Counting and sampling minimum cuts in weighted planar graphs

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Based on joint works with Adam Friedlander and Zach Langley.

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MSRI, Berkeley, January 13, 2012

Problem 1: Counting minimum (s,t)-cuts

Input: a positively weighted (directed) planar graph $G=(V,E)$ and two vertices s,t

Output: the number of minimum (s,t)-cuts of G

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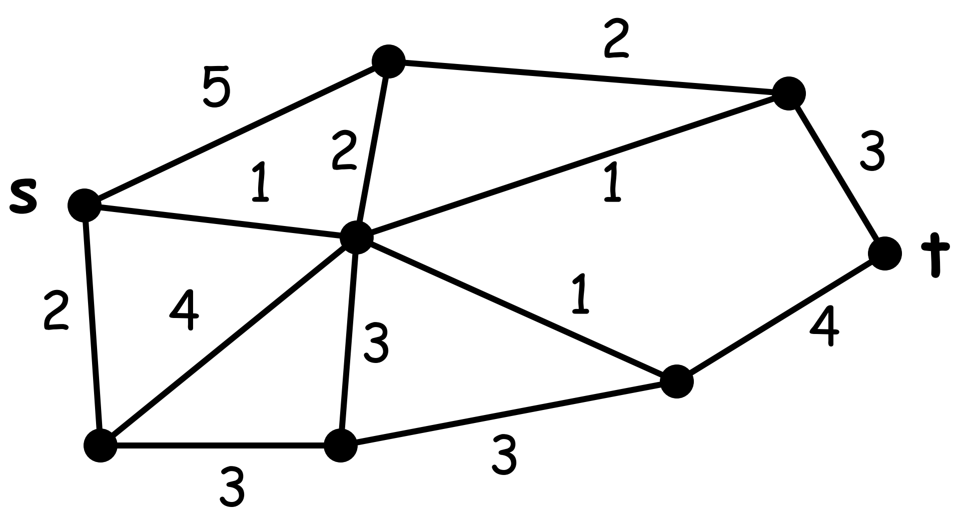

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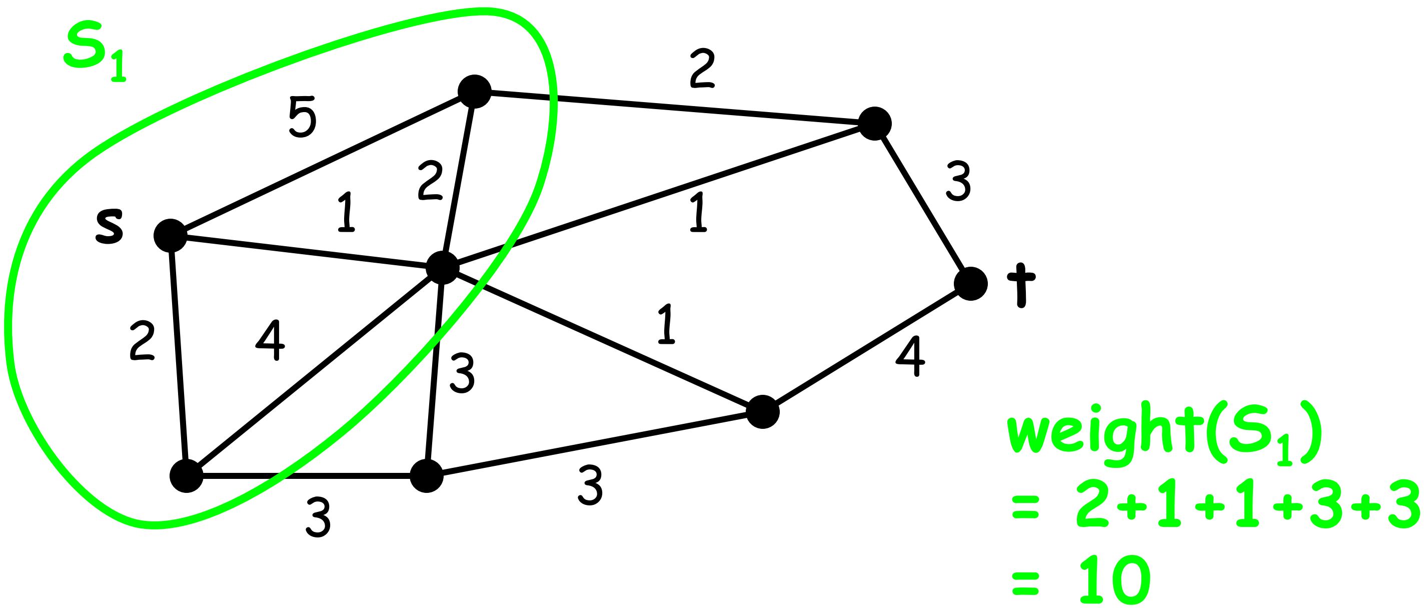

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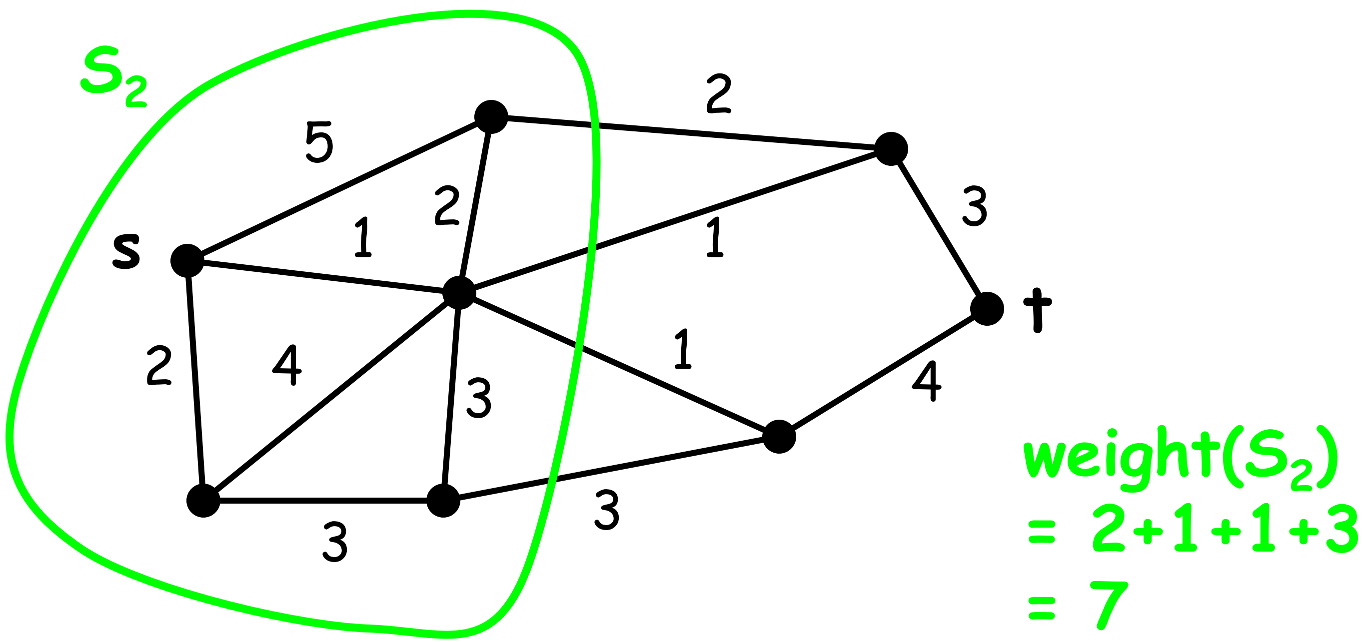

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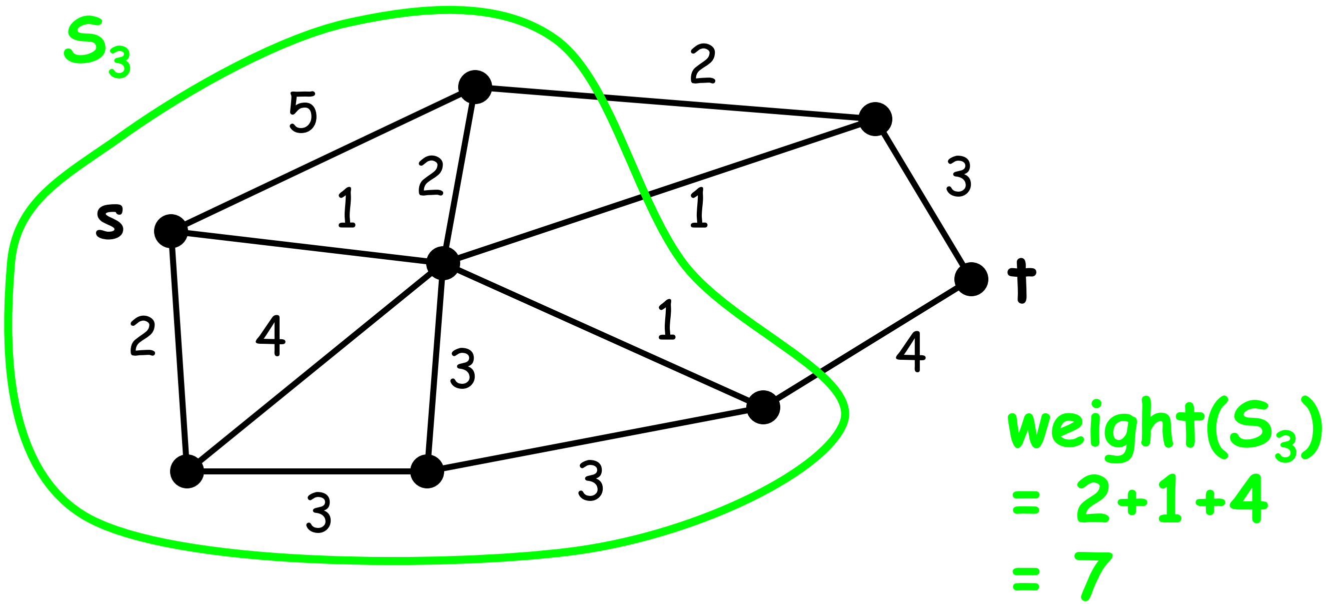

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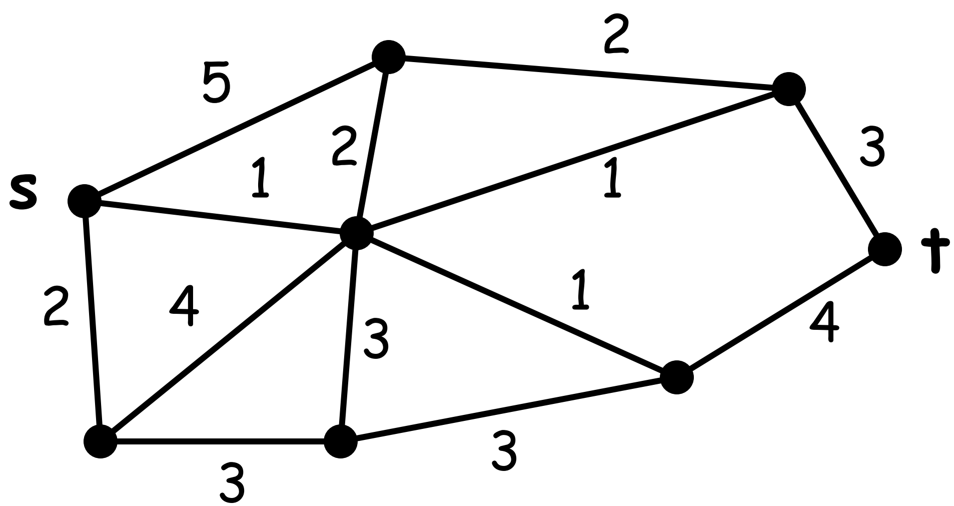

Min (s,t)-cut weight: 7

Number of min (s,t)-cuts: 5

Motivation & related work

Earliest cut-counting works: **network reliability problems**

- Min-cuts / Disconnecting two vertices:
  - number of min (s,t)-cuts useful in estimating the probability of disconnecting the network, e.g., [Colbourn '05]
  - efficient poly-time counting in (unweighted) planar (multi-) graphs when s,t are on the same face [Ball and Provan '83]

Other, recent, motivation: **computer vision**

- e.g.: [Boykov & Veksler '06], [Boykov, Veksler, & Zabih '01], [Vicente, Kolmogorov, & Rother '08], [Zeng, Samaras, Chen, & Peng '08]

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- the simplest case: image segmentation where image represented by a planar graph

- user selects two points, the graph cut represents the segmentation

- currently in use only min-cut algorithms (optimization version), using an arbitrary min-cut

http://path.upmc.edu/cases/case123.html

- many advantages of counting (and the related sampling) versions, e.g.:
    - statistical tests
    - user can choose from several cuts
    - can be used to compute the partition function that can be used to estimate model parameters

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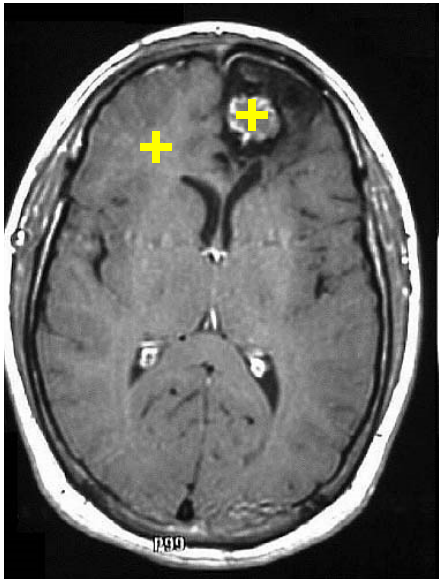

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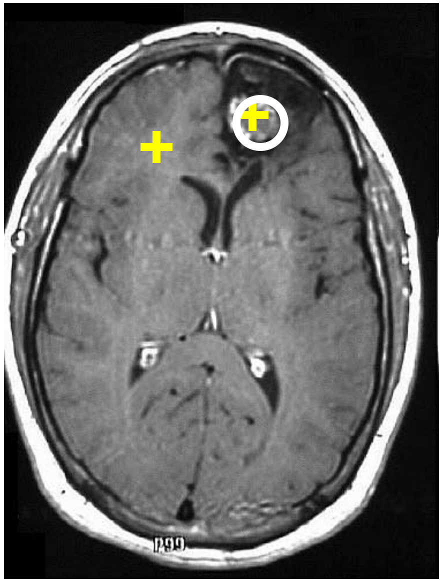

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Problem 2: Single-source-multi-sink cuts

Input: a positively weighted planar graph $G=(V,E)$,
a source vertex s and a set of sink vertices T ($s \notin T$)

Output: the number of contiguous minimum (s,T)-cuts

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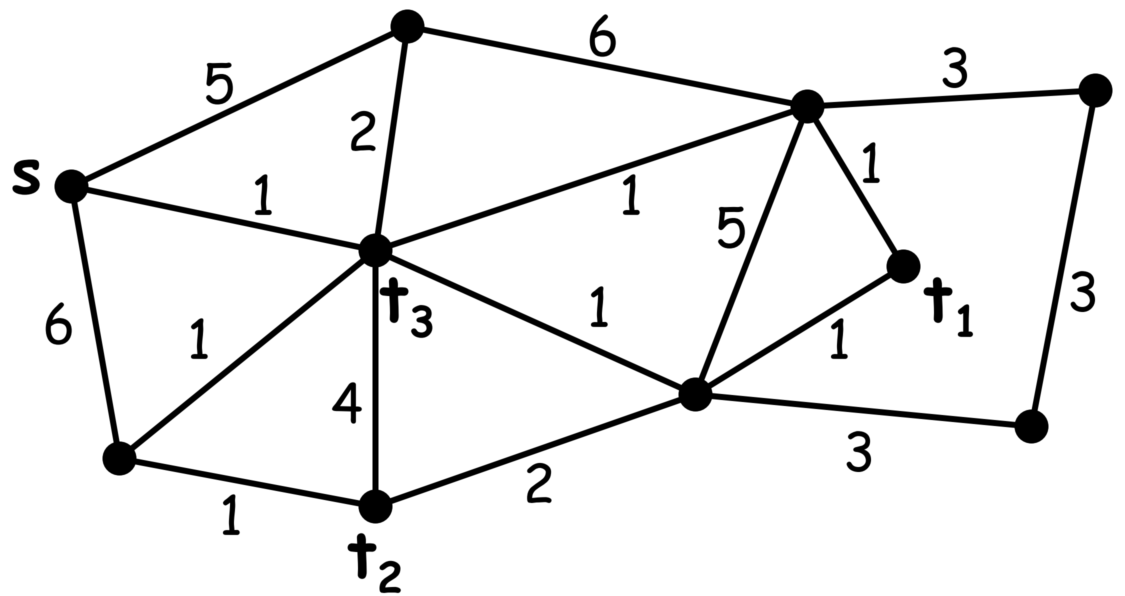

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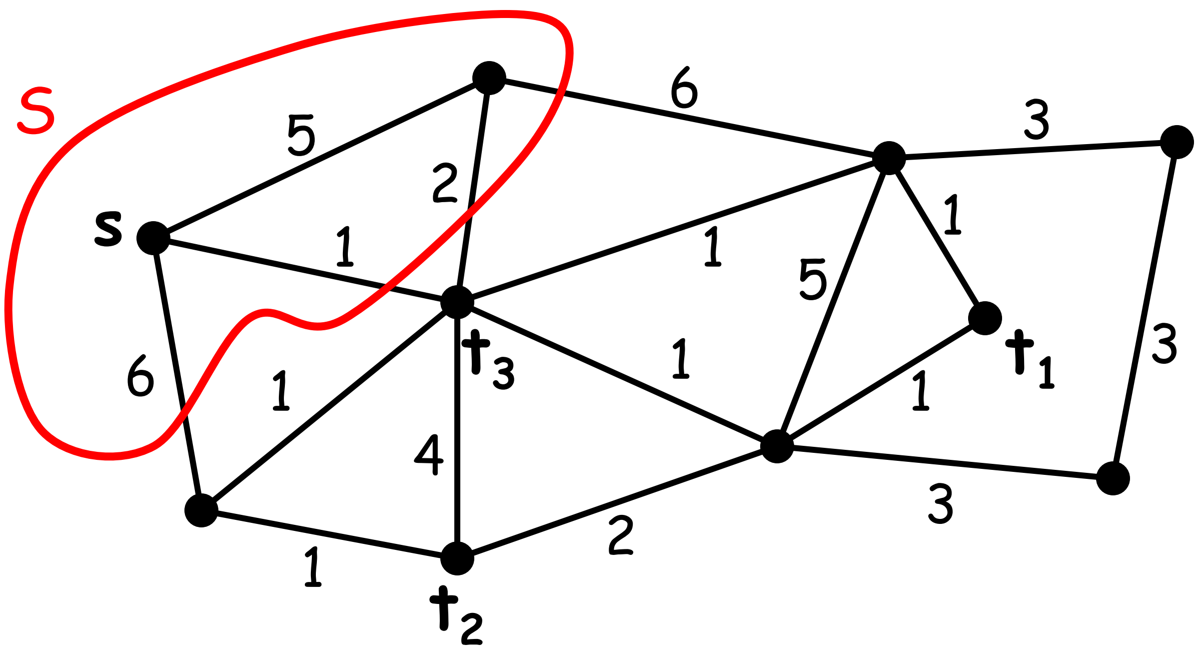

Weight(S) = 6+1+2+6 = 15

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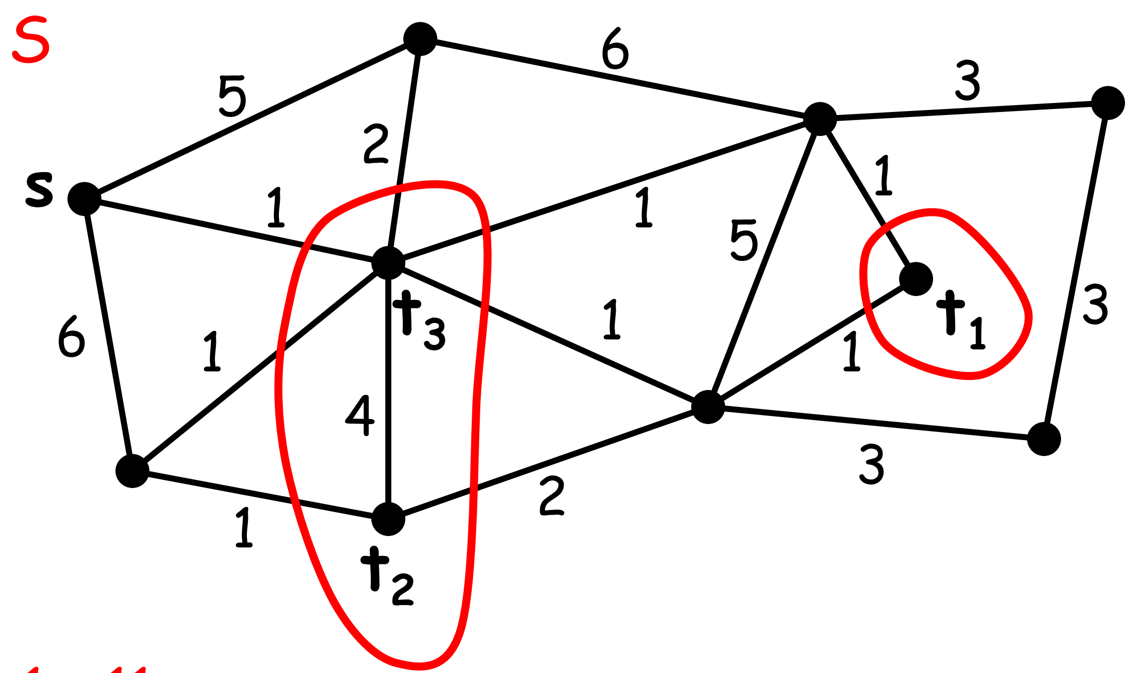

Weight(S) =
1+1+1+2+1+1+2+1+1 = 11

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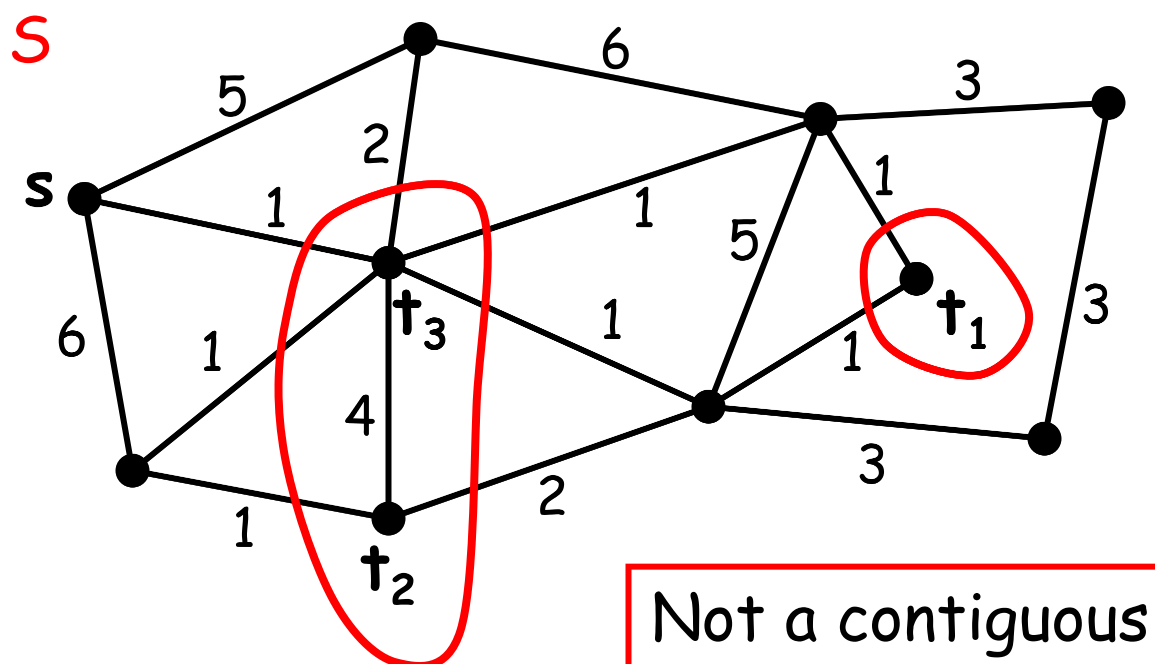

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Not a contiguous cut !

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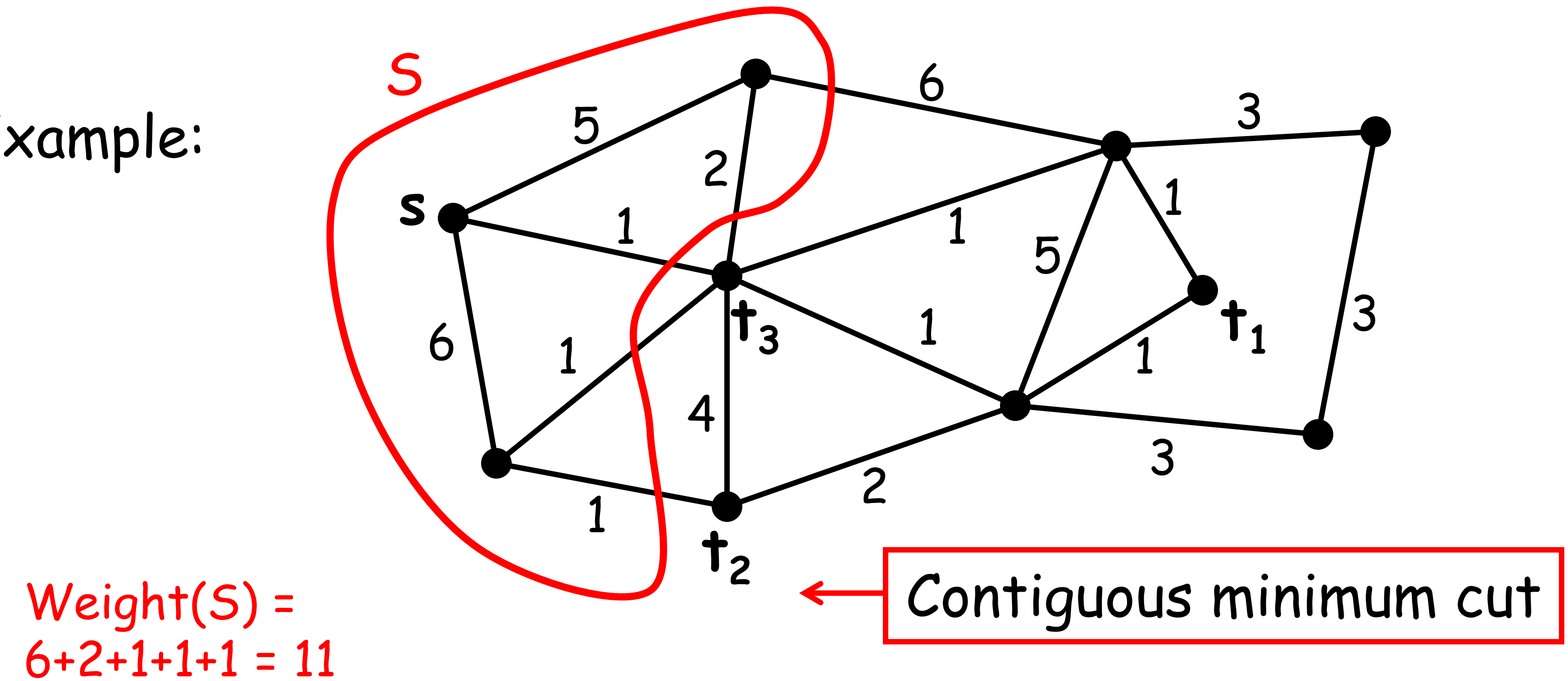
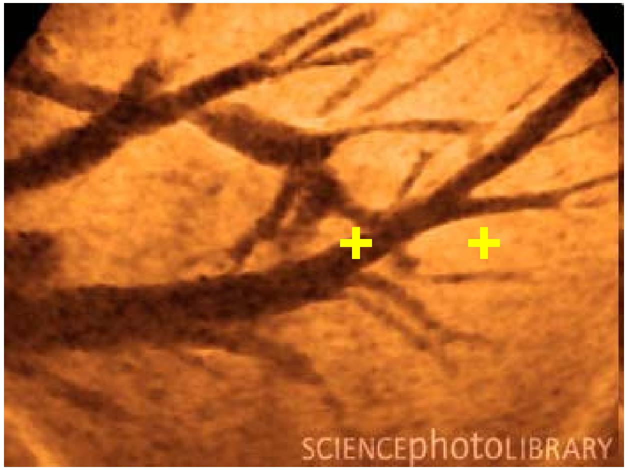


Motivation & related work – con't

Back to image segmentation:

- what about thin objects ?

    - user selects several points to guide the segmentation algorithm



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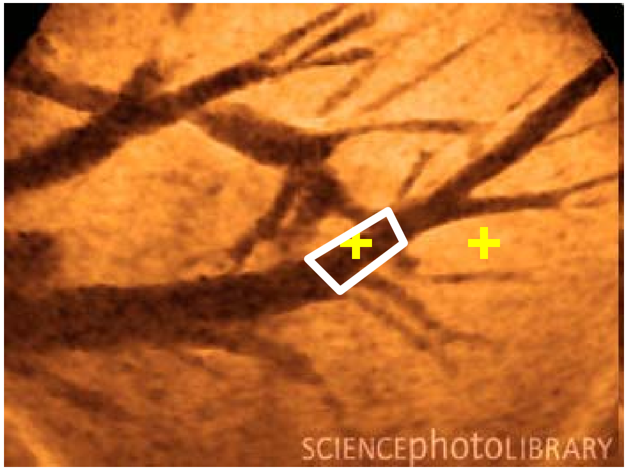

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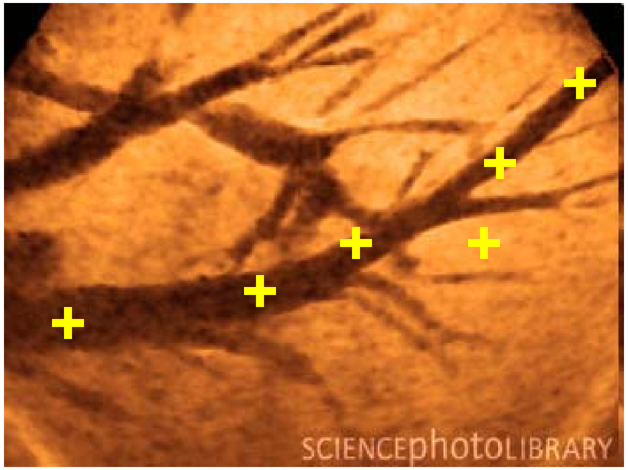

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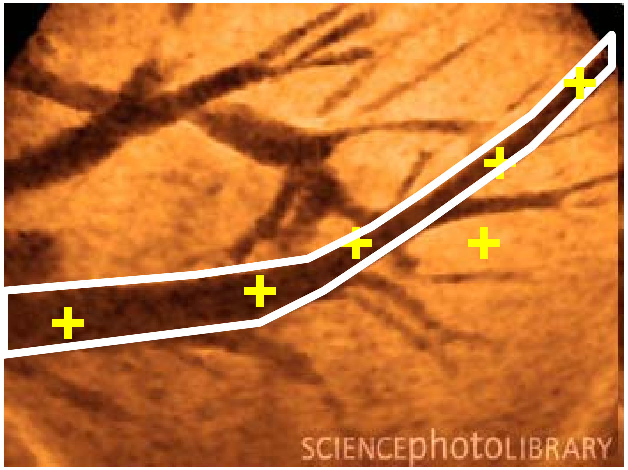

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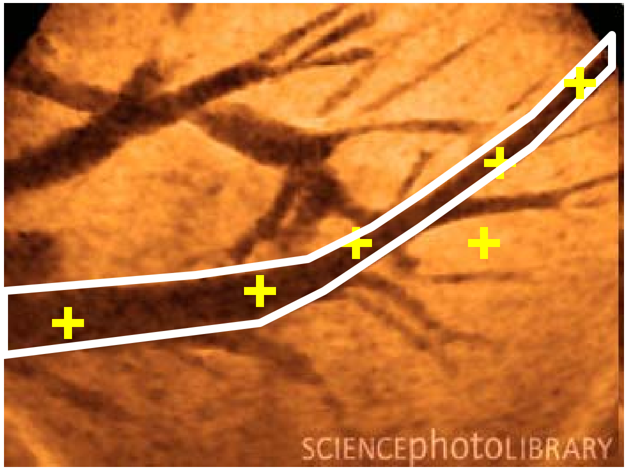
    - contiguity requirements [connectivity priors]



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Motivation & related work - con't

Recent progress in flow/cut algorithms for planar graphs [optimization problems, not counting/sampling]:

- [Borradaile & Klein '09]: O(n log n) single-source-single-sink acyclic max.flow

- [Borradaile, Sankowski, & Wulff-Nilsen '10]: fast computation of a minimum single-source-single-sink cut, for a sequence of given source-sink pairs

- [Italiano, Nussbaum, Sankowski, & Wulff-Nilsen '11]: undirected planar graphs, algos below O(n log n) time barrier

- [Borradaile, Klein, Mozes, Nussbaum, & Wulff-Nilsen '11]: $O(n \log^3 n)$ multi-source-multi-sinks max. flow

Motivation & related work - con't

Hardness results in general graphs:

- [Ball & Provan '83]: counting minimum (s,t)-cuts is #P-complete

- [Dyer, Goldberg, Greenhill, Jerrum '03]: AP-reduction between counting independent sets in bipartite graphs and counting upper sets of a poset

Our contributions

Thm 1: An $O(dn + n \log n)$ algorithm counting all minimum (s,t)-cuts in weighted planar graphs, where n is the number of vertices, and d is the distance from s to t in the unweighted graph.

Thm 2: An $O(n^3)$ algorithm counting all contiguous minimum single-source multi-sink cuts in weighted planar graphs.

[Can be used to find a contiguous minimum (s,T)-cut.]

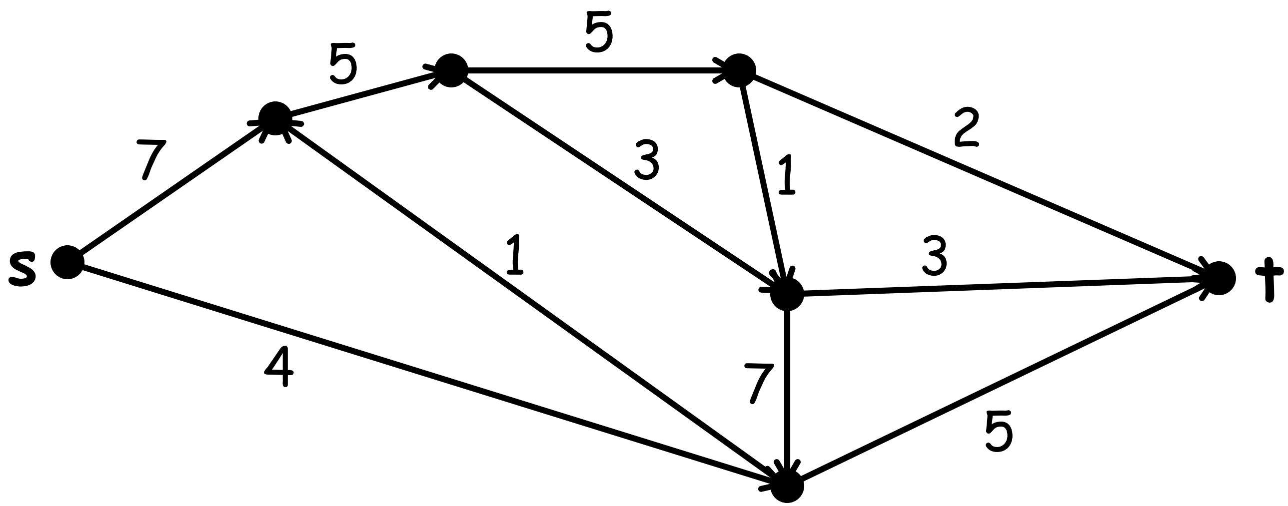
In both cases:

After this preprocessing time, we can produce a uniformly random minimum (s,t)-cut, resp. contiguous minimum (s,T)-cut, in linear time.

Review of network flows

Flow network:

- a directed graph with positive capacities on the edges, and
- two vertices s (the source) and t (the sink)



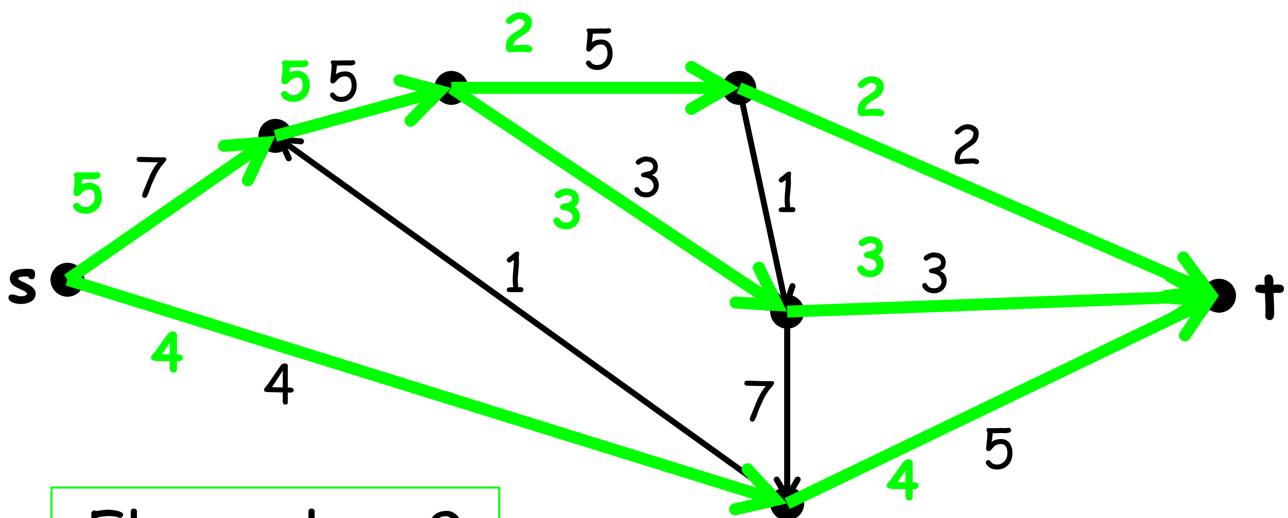
Review of network flows

Flow f: flow amount on every edge satisfying:

- for every edge e: flow amount f(e) $\leq$ capacity c(e), and
- for every vertex v (except s,t):

  flow amount into v = flow amount out of v

- flow value: amount out of s minus amount into s

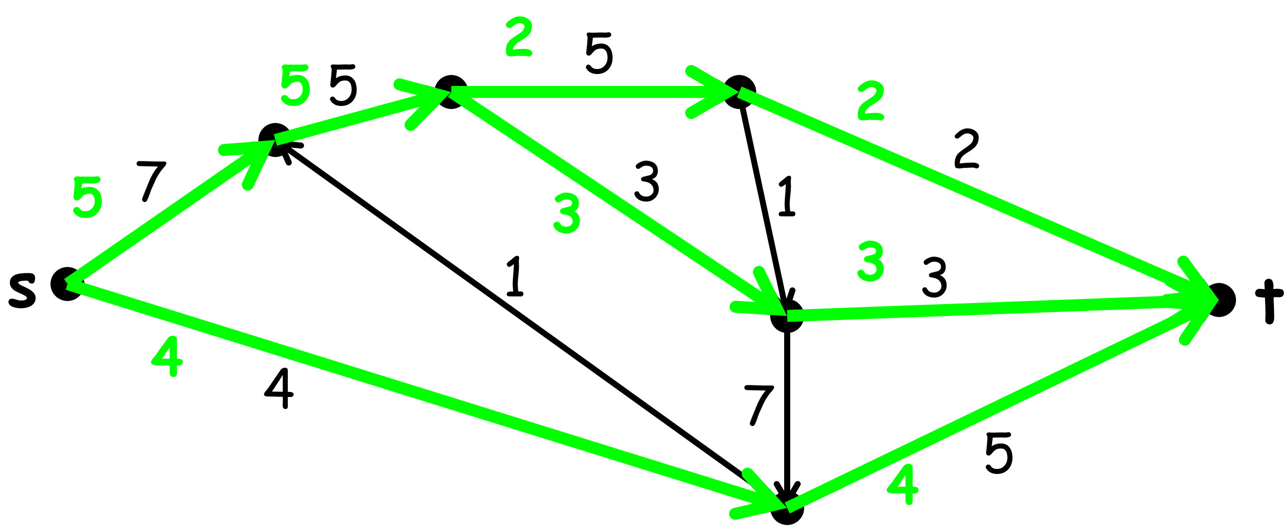


Flow value: 9

Review of network flows

Residual graph of a flow f:

- **forward edges**: weight = capacity – flow
- **backward edges**: weight = flow



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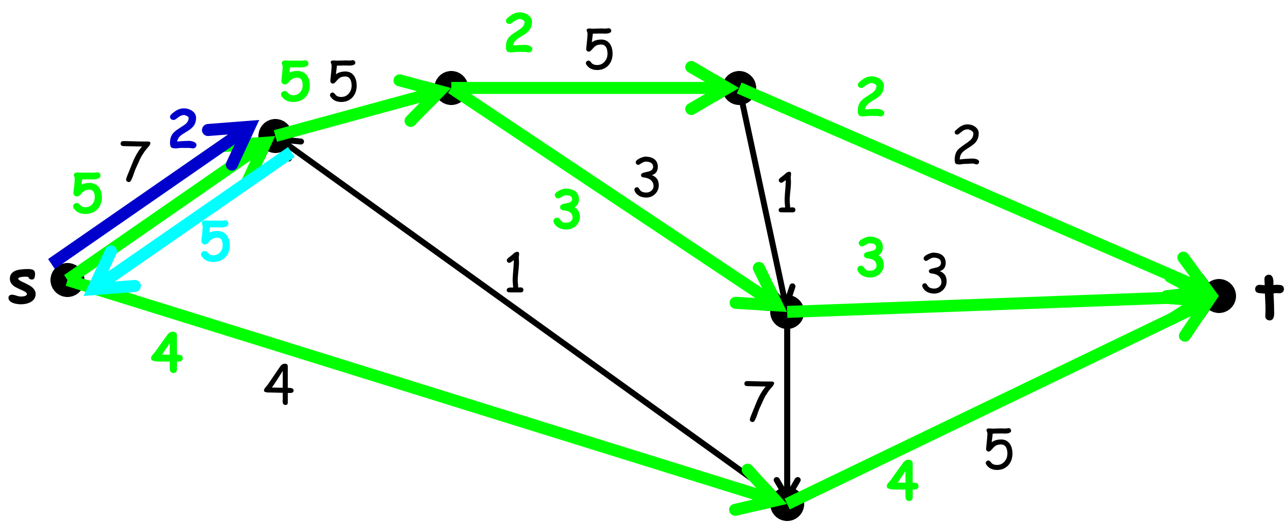

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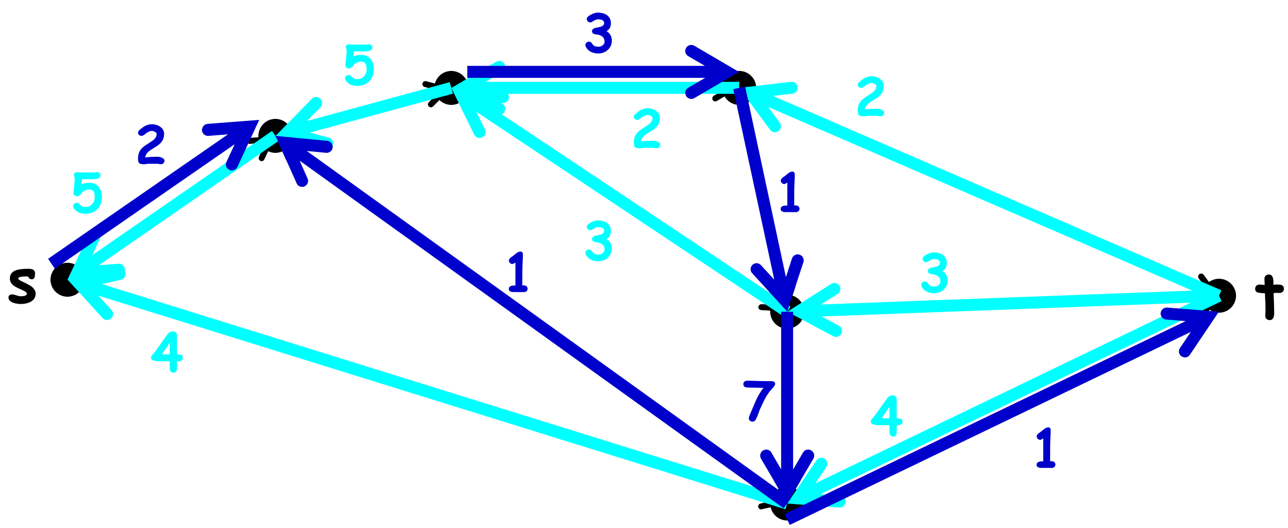
(only edges with positive weight)

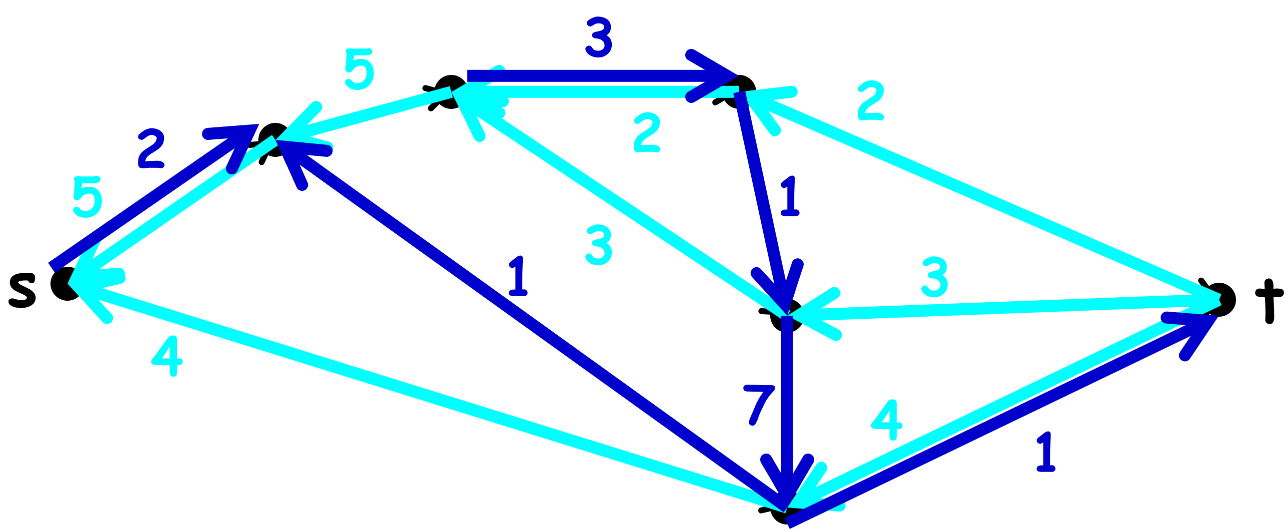


Review of network flows

Ford-Fulkerson Thm:

value of max s-t flow = value of min s-t cut

Note: flow is max iff no s-t path in the residual graph

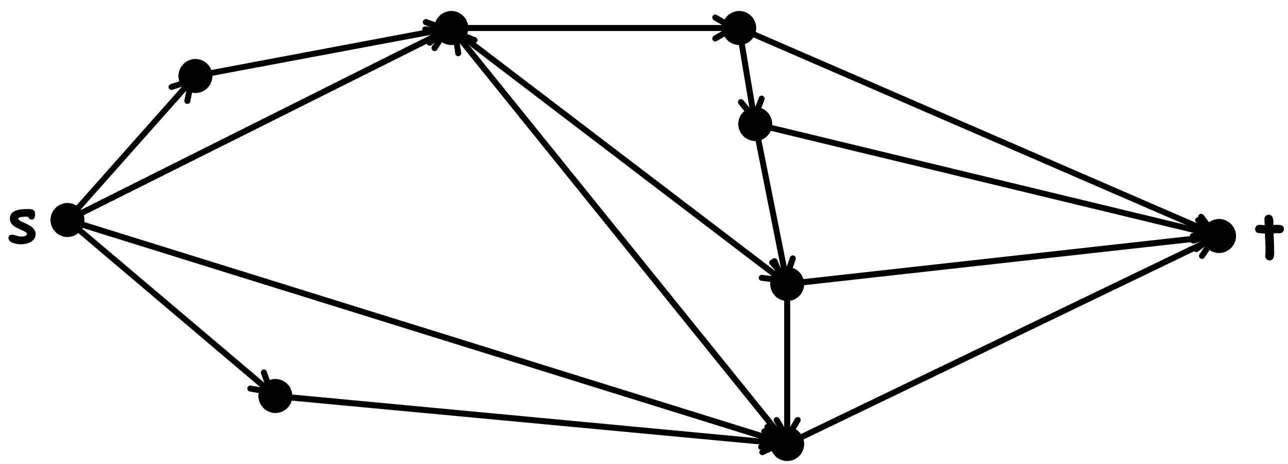


Ball & Provan's reduction

Given an unweighted (multi-)graph and vertices s,t:

1. Find a max flow and construct the residual graph
2. Contract strongly connected components
3. Compute # "forward-cuts" in the DAG

(forward-cuts = upper set / maximal antichains in the poset)



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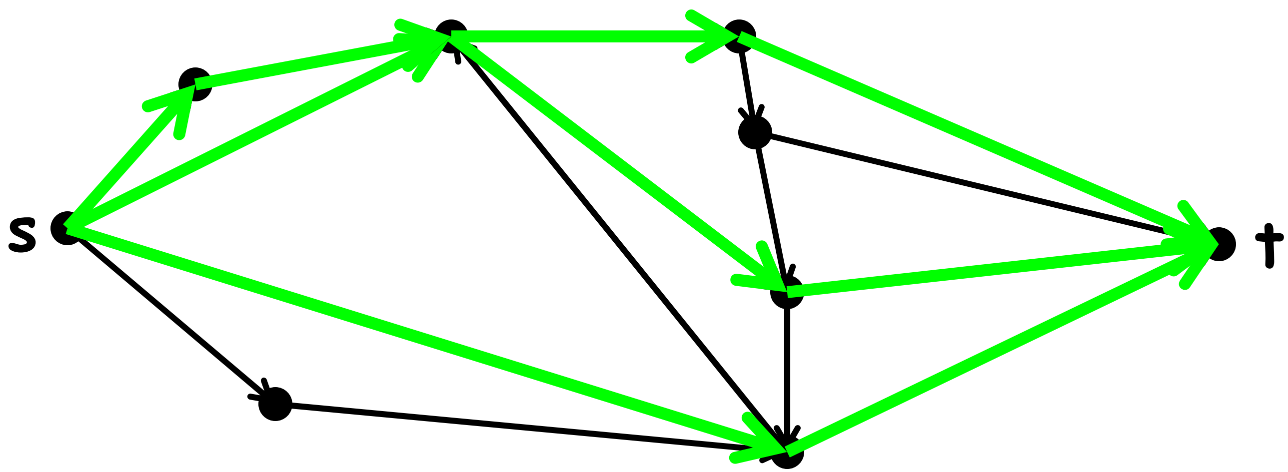

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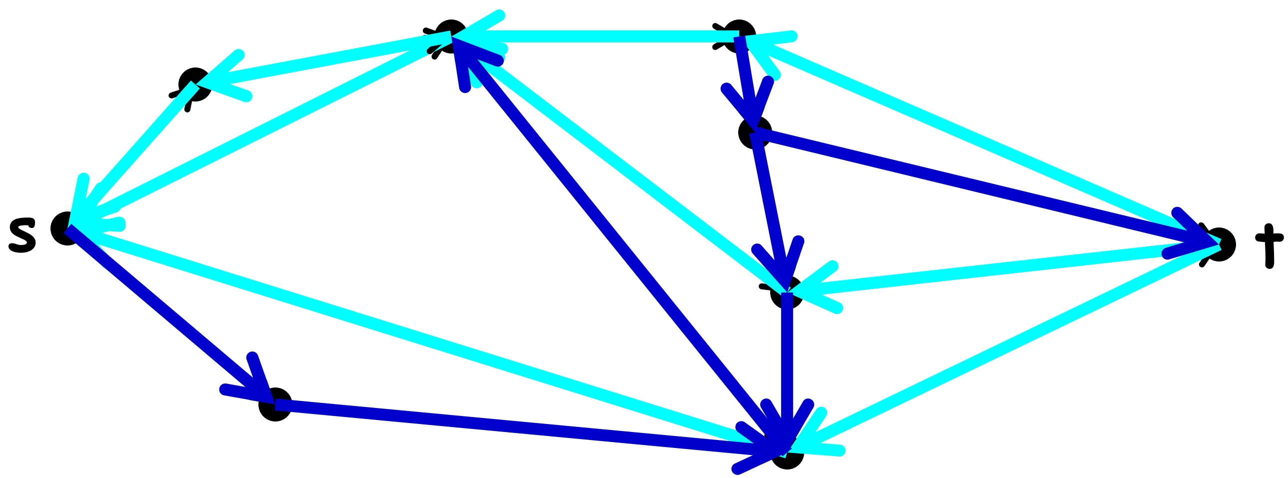

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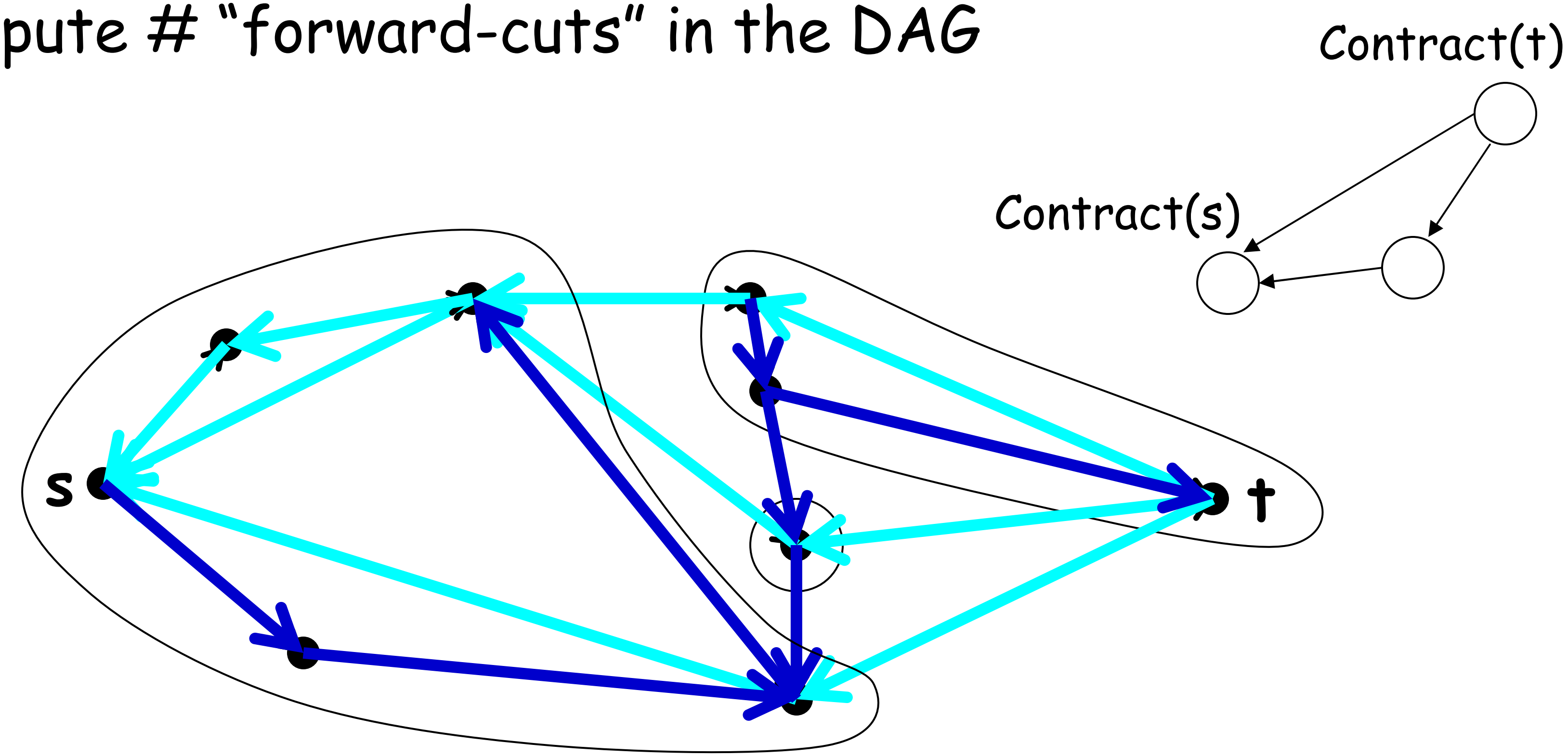

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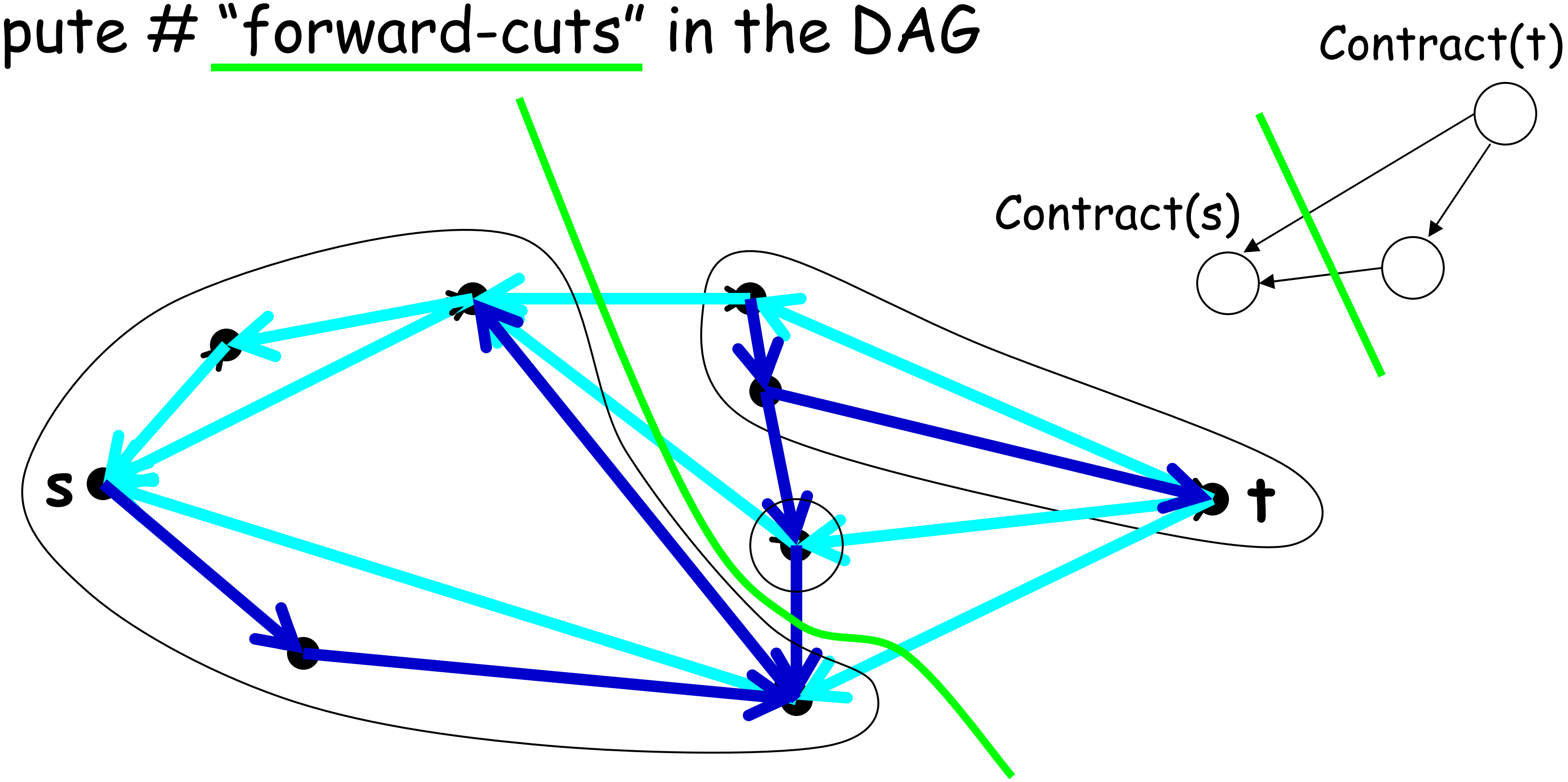

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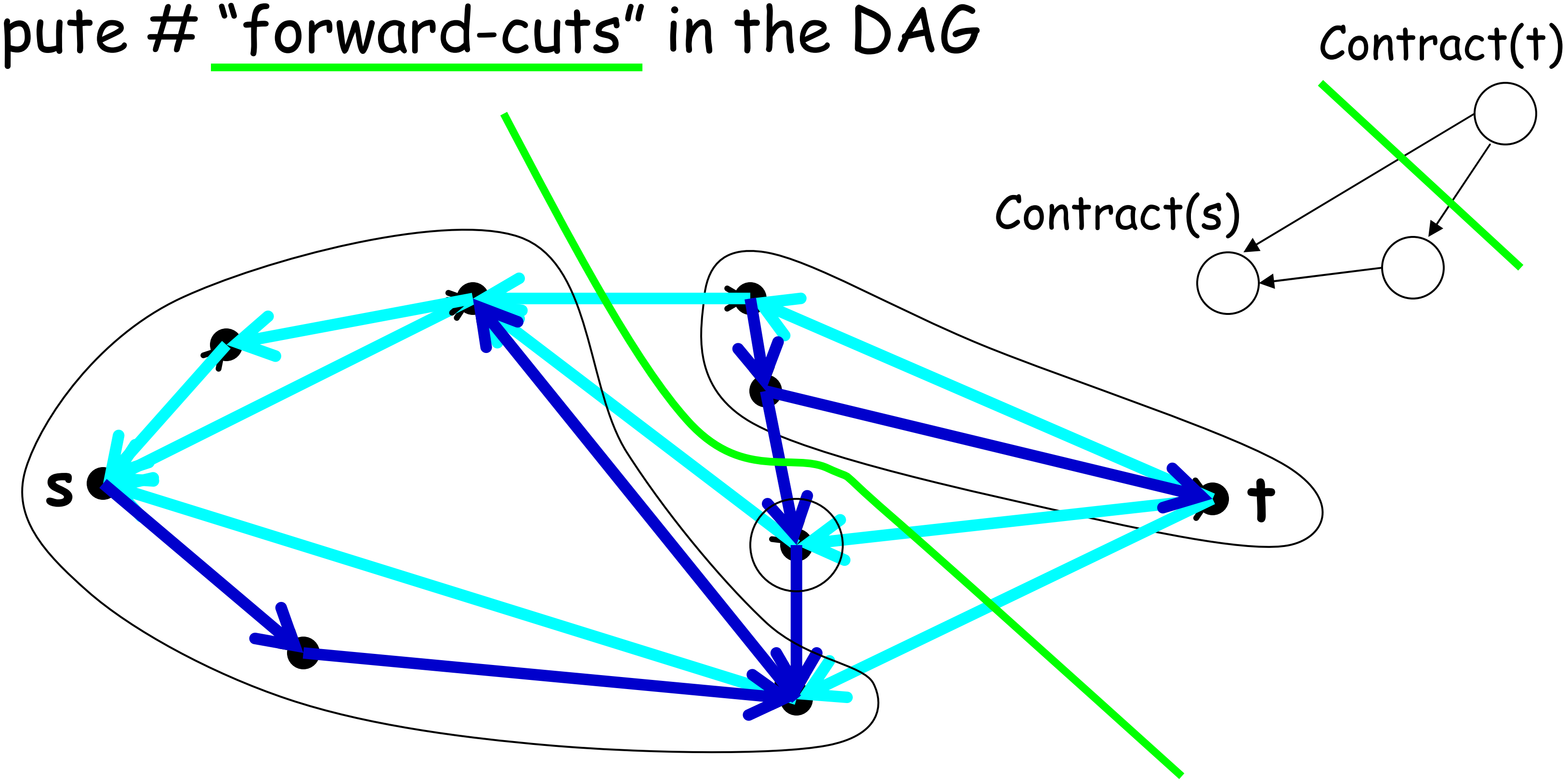

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"Forward-cut:" a set of vertices S such that:

- contains Contract(s) and not Contract(t), and
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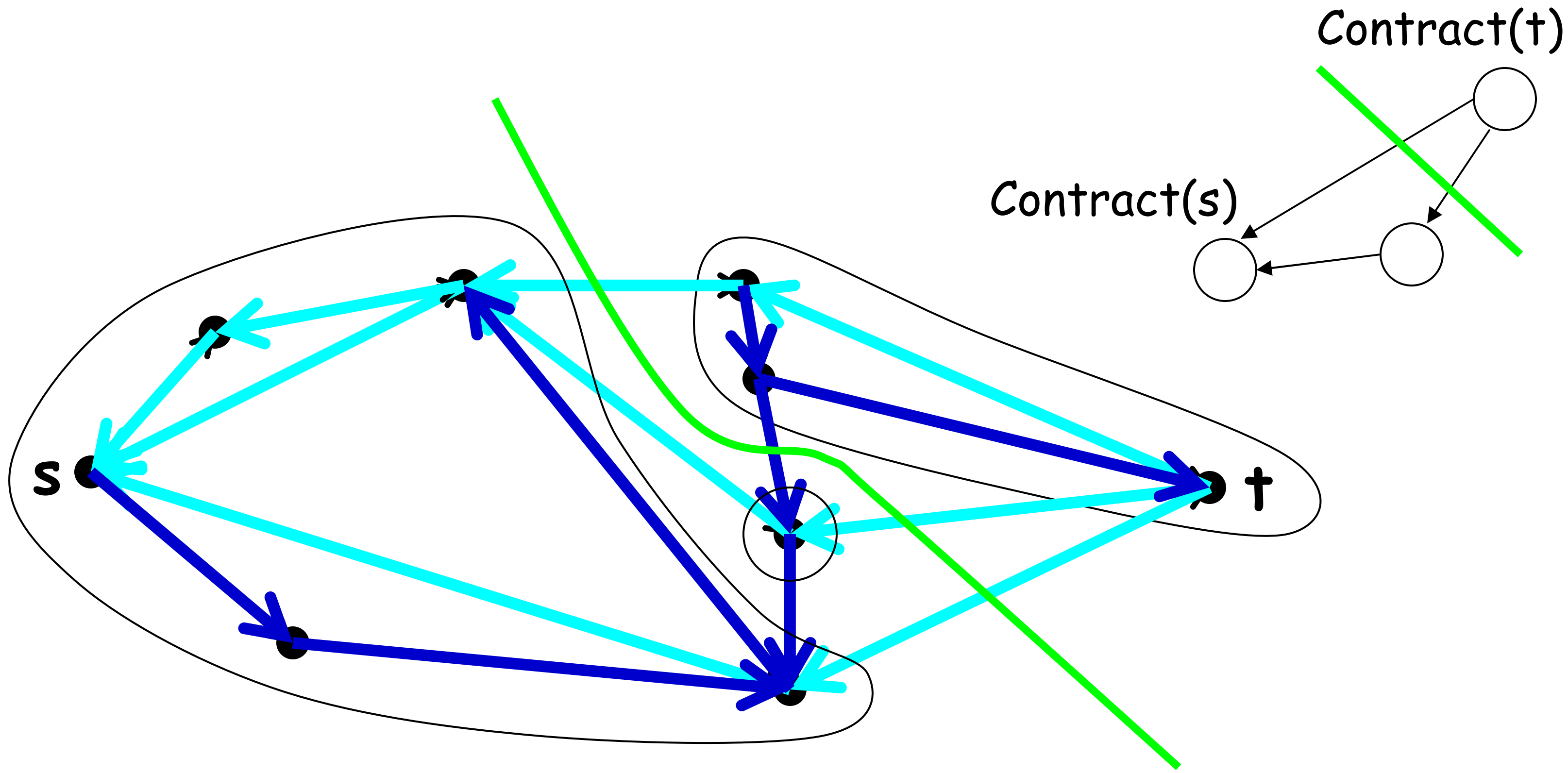

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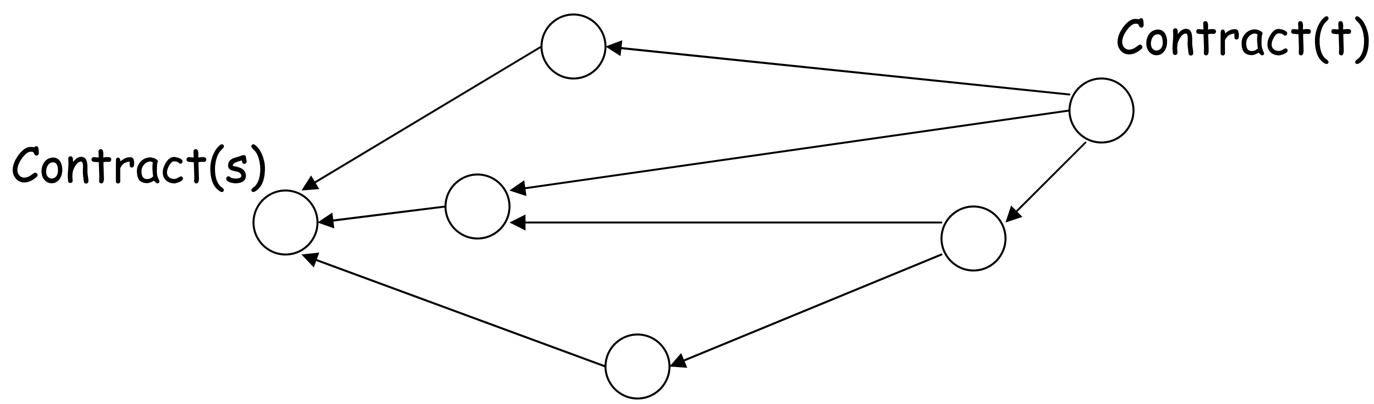

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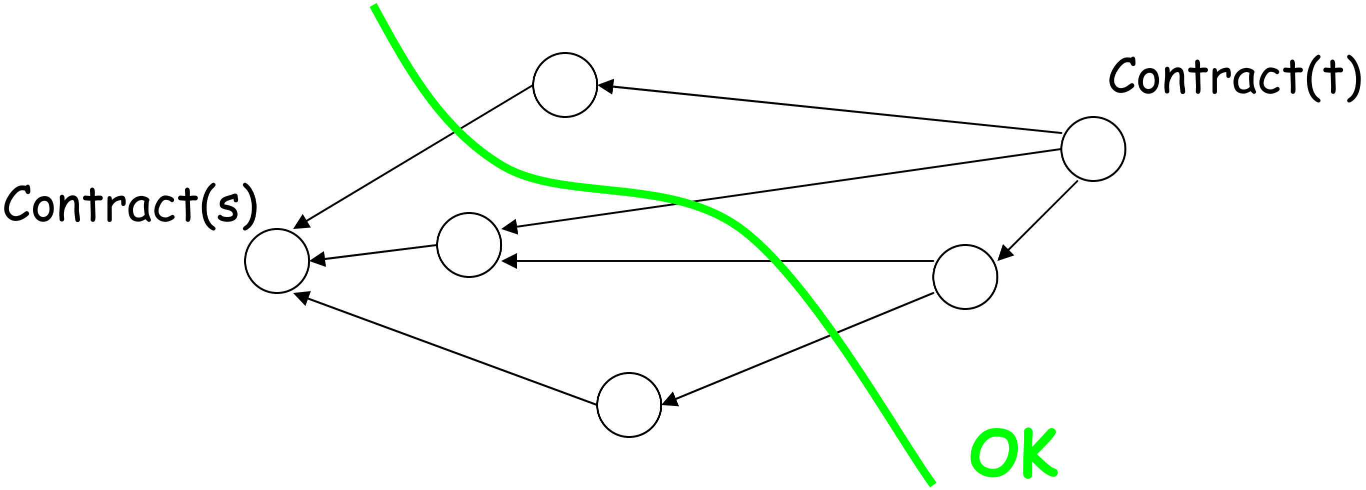

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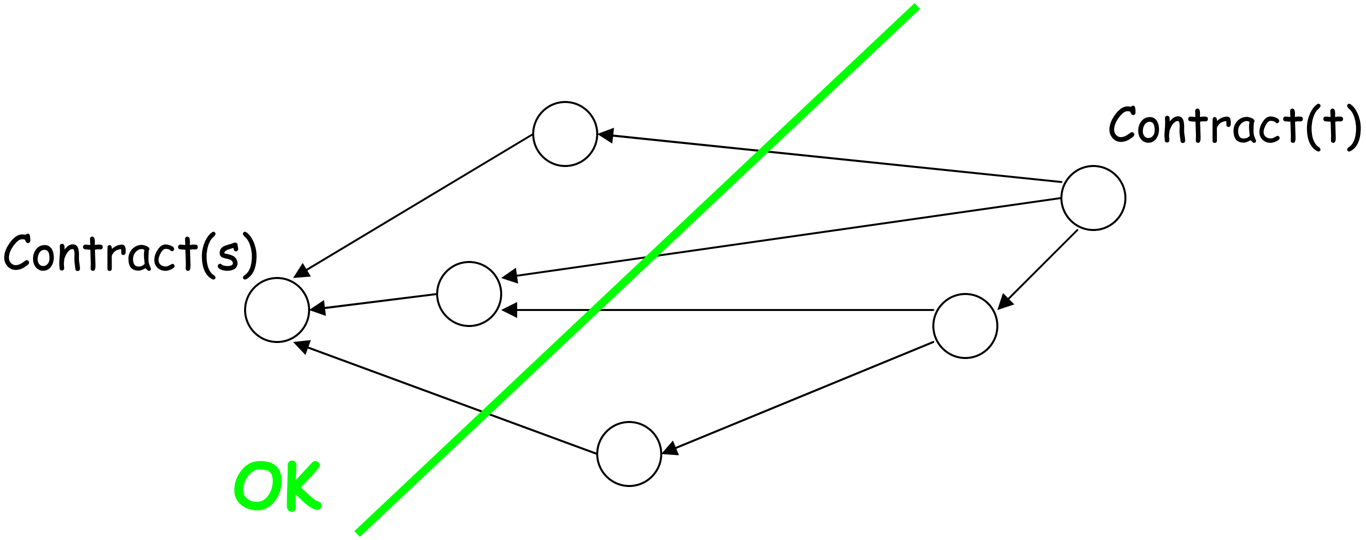

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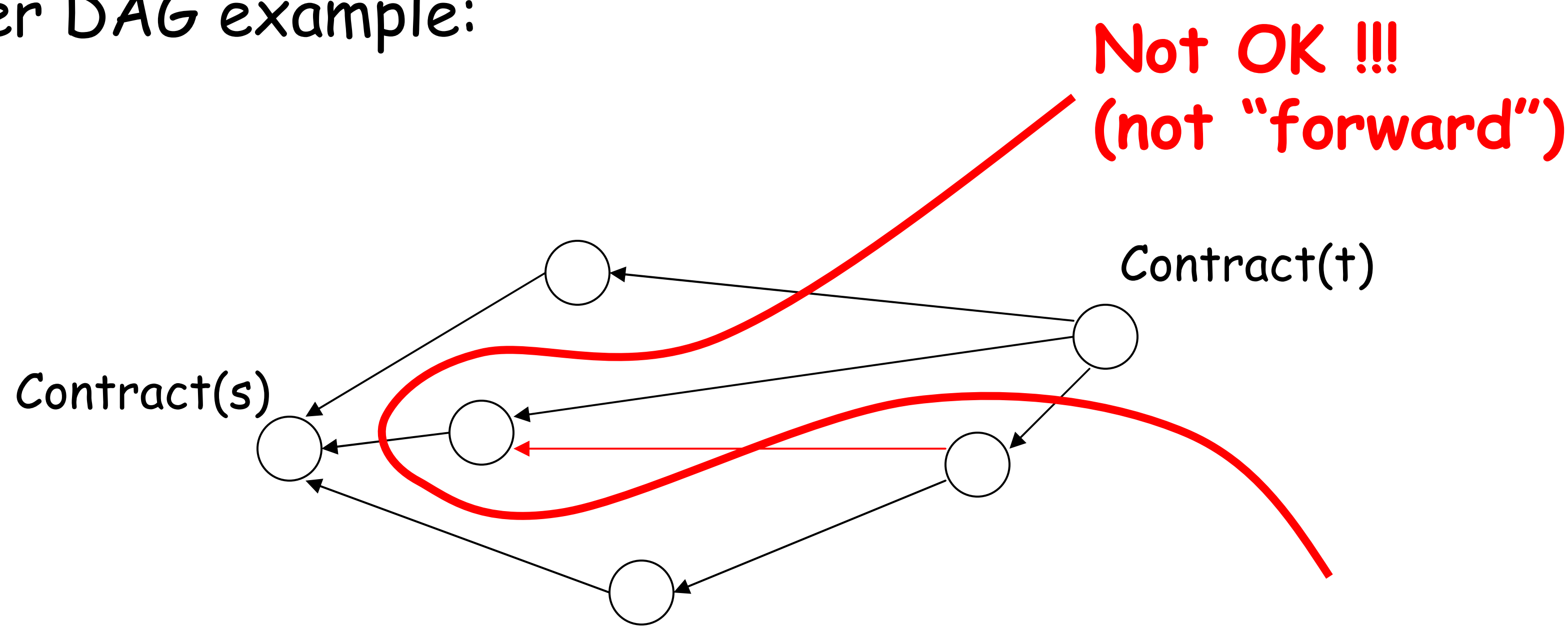

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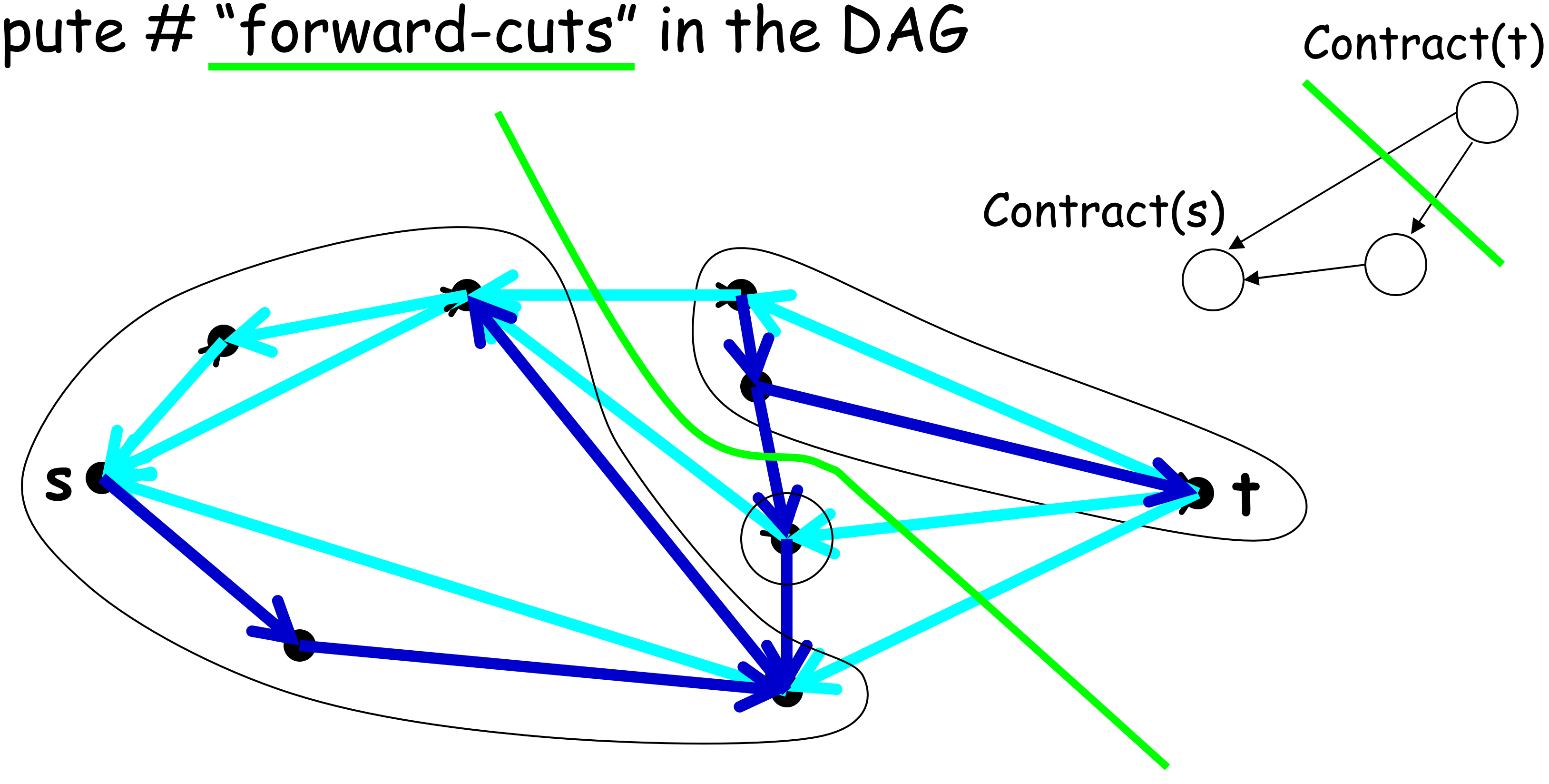

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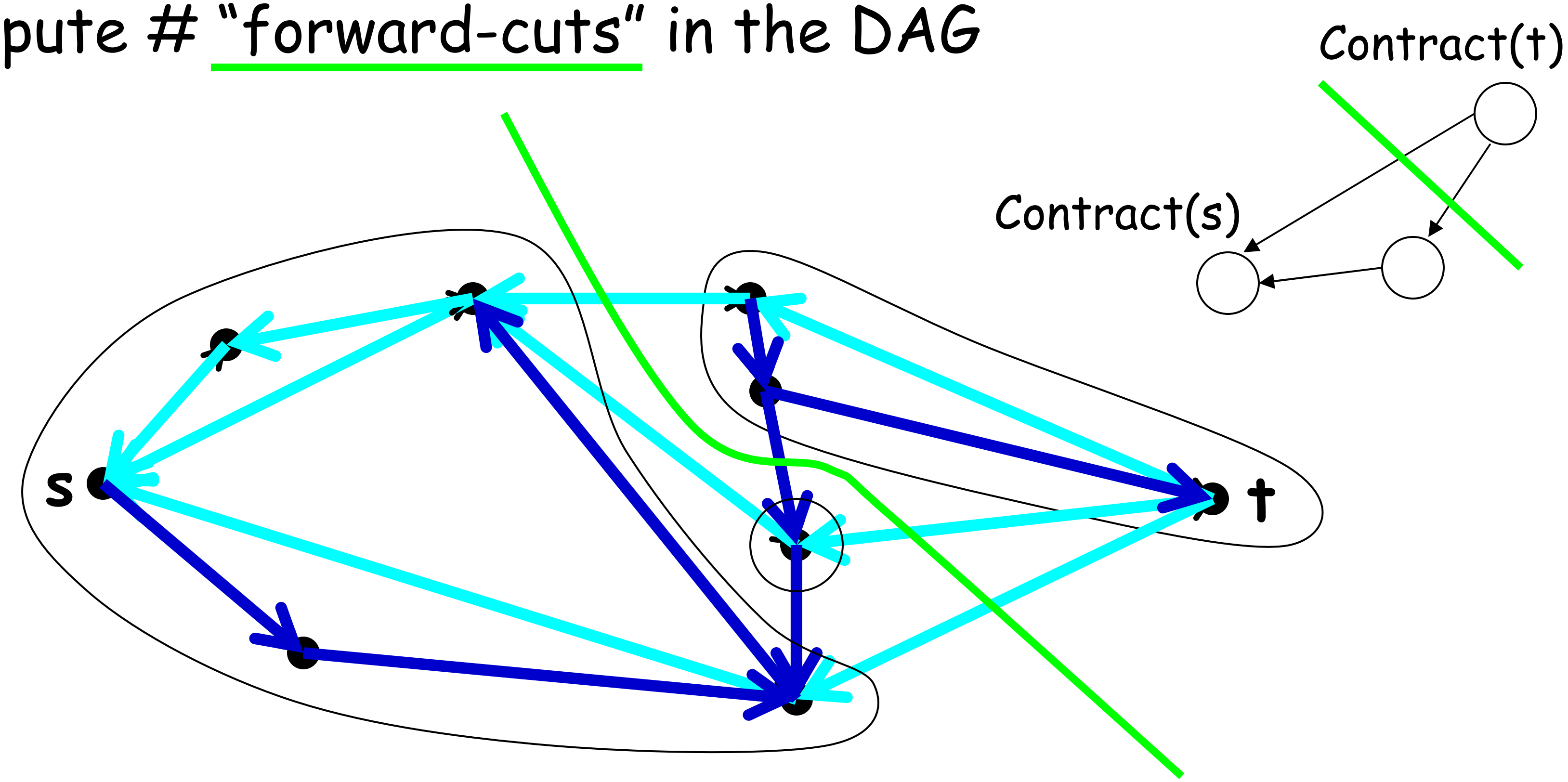
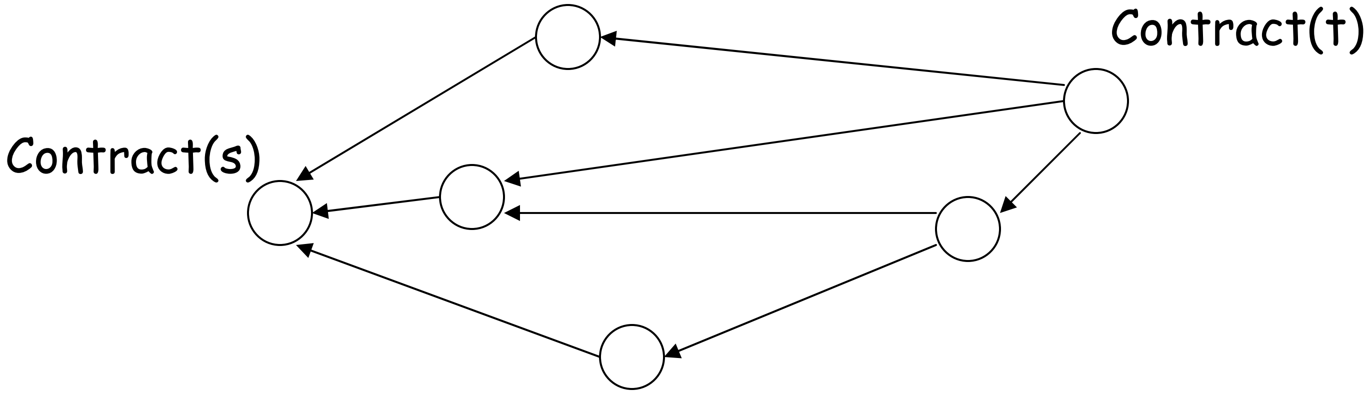


"Forward-cuts" in planar DAGs

Observe: Planar input graph -> planar DAG

Goal: count "forward-cuts" (or maximal antichains)



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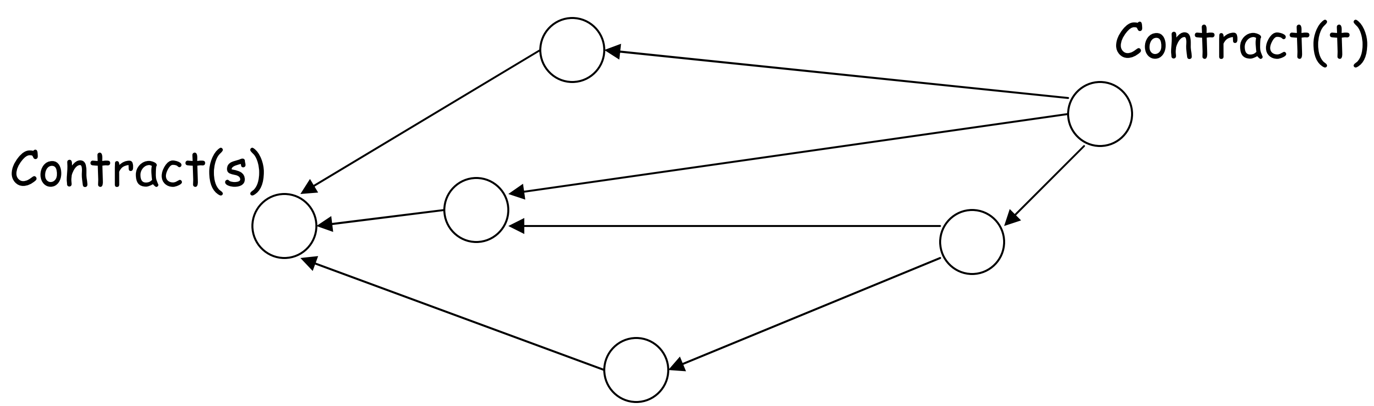
Ball & Provan's algorithm for both s,t on the outer face:

- split the outer face into the "top" and the "bottom" face
- count all "top"-"bottom" paths in the dual graph



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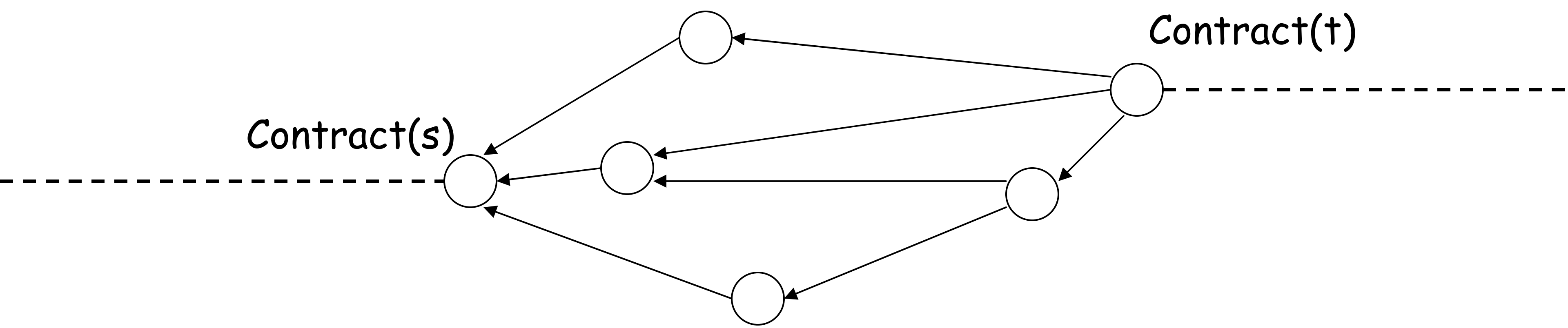

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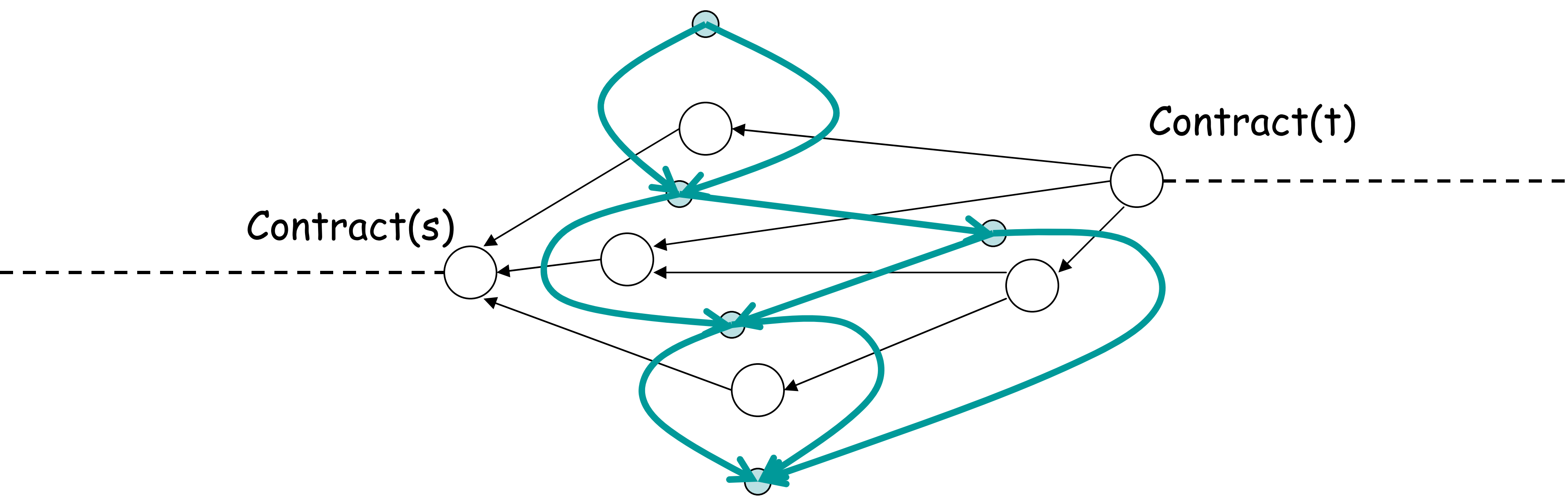

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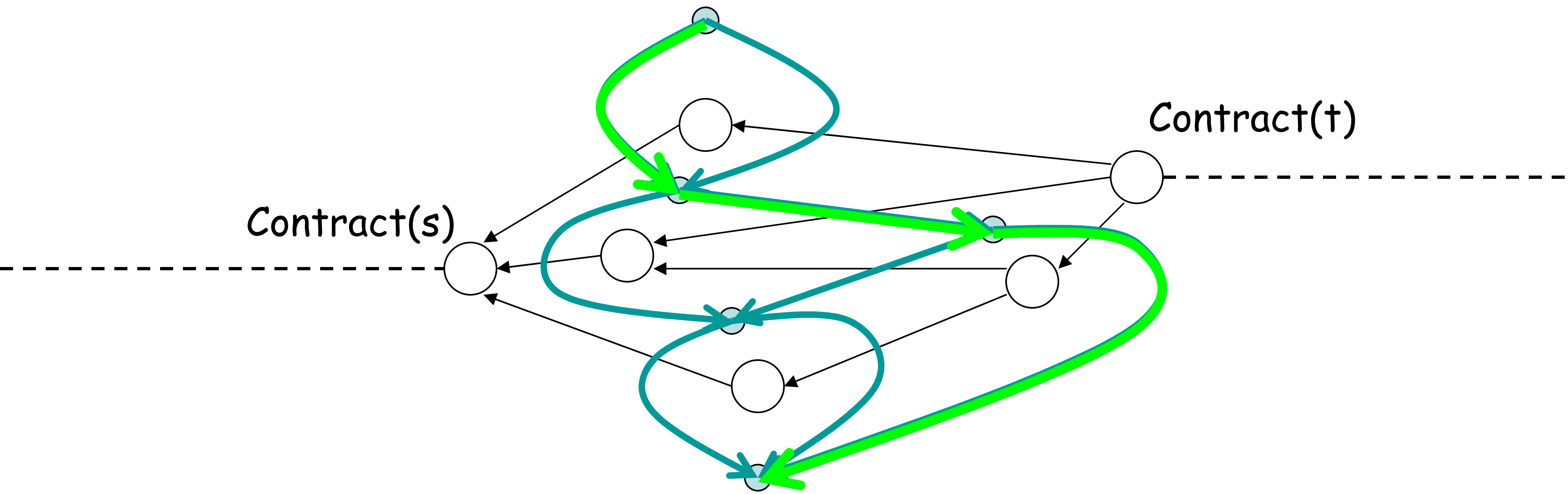

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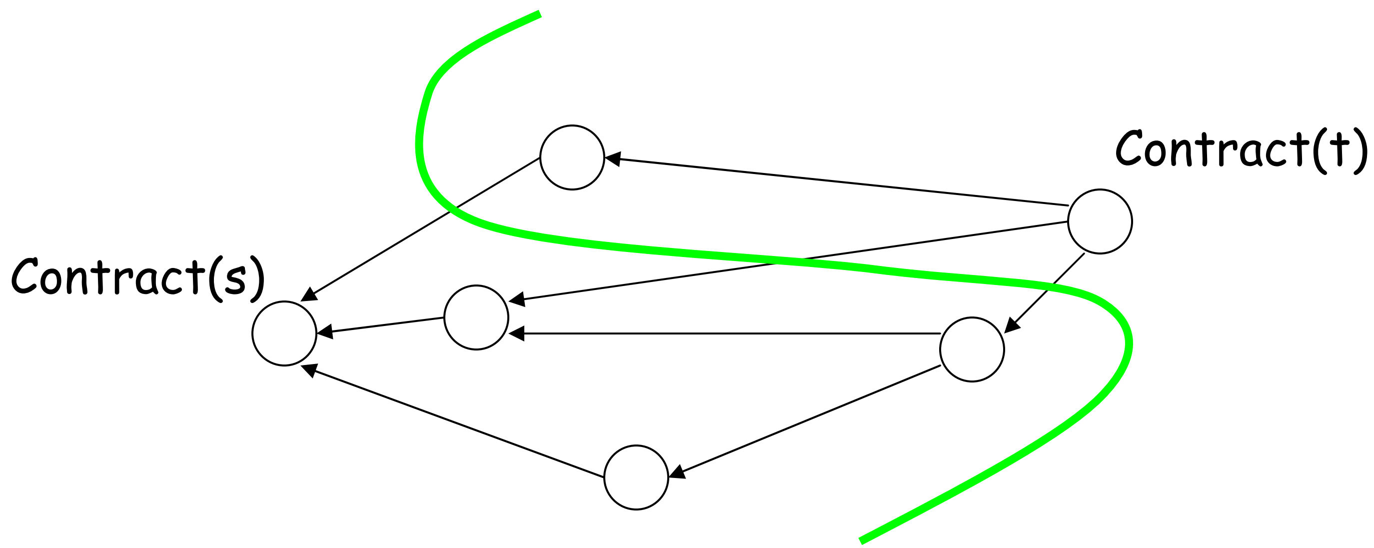

“Forward-cuts” in planar DAGs

Observe: Planar input graph -> planar DAG

Goal: count “forward-cuts” (or maximal antichains)

Ball & Provan’s algorithm for both s,t on the outer face:

- split the outer face into the “top” and the “bottom” face
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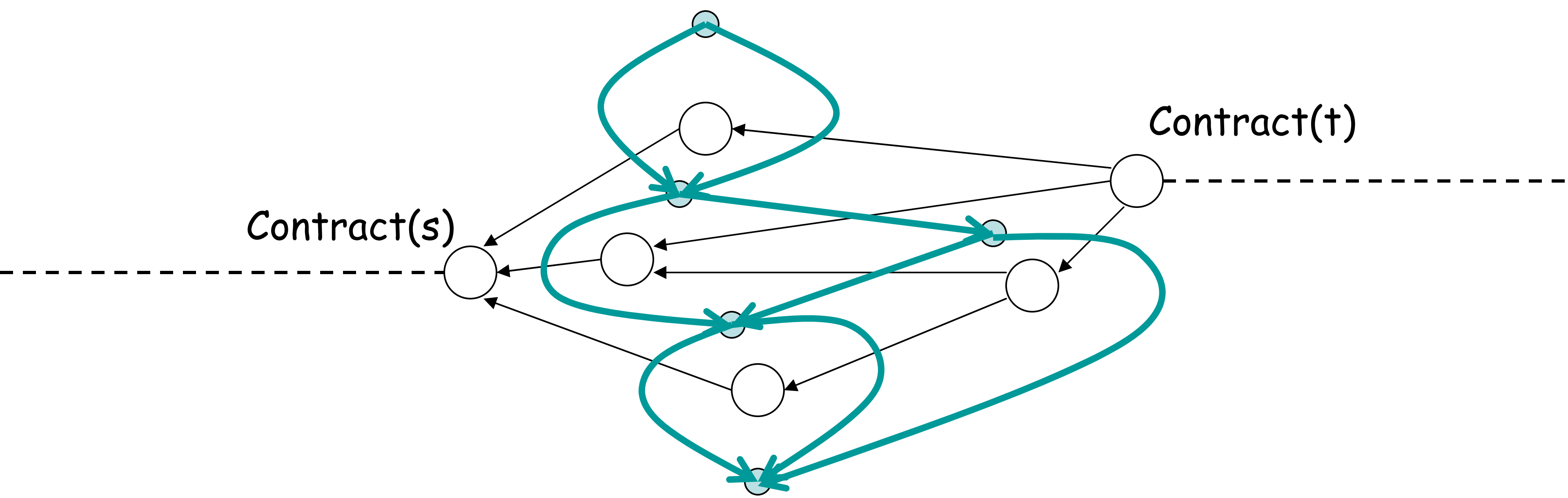

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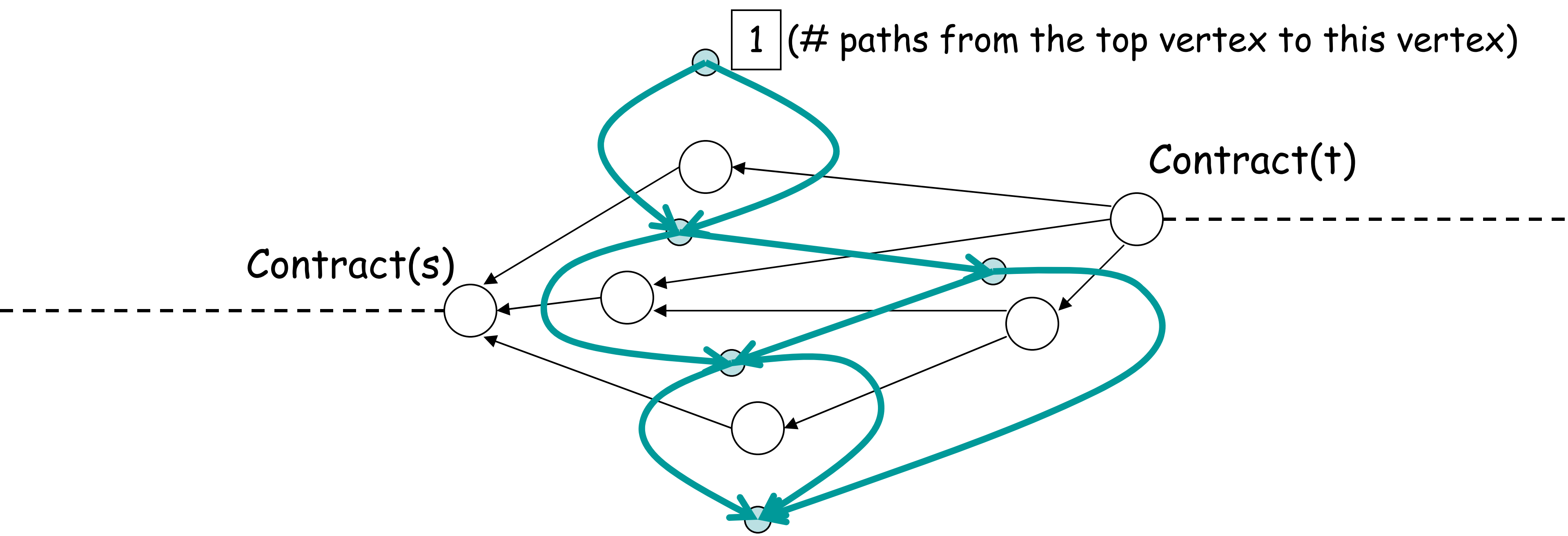

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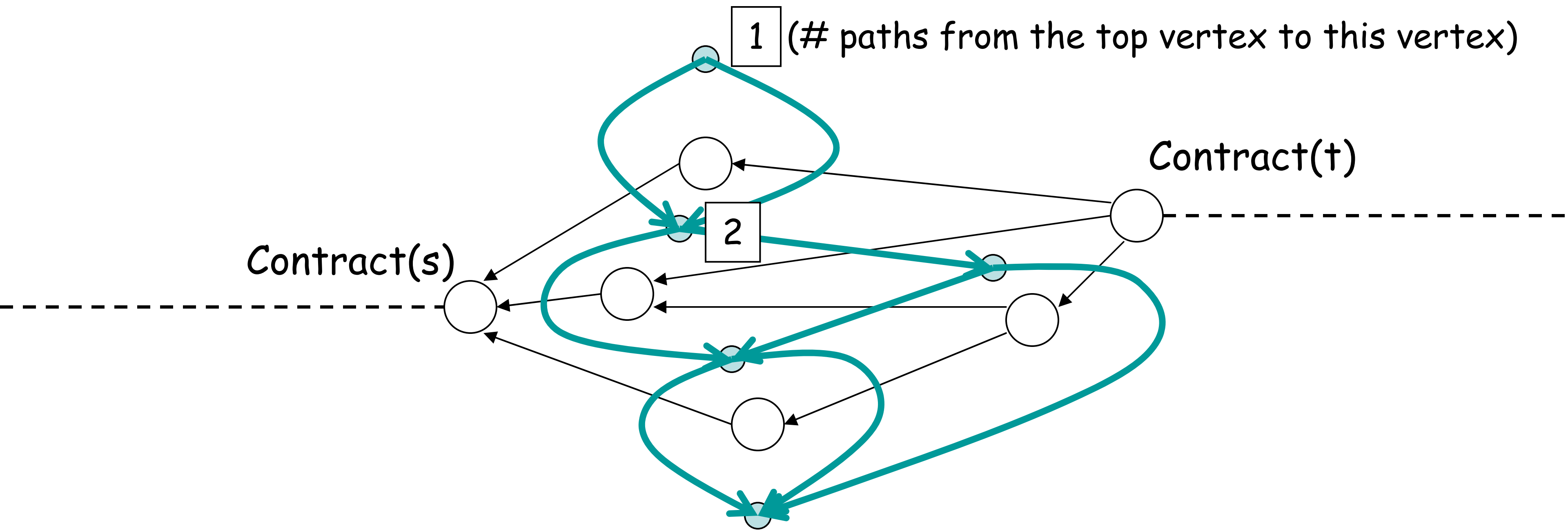

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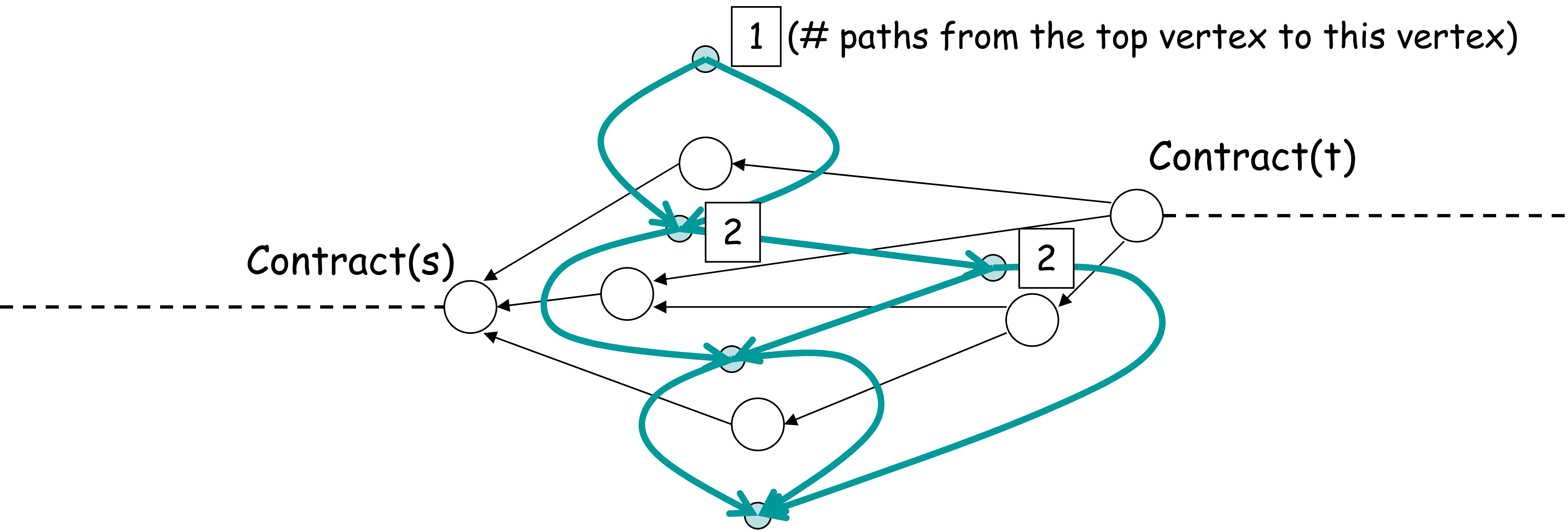

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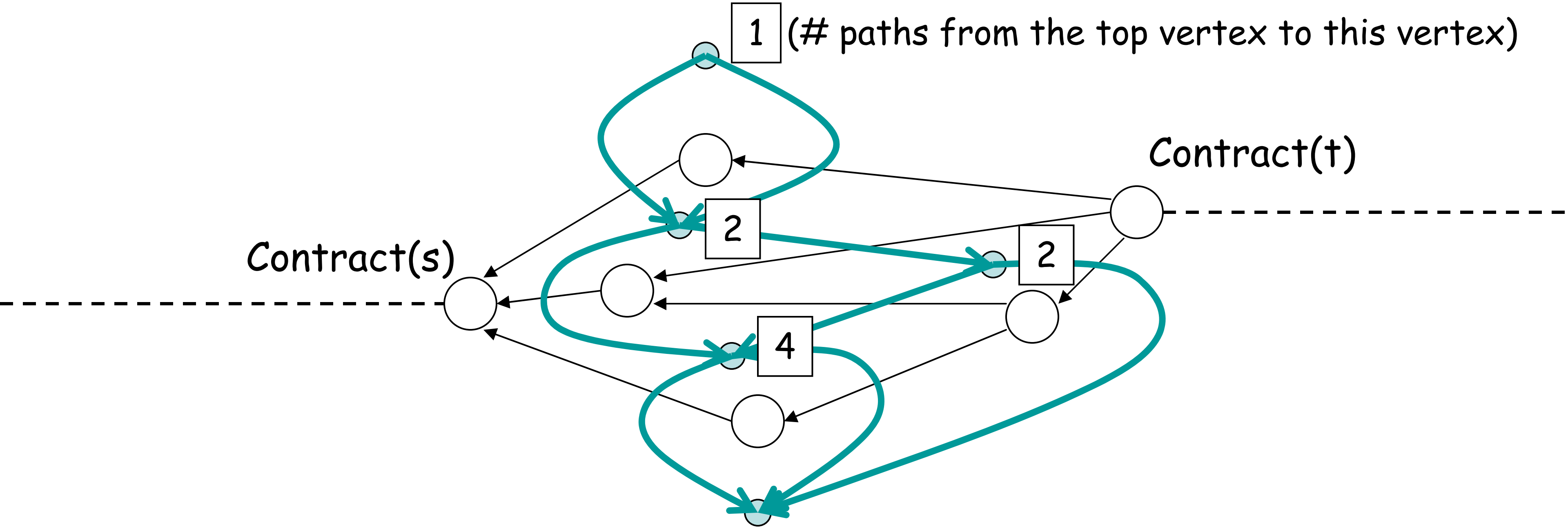

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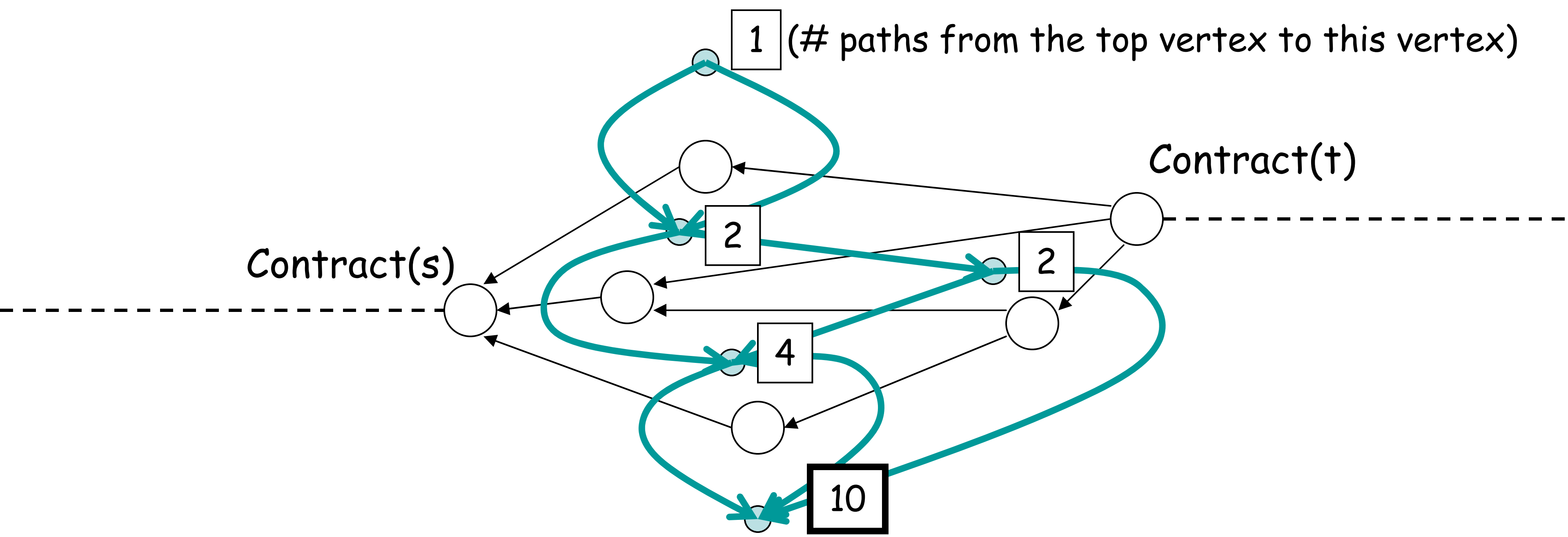
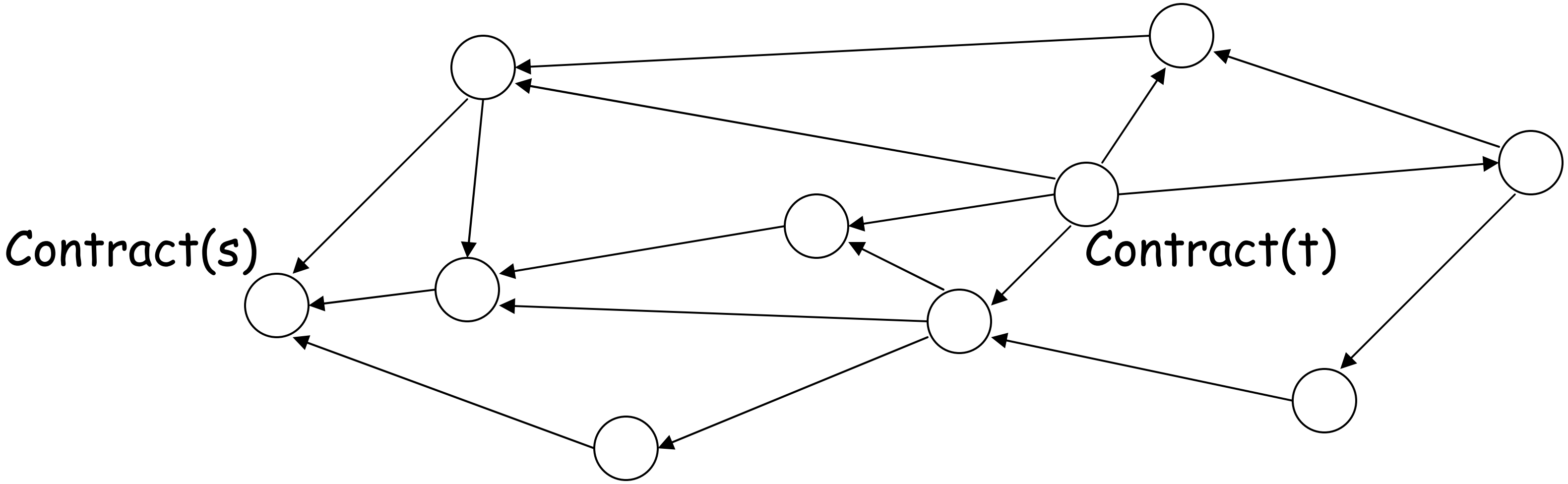


“Forward-cuts” and different faces

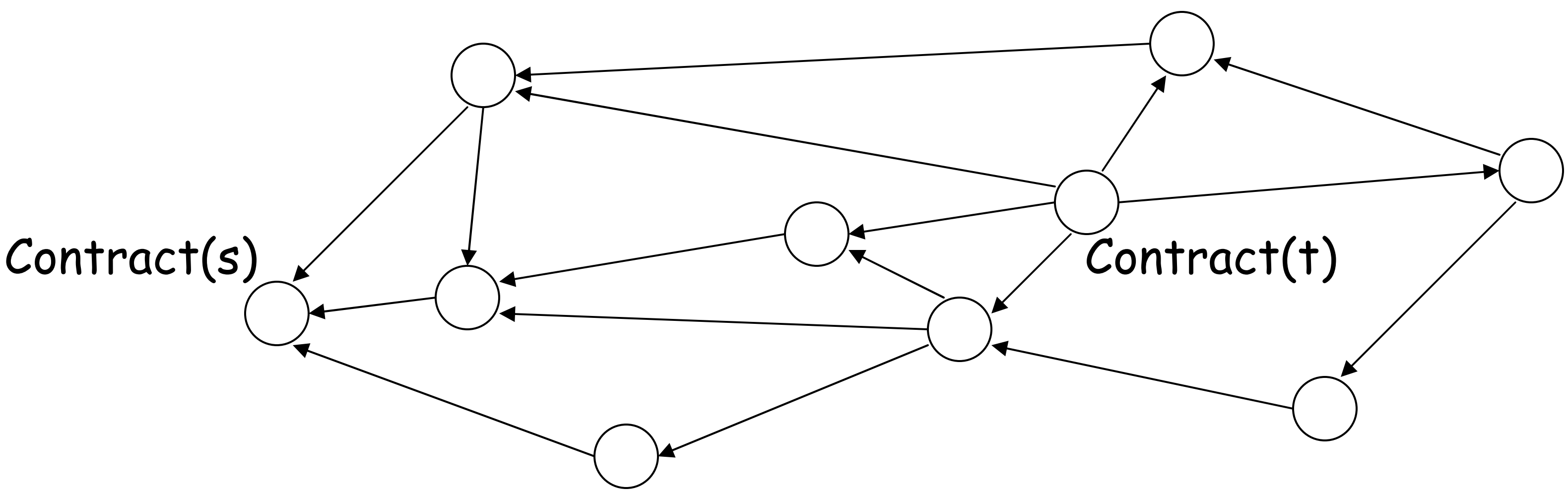
What if s,t are on different faces ?



“Forward-cuts” and different faces

What if s,t are on different faces ?

- find a contract(t)-contract(s) path
- construct the dual, except no edges cross the path
- sum # paths between faces sharing an edge on the path



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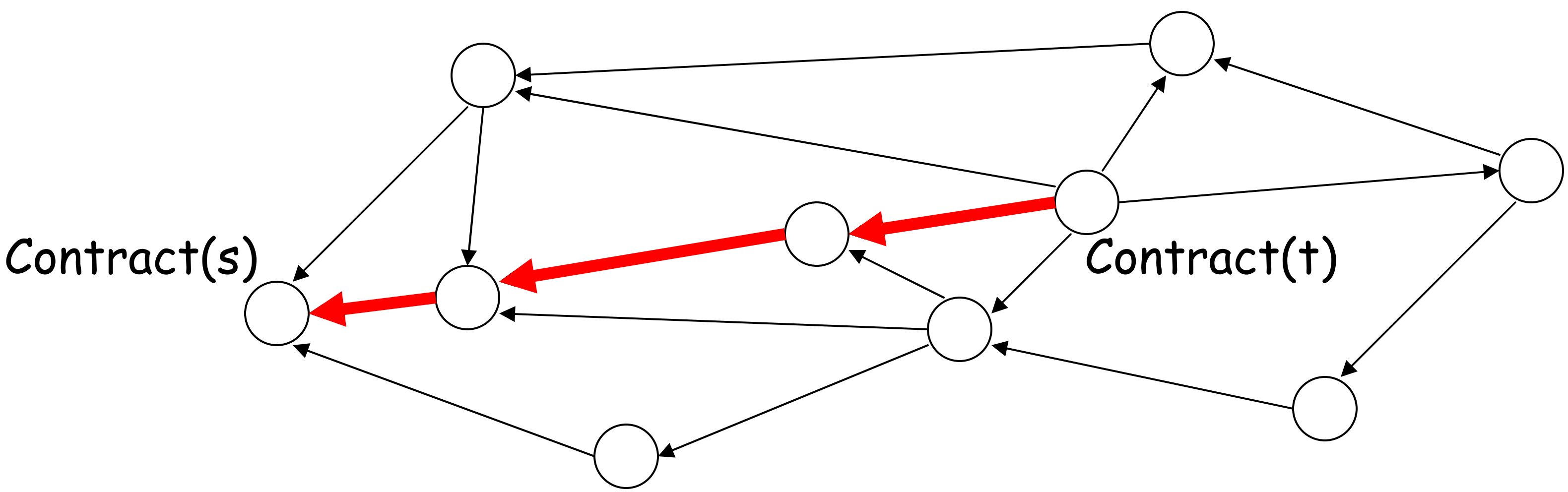

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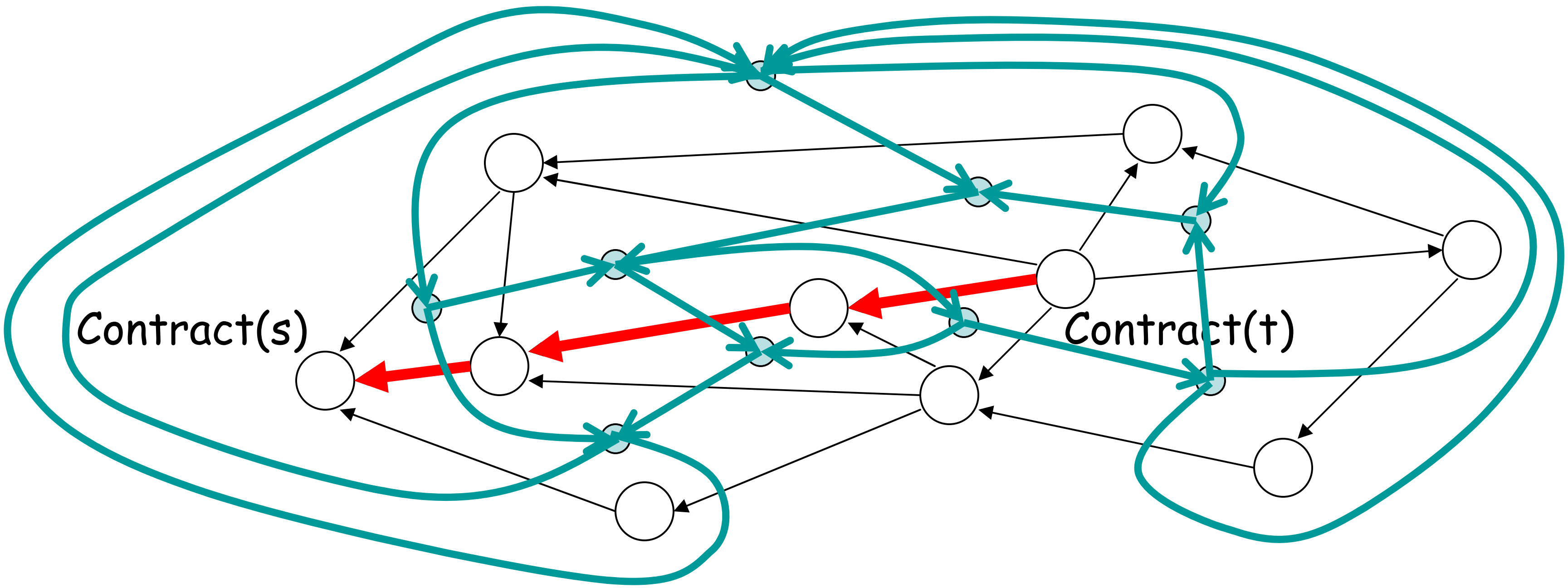

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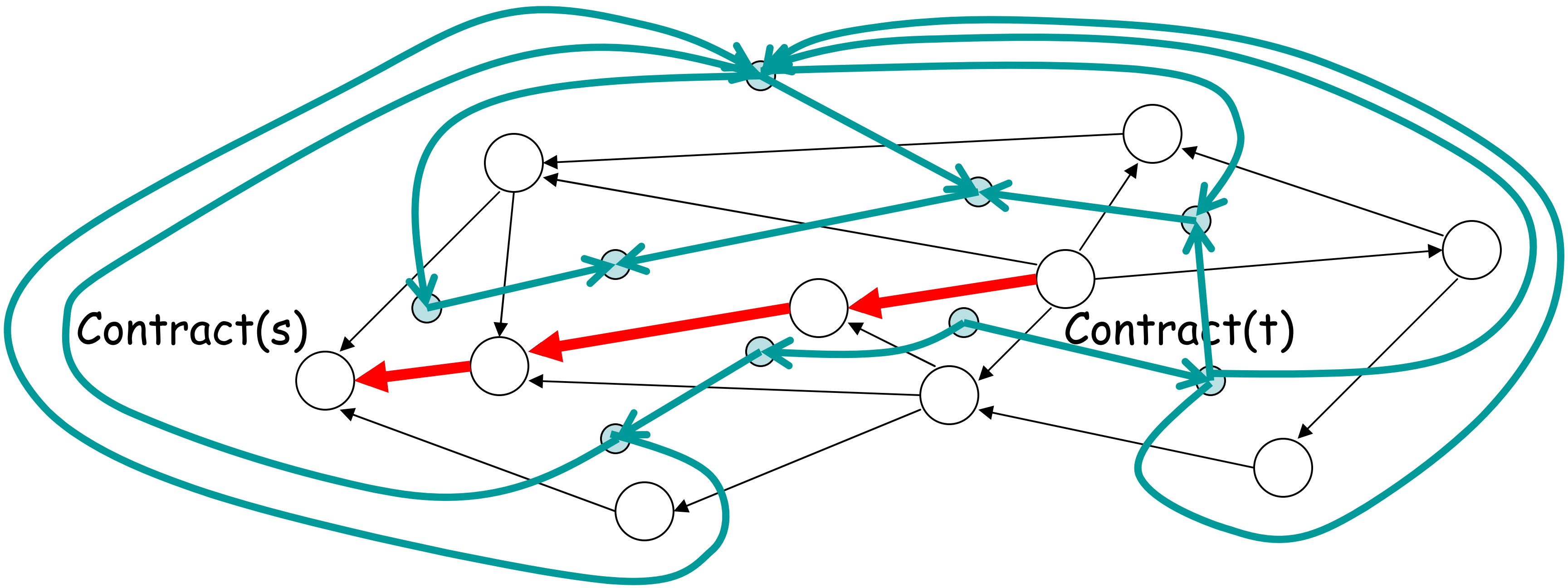

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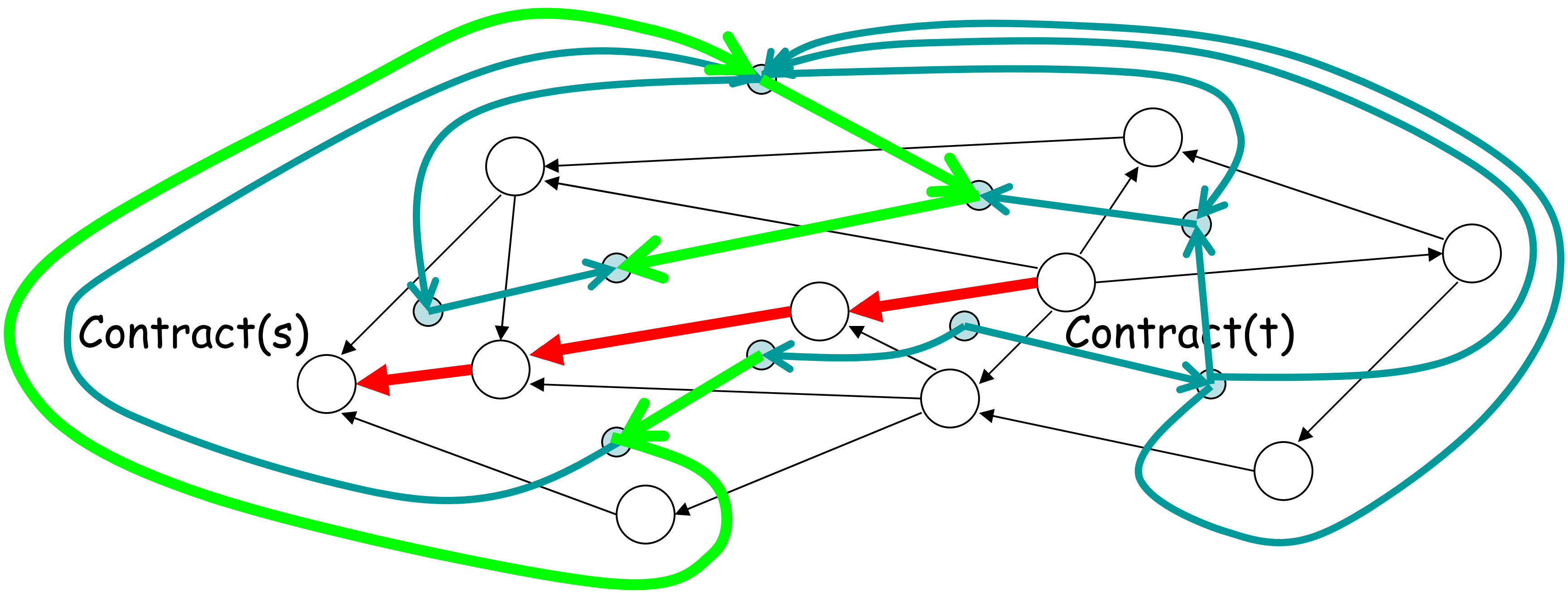

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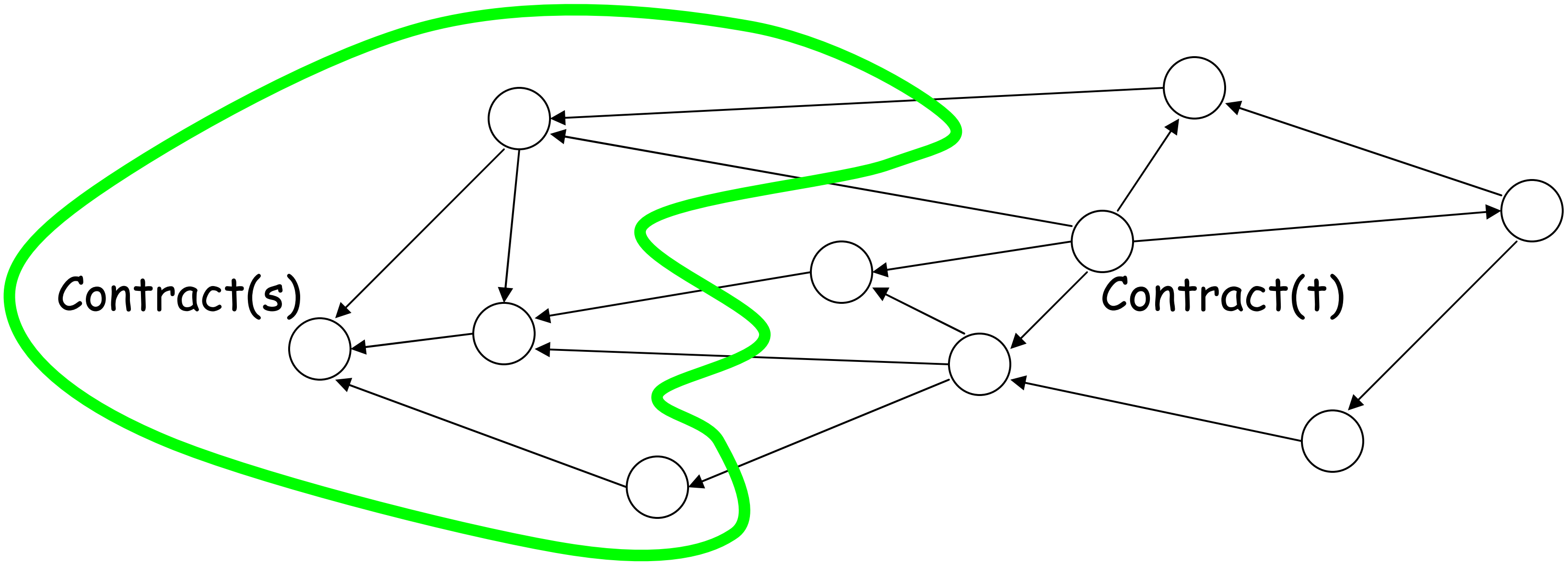

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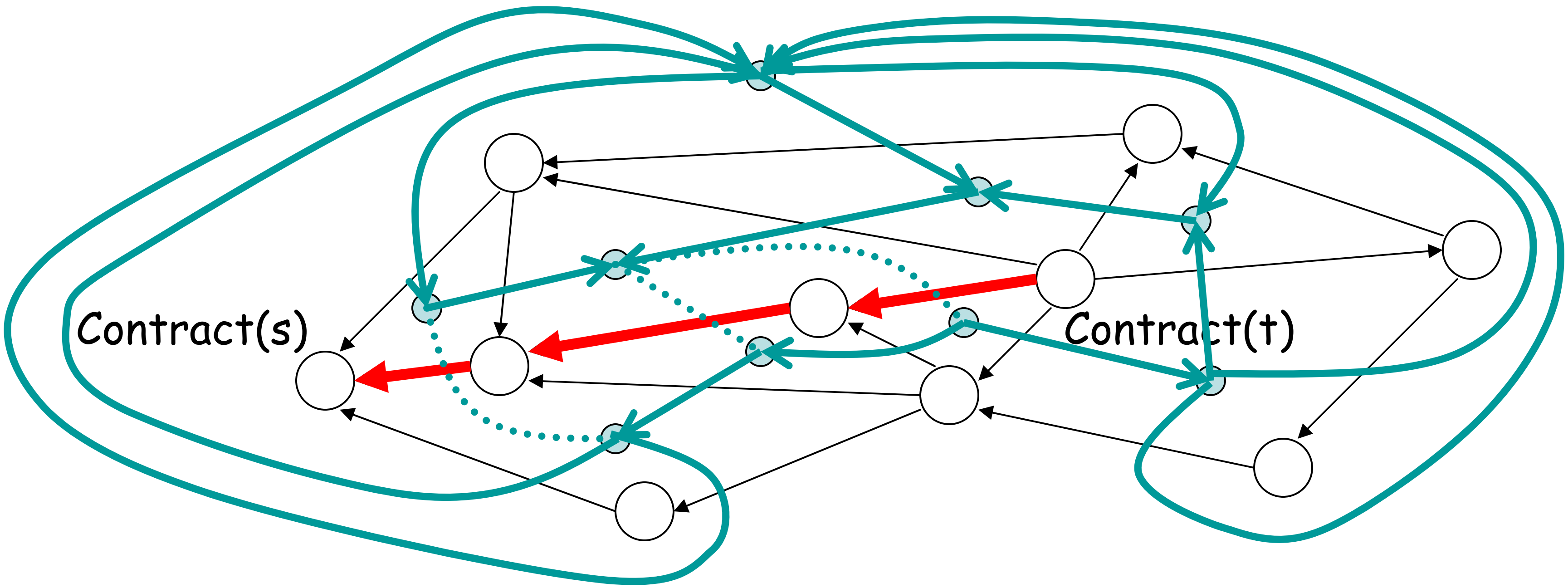
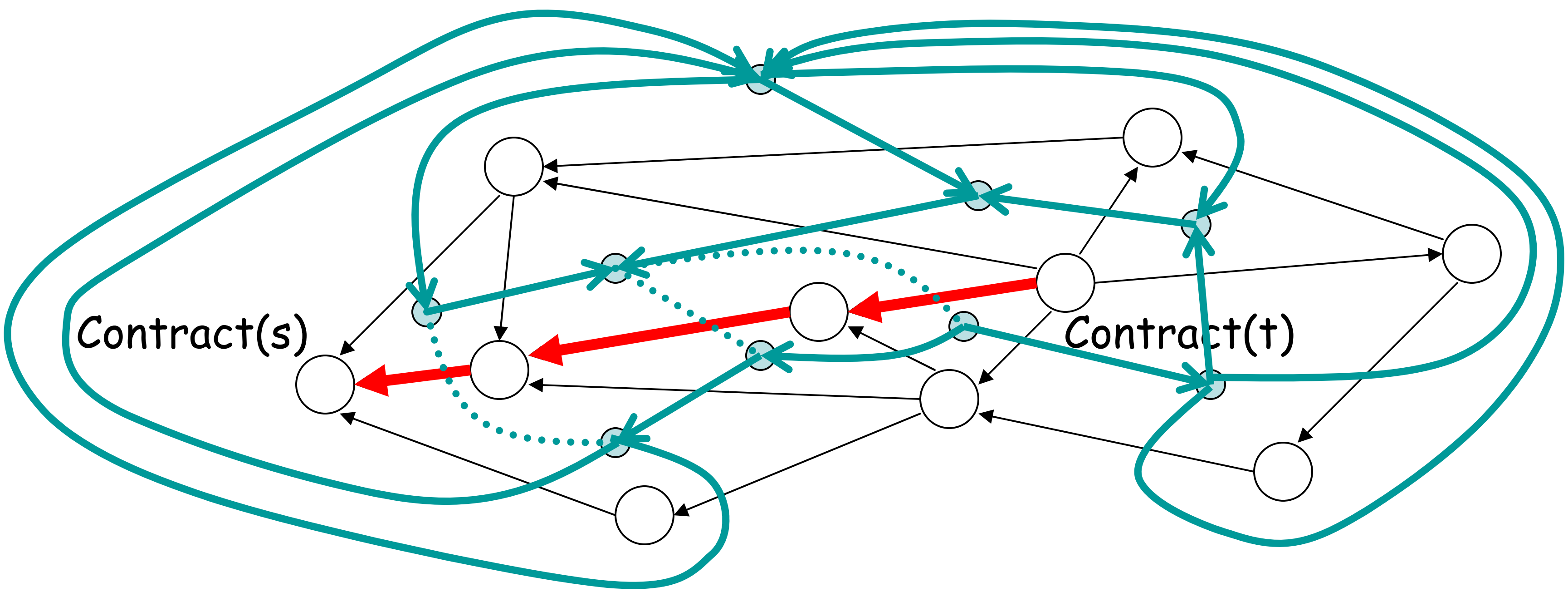


Sampling minimum (s,t)-cuts

- Choose a red edge proportionally to the corresponding path count.
- Starting at the corresponding **end** vertex, choose a predecessor vertex proportionally to the stored value
- Continue until get to the start vertex



Running time

Reduction to forward cuts:

- $O(n \log n)$ to find a (acyclic) max-flow in planar graphs [Borradaile-Klein '09]
- $O(n)$ to find and contract the strongly connected components

Counting forward cuts:

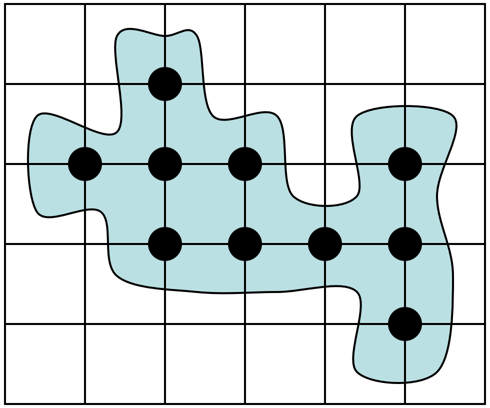
- $O(n)$ find the path, construct the dual graph
- $O(n)$ compute #paths between two end-points in the dual
- $O(dn)$ overall computation of paths, at most d end-point pairs where d = length of the s-t path

TOTAL: $O(dn + n \log n)$

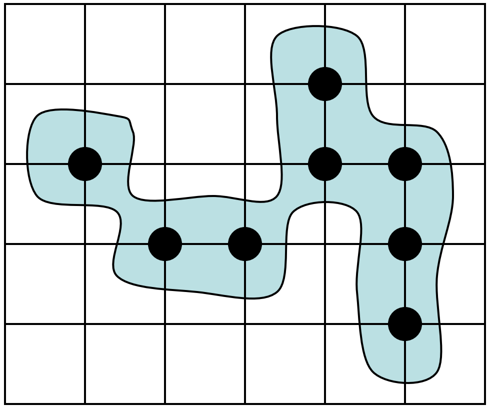
Contiguous cuts

Contiguous set of vertices: can be separated from the other vertices by a simply connected planar region that intersects every edge at most once

Examples:



contiguous



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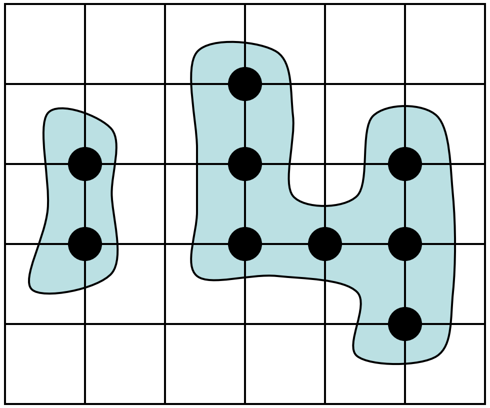
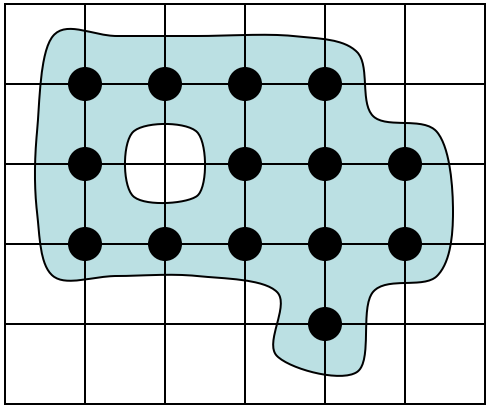


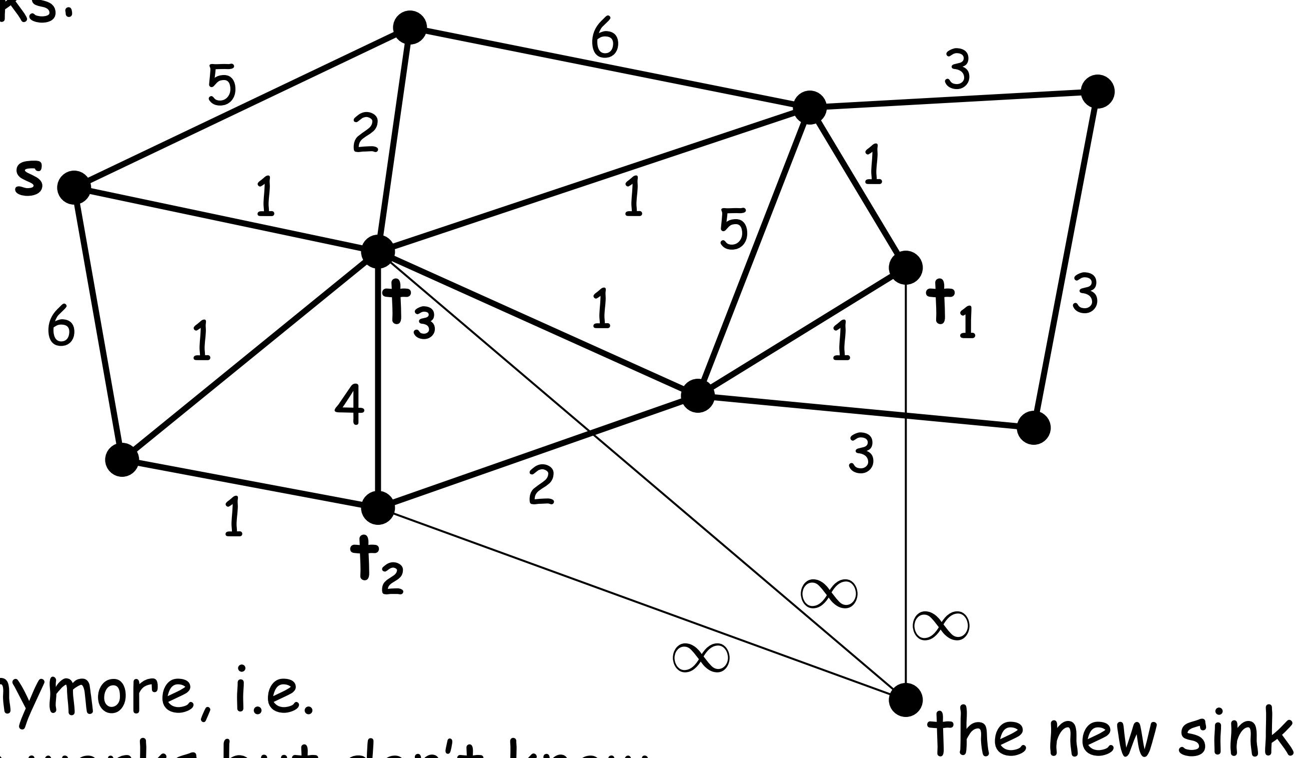
not contiguous



not contiguous

Back to the Ball & Provan's reduction

- Every minimum cut separating s and t is contiguous
- For multiple sinks:

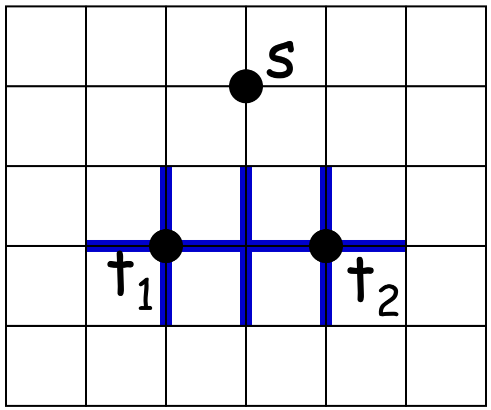


  - not planar anymore, i.e.
    BP reduction works but don't know
    how to count forward cuts
  - does the reduction "preserve" contiguous cuts ?

Back to the Ball & Provan's reduction

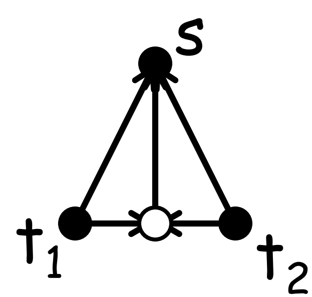
Does the reduction preserve contiguous cuts ?

NO...



All edges weight ∞, except for blue edges: weight 1.

Blue edges used up to their capacity, other edges not ->
Only blue edges survive in the contracted residual graph:



Back to the Ball & Provan's reduction

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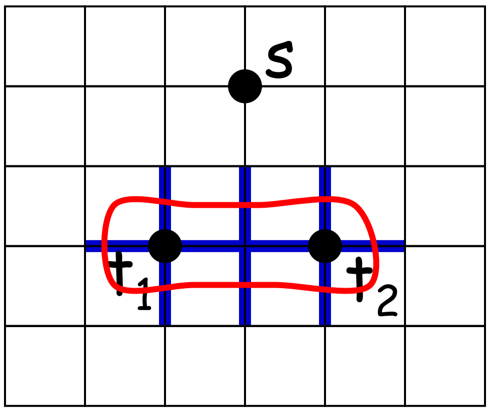
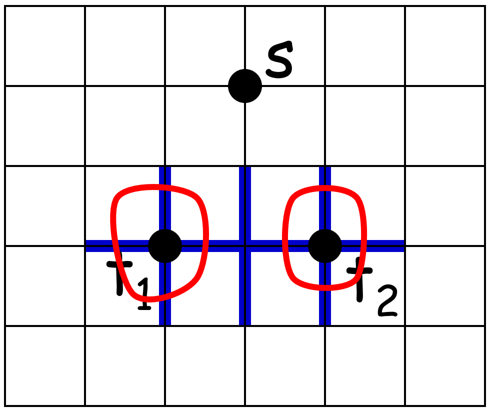
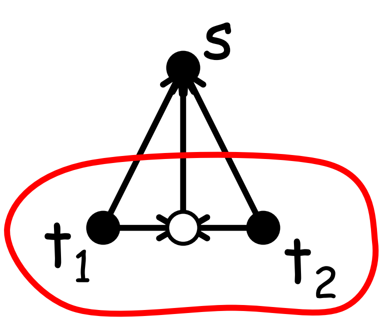
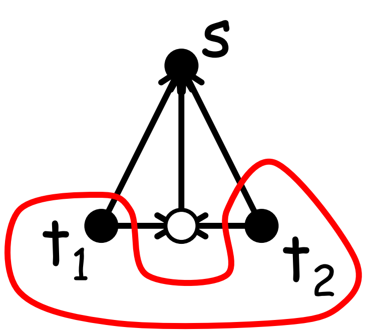






both contiguous



only bottom contiguous

The new reduction and counting

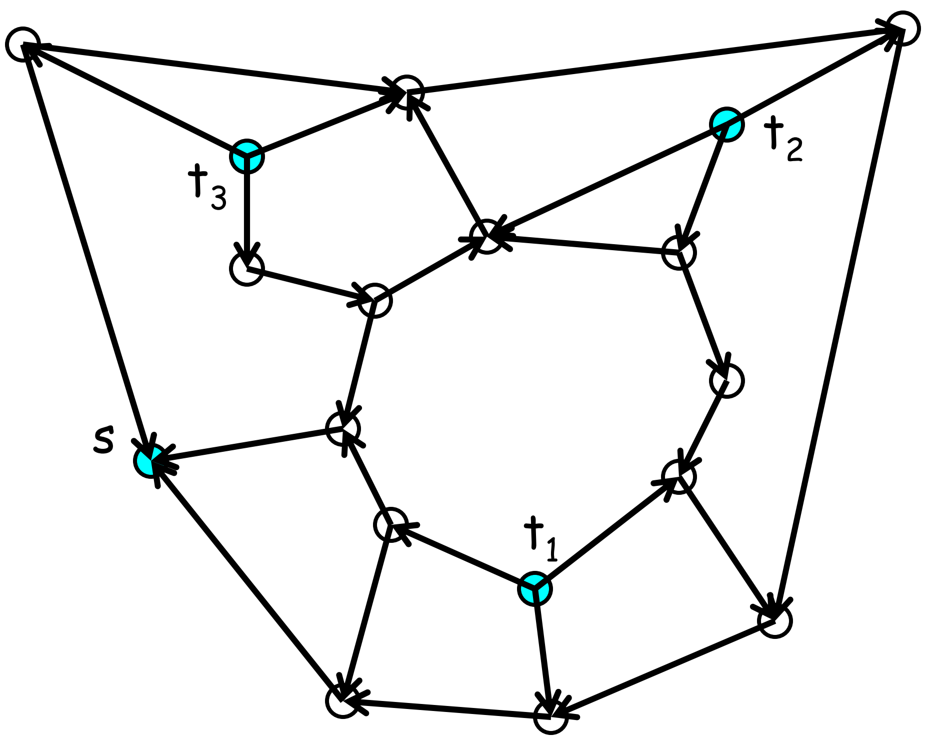
- A new reduction: creates a different DAG where contiguous forward (T',s')-cuts are in bijection with contiguous (s,T)-cuts in G

- Computing the number of contiguous forward (T',s')-cuts: dynamic programming across a tree-structure connecting T' with s' in the dual graph

Tours in the dual graph

Cut is not a simple cycle !

-> a tour in the dual graph.



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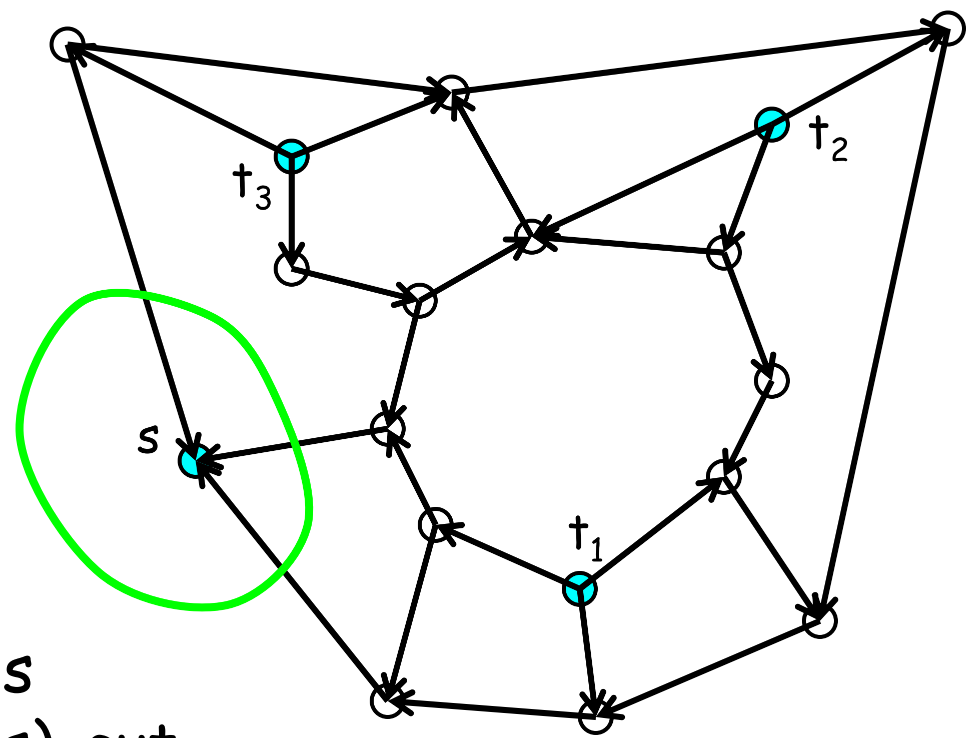

A contiguous forward (T,s)-cut

Tours in the dual graph

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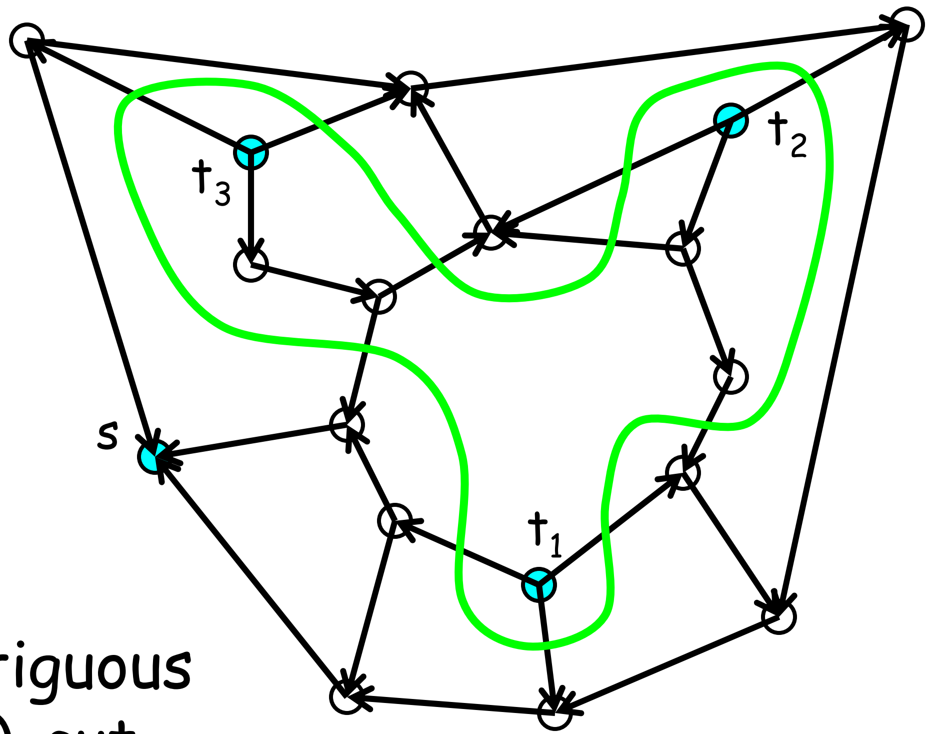
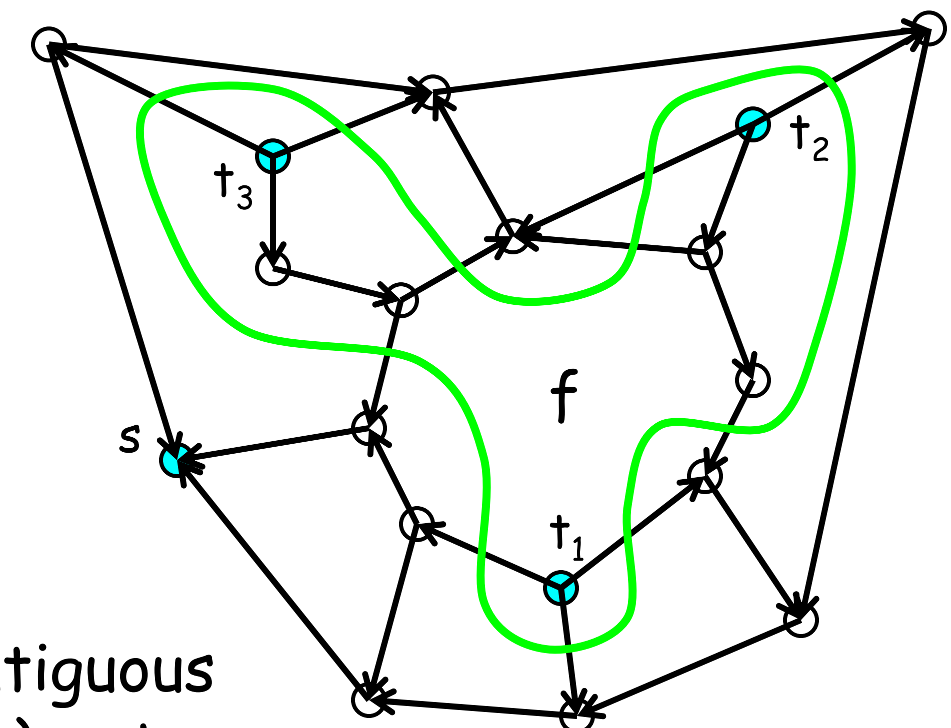


Another contiguous forward (T,s)-cut

Tours in the dual graph

A non-crossing tour:

For every face visited by the tour, its edges must be "cut" in the clockwise order when traversing the tour.



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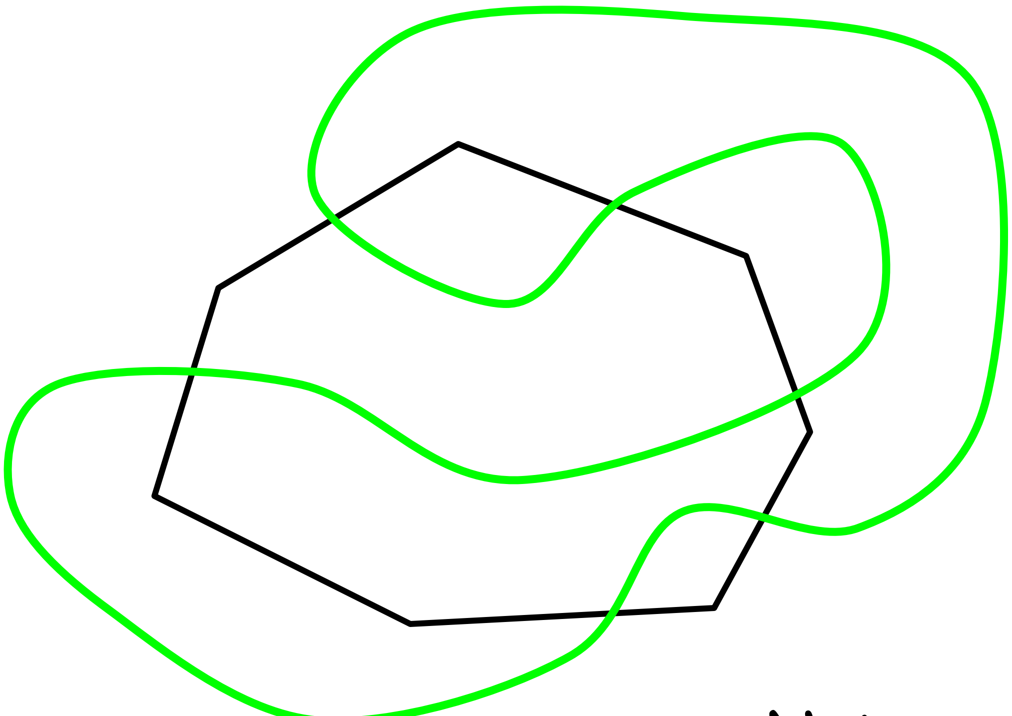
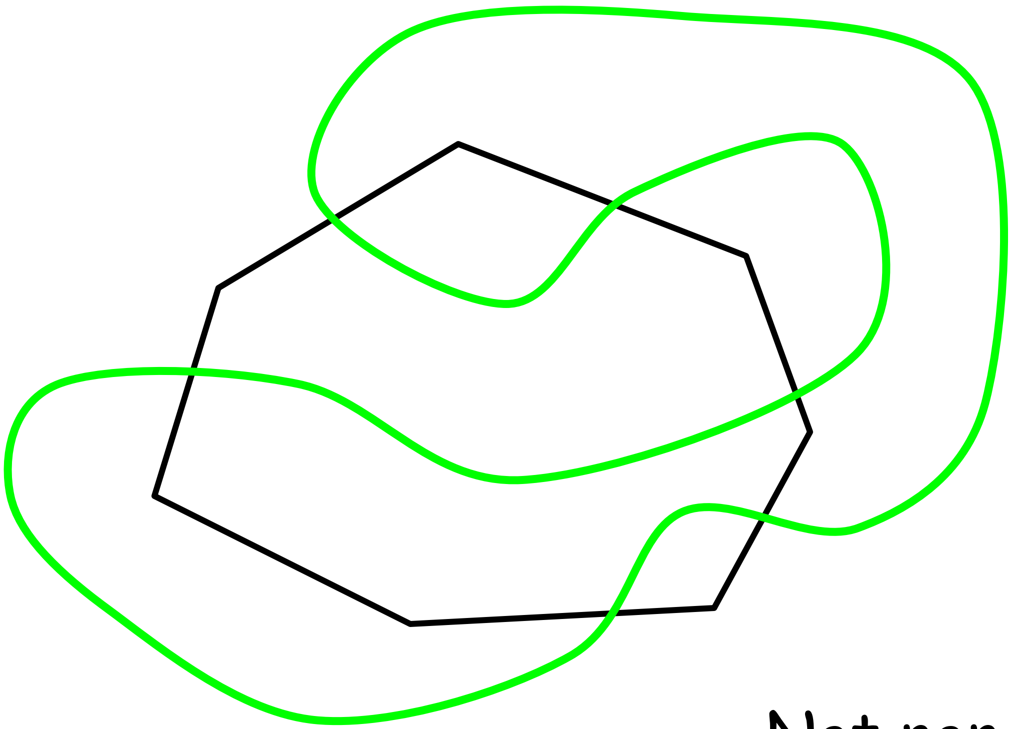


Not non-crossing !

Tours in the dual graph

Lemma:

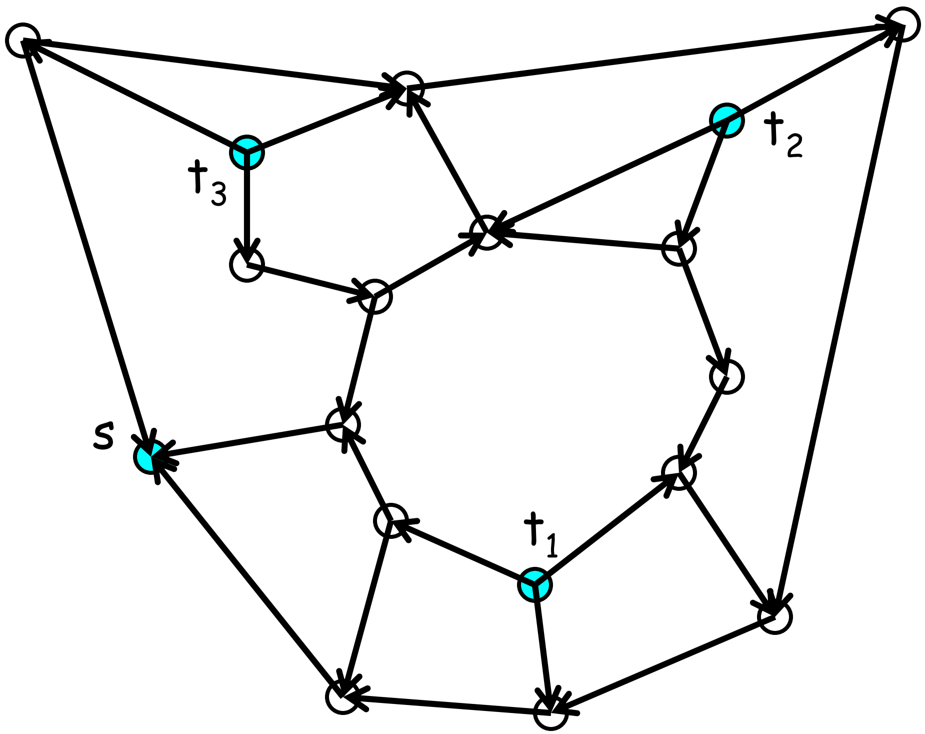
Contiguous forward (T,s)-cuts are in 1-1 correspondence with non-crossing tours in the dual of the graph obtained from the new reduction.



Not non-crossing !

Counting via dynamic programming

Goal: count non-crossing tours



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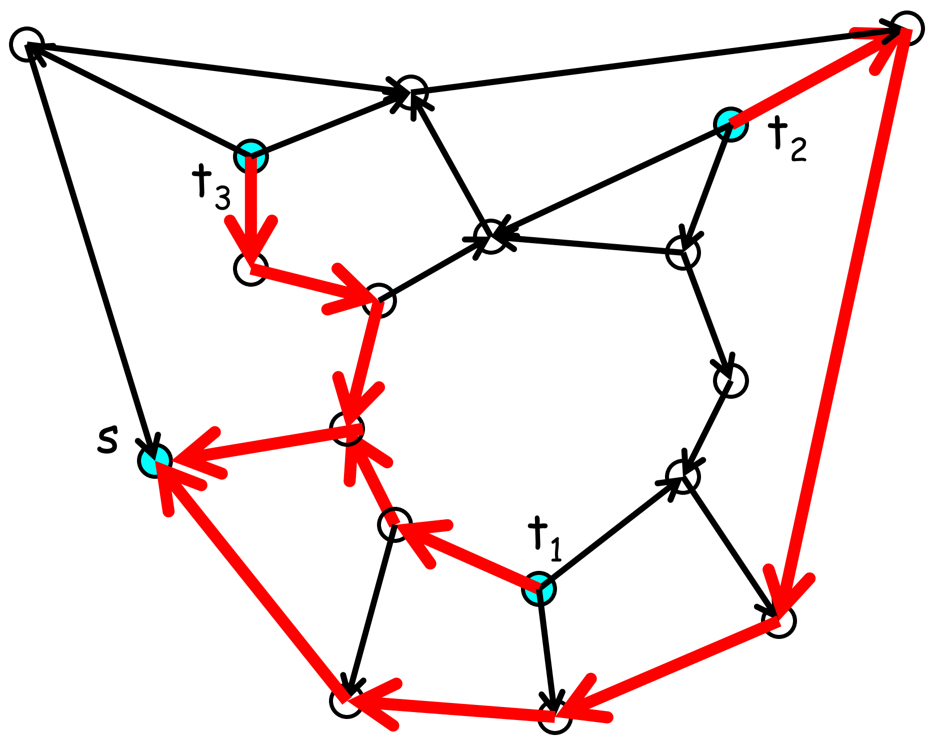
Idea: build a tree from T to s:



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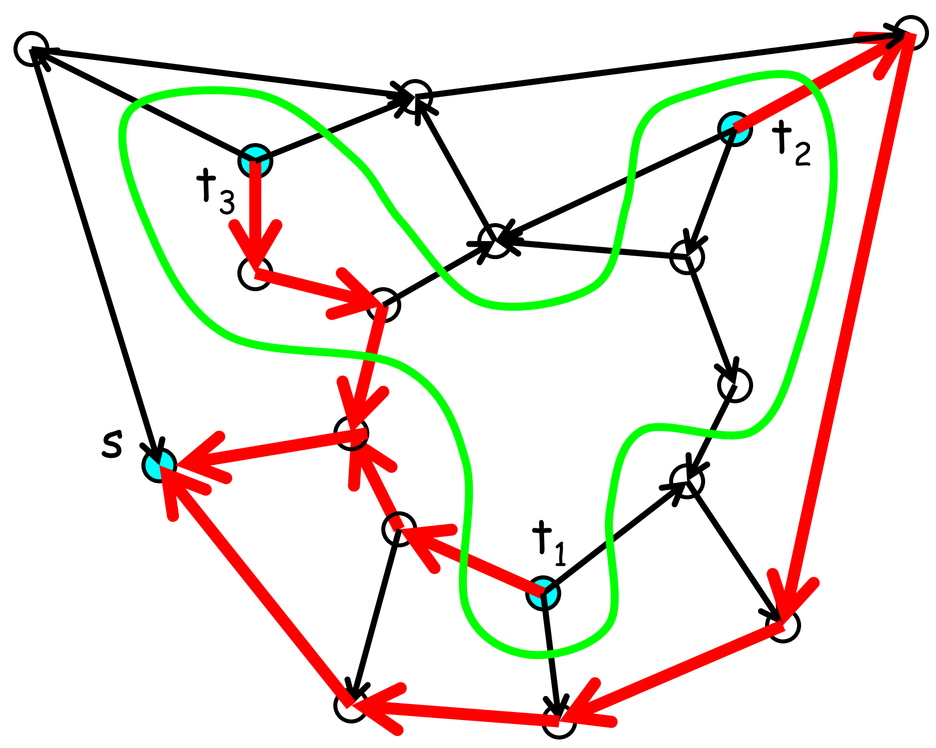
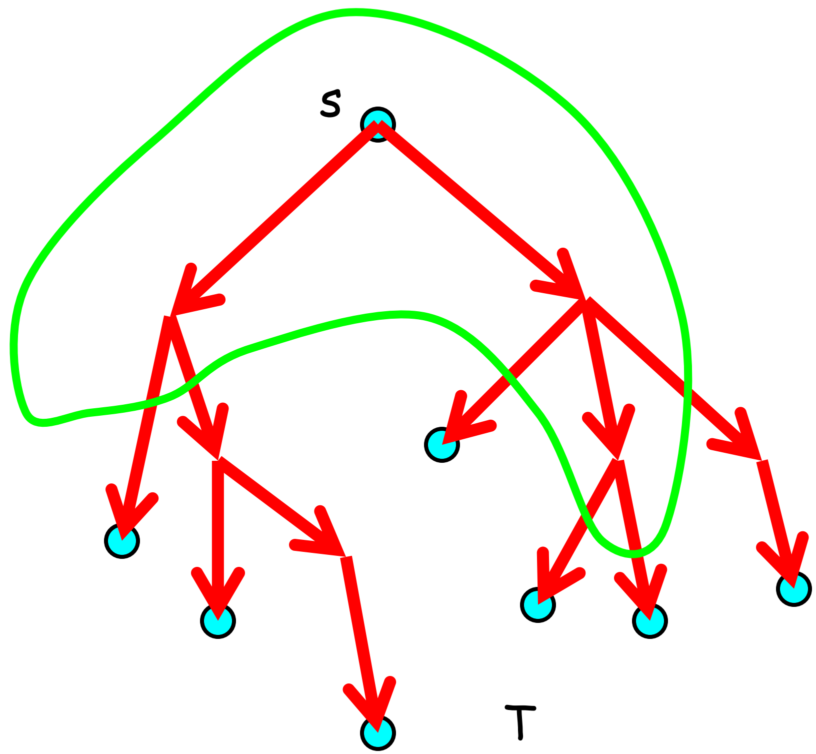


Counting via dynamic programming

Goal: count non-crossing tours

Idea: build a tree from T to s:

Then, compute #paths
between a "left" and
a "right" edge in a "wedge".



Generalize to larger wedge distance (of the left and the right edge) via dynamic programming.

Open Problems

Many open problems:

- multi-source multi-sink min cuts: counting (contig or not)
- multi-source multi-sink contig. min cuts: find
- other notions of contiguity

- graphs arising in computer vision (e.g., high-dimensional grids)
- non-planar graphs ? (unweighted or weighted)
- sampling all cuts proportionally to their weights