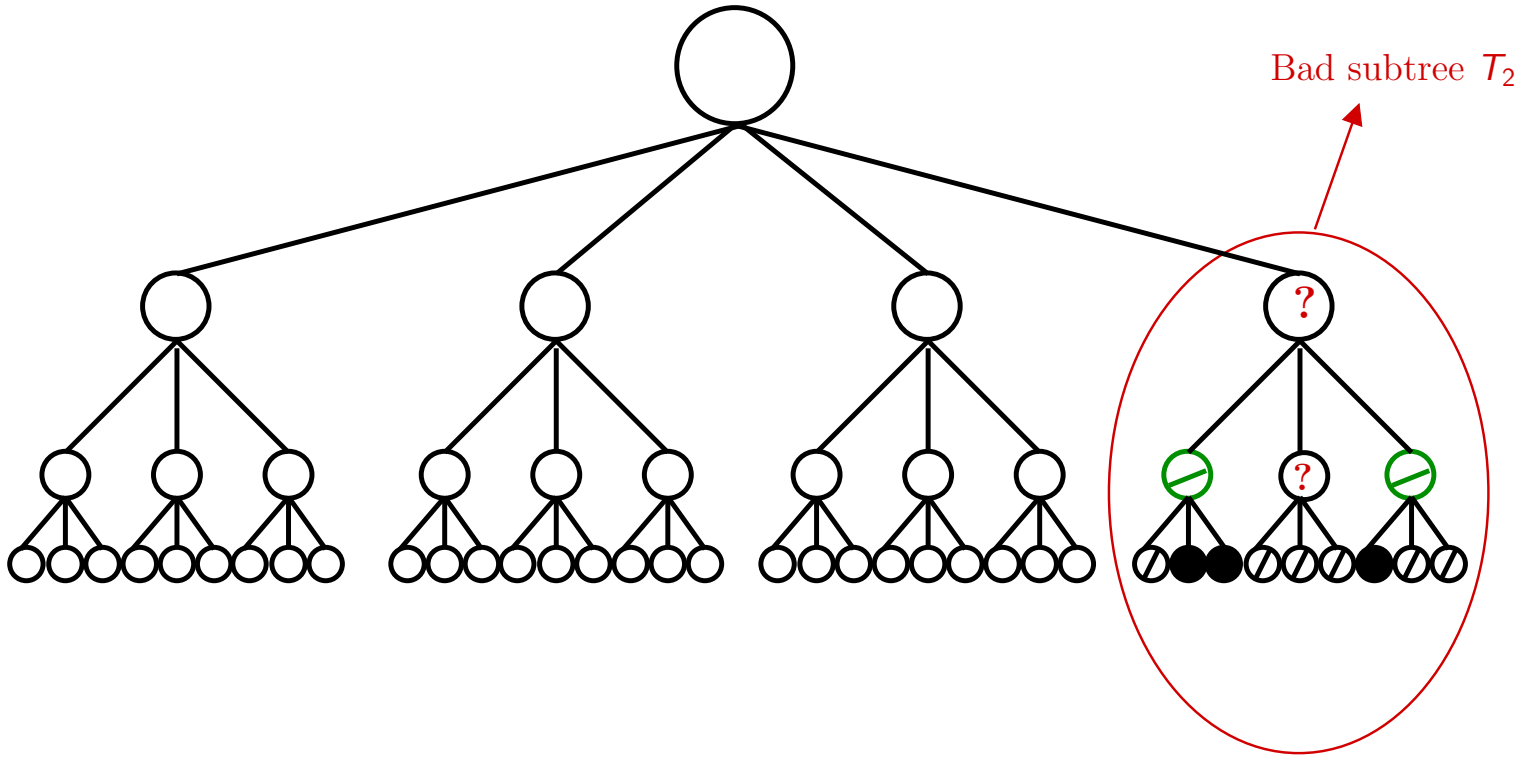

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- Bad subtrees $\Rightarrow$ if $\omega \leq \omega^*$, $\mathbb{P}^1_{T_3}[\sigma_\rho = 1 \mid \sigma(L)] = o_k(1)$.
- $\mathbb{E}^1_{T_3}[\mathbb{P}[\sigma_\rho = 1 \mid \sigma(L)]] < 1/2$.

Random Constraint Satisfaction

- Random 3-SAT: n Boolean variables $x_1, \cdots, x_n$.
- Choose $m = \alpha n$ constraints, or "clauses" uniformly and independently. Negate variables w.p. $1/2$.

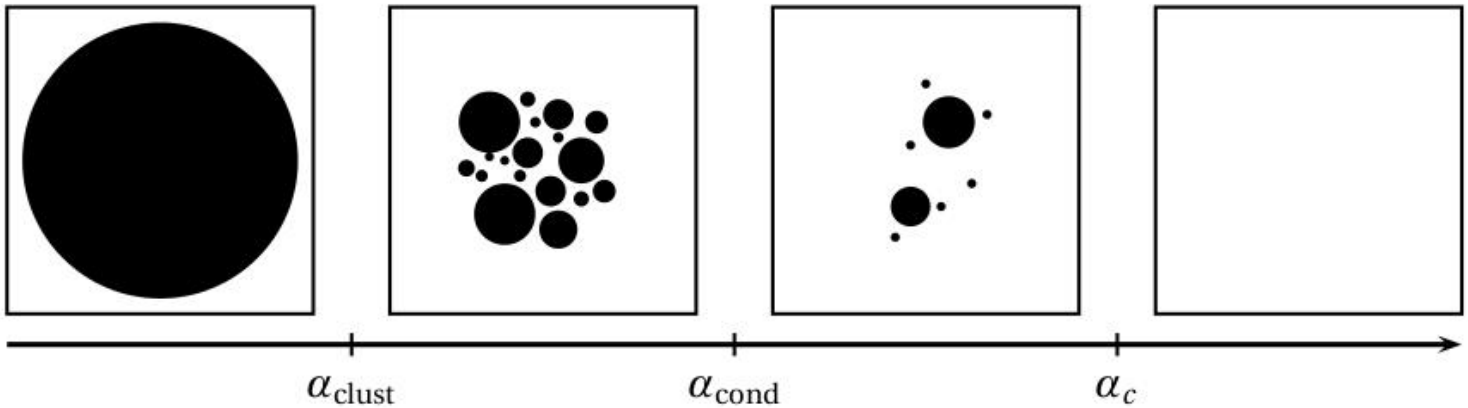
$$f = (x_4 \vee \overline{x_2} \vee x_{12}) \wedge (x_{27} \vee x_8 \vee \overline{x_1}) \wedge \cdots$$

- Intuitively, larger α harder to satisfy.
- [Friedgut '99] There is an $\alpha_c(n)$ s.t. for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}[f \text{ is satisfiable}] = \begin{cases} 1 & \text{if } \alpha < (1-\varepsilon)\alpha_c(n) \\ 0 & \text{if } \alpha > (1+\varepsilon)\alpha_c(n) \end{cases}$$

Geometry of the Solution Space

[Krzakala, Montanari, Ricci-Tersenghi, Semerjian, Zdeborova-'06]



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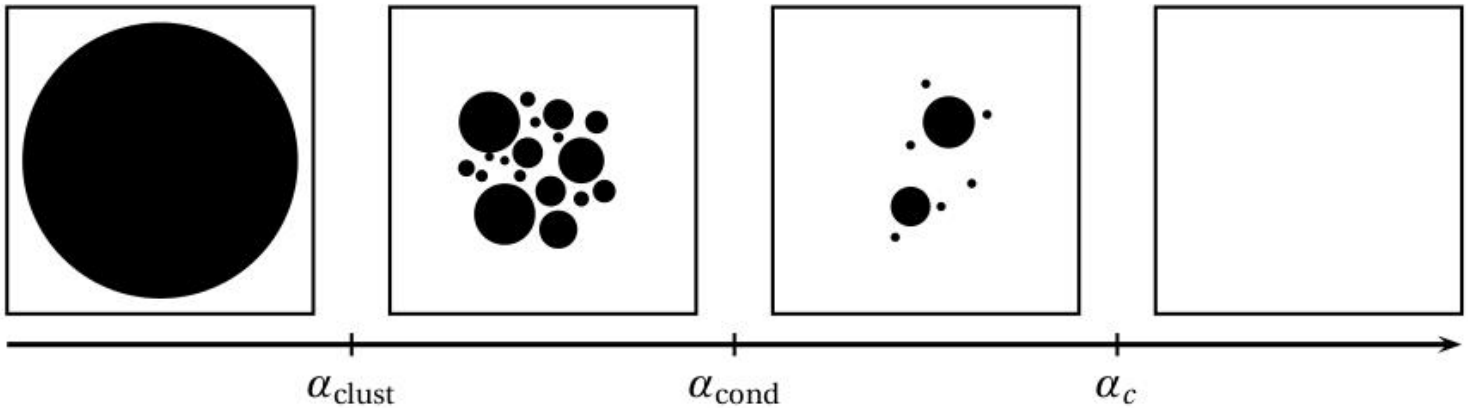

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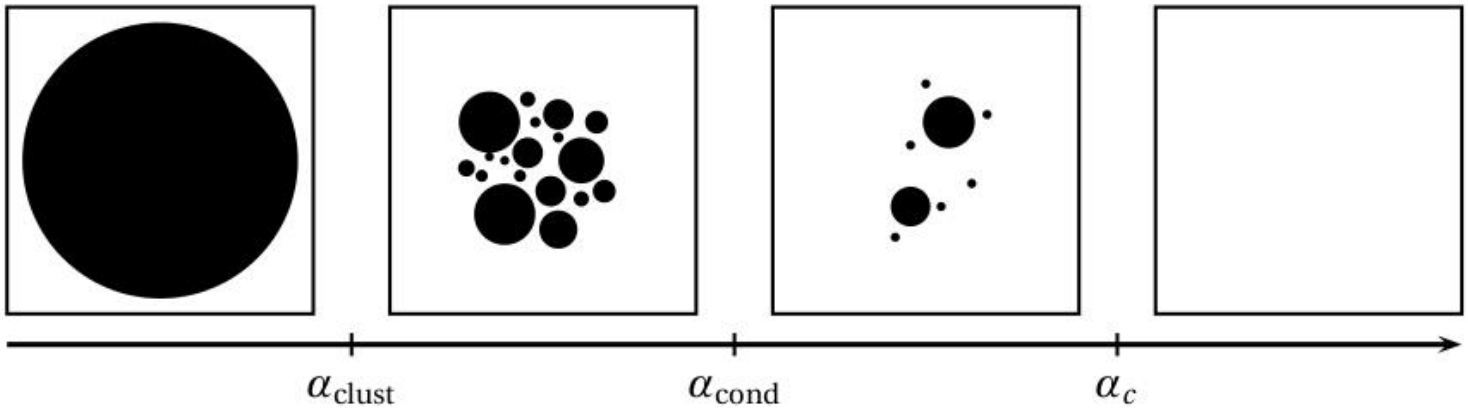

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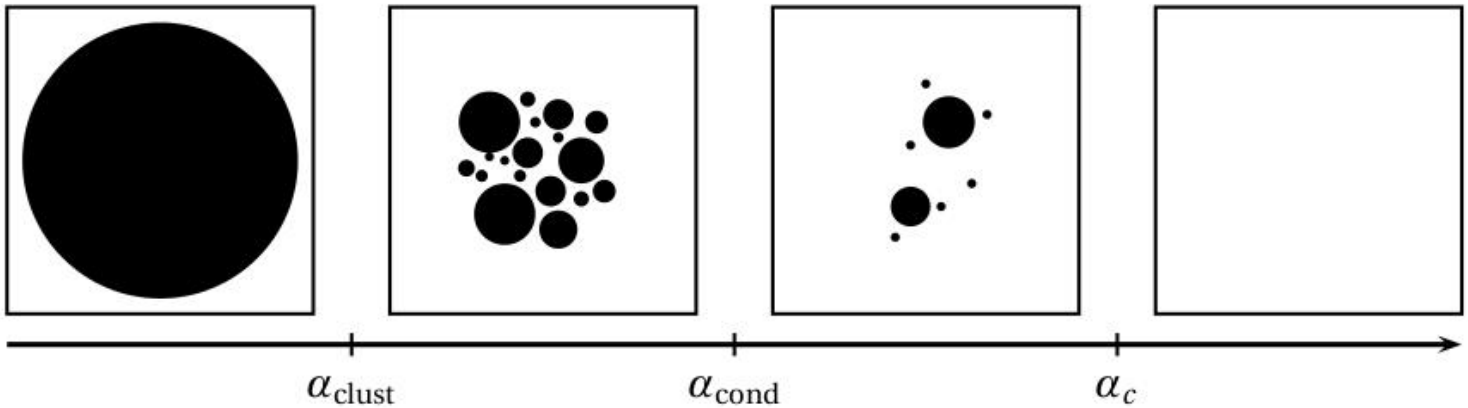

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Clustering and Algorithmic Efficiency

- [Achlioptas-Coja-Oghlan-Ricci-Tersenghi '10], [Montanari-Restrepo-Tetali '09] Clustering for random colorings of $G(n, k)$ and that it coincides with reconstruction in the tree (threshold bound [Sly '09]).

- Clustering coincides with persistence of long-range correlation of spins of vertices for sparse random graph.

- At clustering, energy barriers between clusters are bottlenecks for Glauber dynamics. Local algorithms conjectured to be efficient upto reconstruction threshold.

Glauber Dynamics on Trees

- [Berger-Kenyon-Mossel-Peres '05], [Ding-Lubetzky-Peres '10], [Martinelli-Sinclair-Weitz '04] For the Ising model, $T_{\text{relax}} = O(n)$ in non-reconstruction regime and slower in reconstruction regime.
- [Tetali-Vera-Vigoda-Yang '10] Phase transition of T_{relax} at reconstruction threshold for colorings of tree with free boundary.
- [Martinelli-Sinclair-Weitz '04] For the hardcore model, $T_{\text{relax}} = O(n)$ for all λ on the tree with free boundary.
- [Restrepo-Stefankovic-Vera-Vigoda-Yang '10] For the hardcore modeel, phase transition in relaxation time for constructed boundary at reconstruction threshold.

Glauber Dynamics on Sparse random Graphs

- Random regular graphs. [Vigoda '99] $O(n \ln n)$ mixing for colorings only well below uniqueness.
- Erdös-Renyi graphs. [Mossel-Sly '08] Polynomial mixing for colorings for $q < k^4$.

Constraint Satisfaction - Optimization

Finding maximum independent set in random sparse graph of average degree k.

- [Frieze '90] There exist independent sets of size $\frac{(2-o(1))n \ln k}{k}$.
- Greedy algorithm yields independent sets of size $\frac{(1+o(1))n \ln k}{k}$ (corresponding to $\pi_1^* n$).
- [Coja-Oghlan - Efthymiou '10] Combinatorial structure of solution space undergoes phase transition when independent sets are of size $\sim \frac{n \ln k}{k}$.