Brownian motion with drift and the Wiener sausage

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Joint work with

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Wiener sausage

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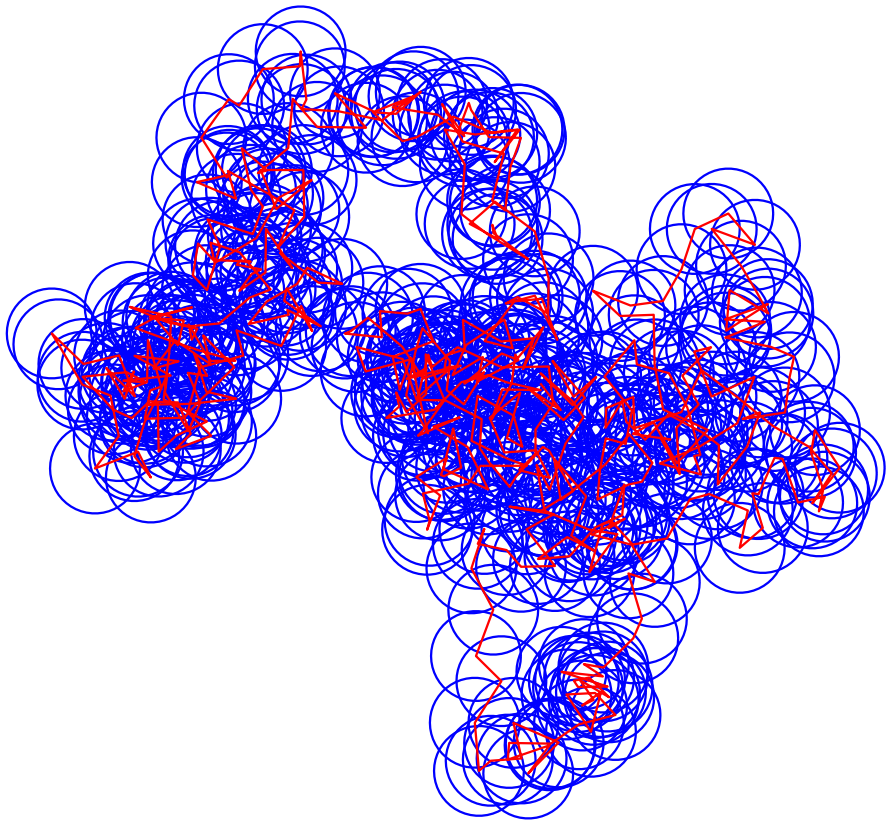

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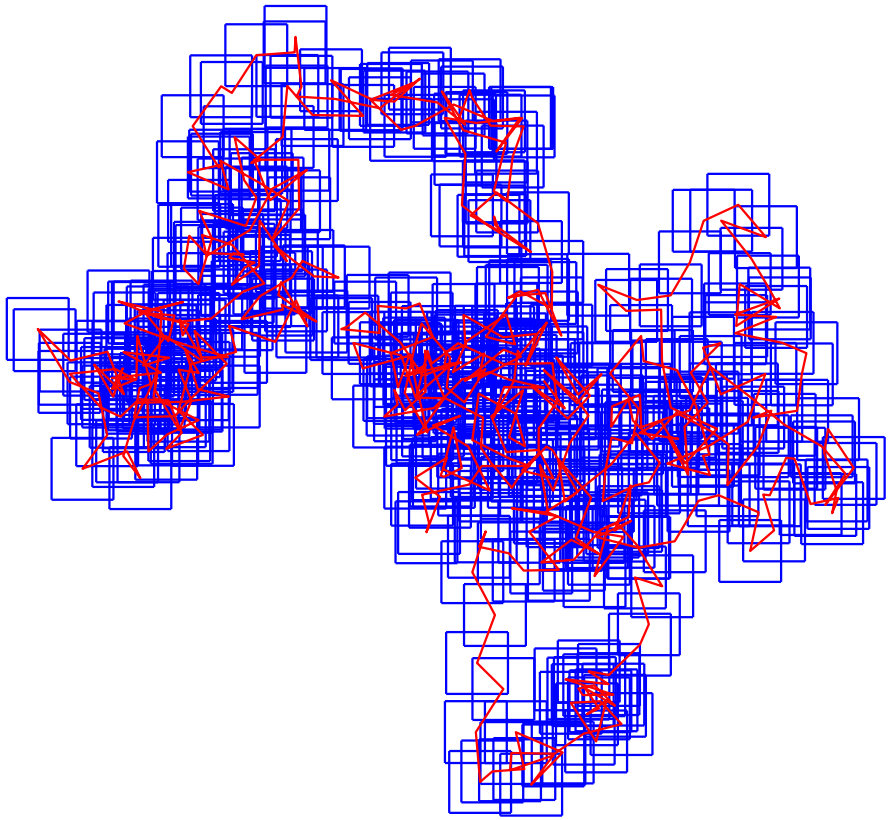
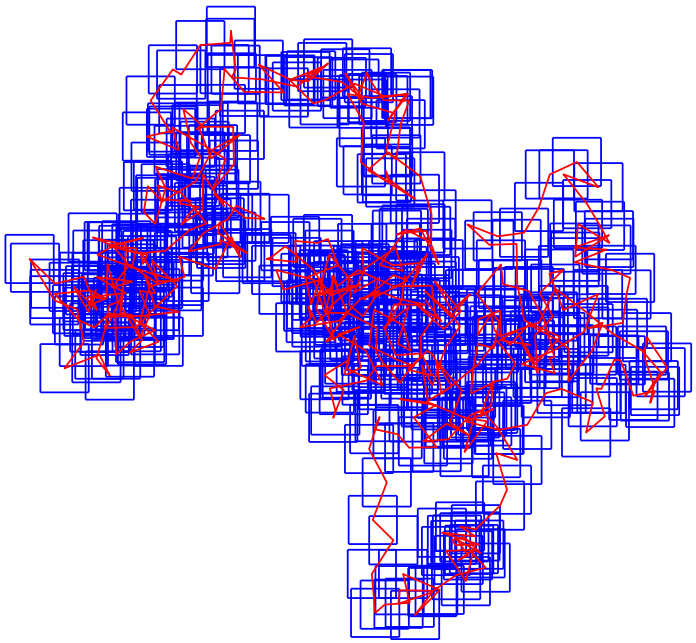
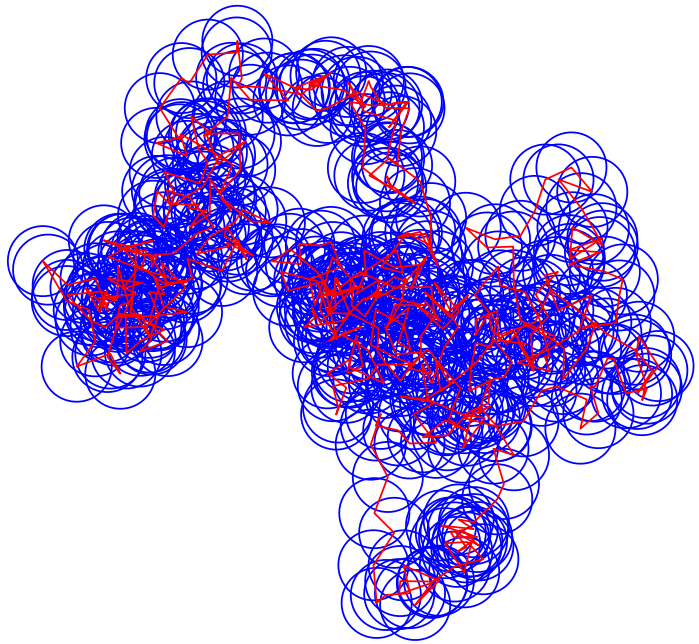


Wiener sausage

Question

Which has bigger expected volume?





Wiener sausage

Theorem (Peres, S. (2011))

Let $(\xi(s))_{s\geq 0}$ be a standard Brownian motion in $d \geq 1$ dimensions and let $(D_s)_{s\geq 0}$ be open sets in $\mathbb{R}^d$ with $\mathrm{vol}(D_s) = c$ for all s. Then for all t we have that

$$\mathbb{E}\left[\mathrm{vol}\left(\cup_{s\leq t}\left(\xi(s)+D_s\right)\right)\right] \geq \mathbb{E}\left[\mathrm{vol}\left(\cup_{s\leq t}(\xi(s)+\mathcal{B}(0,r))\right)\right],$$

where r is such that $\mathrm{vol}(\mathcal{B}(0,r)) = c$.

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In particular this gives that the expected volume of the Wiener sausage with squares is bigger than the expected volume with balls.

A connection to capacity

Spitzer and Whitman(1964) proved that in $d \geq 3$, if $A \subset \mathbb{R}^d$ is an open set with finite volume, then

$$\frac{\mathbb{E}[\mathrm{vol}(\cup_{s \leq t}(\xi(s) + A))]}{t} \to \mathrm{Cap}(A) \text{ as } t \to \infty.$$

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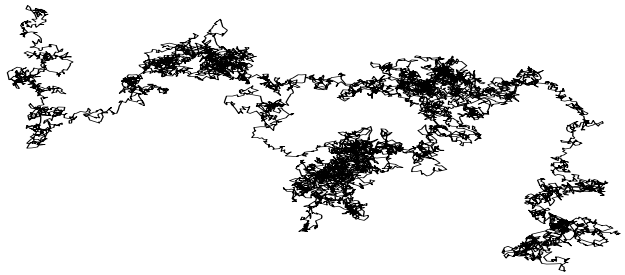
In $d \geq 3$ among all open sets of fixed volume, the ball has the smallest Newtonian capacity.

Planar Brownian motion



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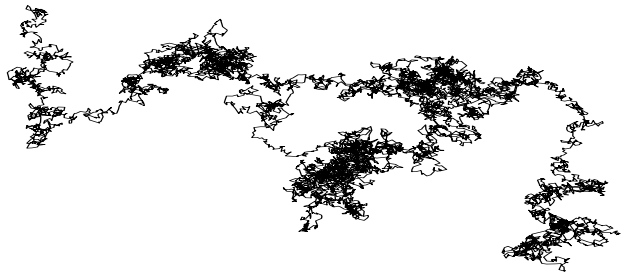

Theorem (Lévy 1940)

Let B be a planar Brownian motion. Then

$$\mathcal{L}(B[0,1]) = 0 \text{ a.s.}$$

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An a.s. property insensitive to the drift:

For any f continuous, $B + f$ is nowhere differentiable a.s.

Cameron–Martin Theorem

Denote by $D[0,1]$ the **Dirichlet space**

$$D[0,1] = \left\{ f \in C[0,1] : \exists g \in \mathbf{L}^2[0,1] \text{ s.t. } f(t) = \int_0^t g(s)ds, \forall t \in [0,1] \right\}.$$

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Theorem (Cameron–Martin 1944)

If $f \in D[0,1]$, then the law of B is mutually absolutely continuous w.r.t. the law of $B + f$.

Hence, if $f \in D[0,1]$, then $\mathcal{L}(B+f)[0,1] = 0$ a.s.

Graversen's result

Theorem (Graversen 1982)

For all $0 < \alpha < 1/2$, *there exists a Hölder*(α) *continuous function* $f : \mathbb{R}_+ \to \mathbb{R}^2$ *s.t.* $\mathbb{E}[\mathcal{L}(B + f)[0, 1]] > 0$.

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We will see: same transition from Hölder(α) for $\alpha < 1/2$ to $\alpha = 1/2$ applies to a large variety of properties of Brownian motion.

Antunović, Peres and Vermesi result

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Theorem (Antunović et al 2010)

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A remaining question

In all these works it was not clear whether for any continuous f

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This was the impetus for our work.

0-1 Laws

Let $(B_t, 0 \leq t \leq 1)$ be a standard Brownian motion in $\mathbb{R}^d$ and let $f : [0,1] \to \mathbb{R}^d$ be a continuous function.

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- $\mathbb{P}(\text{interior of } (B+f)[0,1] \neq \emptyset) \in \{0,1\}$.
- $\dim(B+f)[0,1] = c$ *a.s., where* c *is a positive constant and* dim *is the Hausdorff dimension.*

Beyond the Cameron–Martin theorem

Again the same setting, B is a standard Brownian motion and $D[0,1]$ is the Dirichlet space

$$D[0,1] = \left\{ f \in C[0,1] : \exists g \in \mathbf{L}^2[0,1] \text{ s.t. } f(t) = \int_0^t g(s)ds, \forall t \in [0,1] \right\}.$$

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If $f \notin D[0,1]$, then the law of B and the law of $B + f$ are singular.

As a consequence, when $f \notin D[0,1]$, there is some a.s. property of Brownian motion that fails for $B + f$.

Hitting probabilities

Cauchy–Scwartz inequality gives that if $f \in D[0,1]$, then f is Hölder$(1/2)$.

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Question

Does $B + f$ hit the same sets as B, if f is Hölder(1/2)?

Hitting probabilities

Theorem (Peres and S.)

Let A be a closed set of $\mathbb{R}^d$, for $d \geq 2$, and f a Hölder(1/2) continuous function. If $\mathbb{P}_x(B \text{ hits } A) > 0$, for all $x \in \mathbb{R}^d$, then $\mathbb{P}_x(B + f \text{ hits } A) > 0$, for all $x \in \mathbb{R}^d$.

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In 2 dimensions, if $\mathbb{P}_x(B \text{ hits } A) > 0$, then by neighborhood recurrence, $\mathbb{P}_x(B \text{ hits } A) = 1$. The same is true for $B + f$, if f is Hölder(1/2).

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Concerning the existence of multiple points, $B + f$ behaves in the same way as B, if f is Hölder(1/2).

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Concerning the existence of multiple points, $B + f$ behaves in the same way as B, if f is Hölder(1/2).

(This can fail if f is not Hölder(1/2), e.g. for f fractional Brownian motion.)

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For every $\alpha \geq 0$, the α-Hausdorff content of a metric space E is defined

$$\mathcal{H}_{\infty}^{\alpha}(E) = \inf\{\sum_{i=1}^{\infty}(\text{diam}(E_i))^{\alpha} : E_1, E_2, \dots \text{ is a covering of } E\}.$$

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The **Hausdorff dimension** of E is defined to be

$$\dim E = \inf\{\alpha \geq 0 : \mathcal{H}_\infty^\alpha(E) = 0\}.$$

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Question

Can we provide bounds for $\dim(B+f)[0,1]$*?*

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$$\dim(B + f)[0,1] \geq \max\{2 \wedge d, \dim f[0,1]\} \text{ a.s.}$$

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Theorem (0-1 law for $\mathcal{L}$)

$$\mathbb{P}(\mathcal{L}(B+f)[0,1] > 0) \in \{0,1\}.$$

Proof of the 0-1 law for $\mathcal{L}$

For an interval $I \subset [0,1]$, define $\Psi(I) = \mathcal{L}(B+f)(I)$.

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The limit of $\mathbb{E}[Z_n]$ exists and can be either infinite or finite.

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Letting $n \to \infty$ gives $\mathbb{P}(\Psi([0,1]) = 0) = 0$.

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Since $\Psi(\text{good points}) = 0 \Rightarrow \Psi([0, 1]) = 0$ a.s.

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