Random partitions in a box: limit shape and fluctuations

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Partitions and Young diagrams

Definition

A **partition** of an integer n is a sequence of integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \quad \text{such that} \quad |\lambda| = \sum_k \lambda_k = n$$

Example

$(5, 5, 4, 3, 1, 1, 0, \dots)$ is a partition of 19

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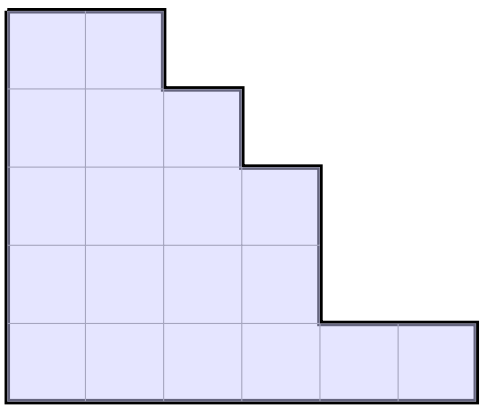


- largest part: 5
- number of parts: 6

Grand canonical ensemble

- Euler: generating function $Z(q) = \sum_{\lambda} q^{|\lambda|} = \prod_{k \geq 1} \frac{1}{1-q^k}$
- Natural probability measure: $\mu_q(\{\lambda\}) = \frac{q^{|\lambda|}}{Z(q)}$

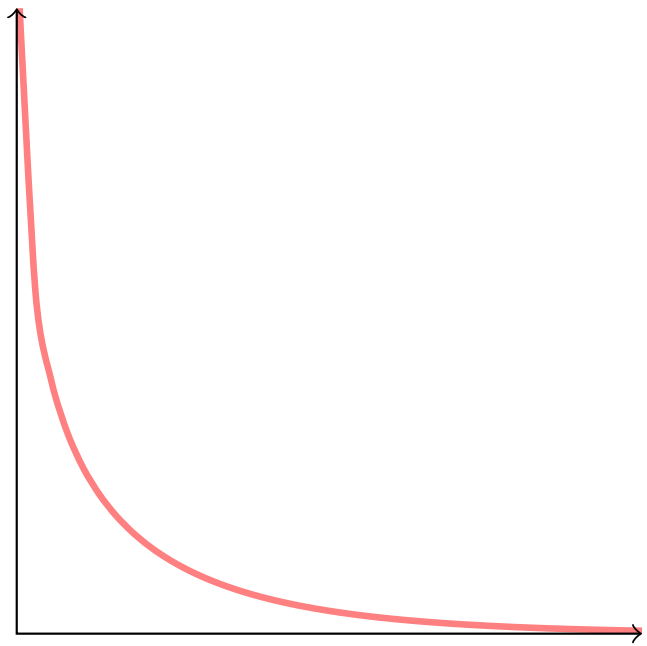
  What does a typical partition of a large integer look like?

$$\text{size of order } n \quad \simeq \quad \log q^{-1} \text{of order } \frac{1}{\sqrt{n}}$$

- Hardy–Ramanujan: $\#\{\lambda : |\lambda| = n\} \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}$
- Erdős–Lehner: asymptotics for the largest part
- Vershik: limit shape when $1/\sqrt{n}$-rescaling $e^{-ax} + e^{-ay} = 1$

Vershik's curve

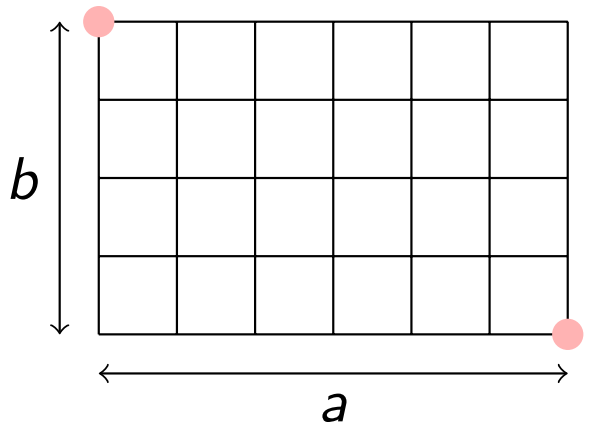
$$e^{-ax} + e^{-ay} = 1$$



Confinement in a box

Instead of looking at all partitions, add constraints:

- number of parts at most a
- parts at most b



Counting path from NW to SE corner, weighted by ther area

$Z_{a,b}(q)$: generating function for partitions fitting in an $a \times b$ box

Generating function

Lemma

$$Z_{a,b}(q) = \binom{a+b}{a}_q = \frac{(a+b)!_q}{a!_q b!_q}, \quad \text{where } a!_q = \prod_{1 \le k \le a} \frac{1-q^k}{1-q}$$

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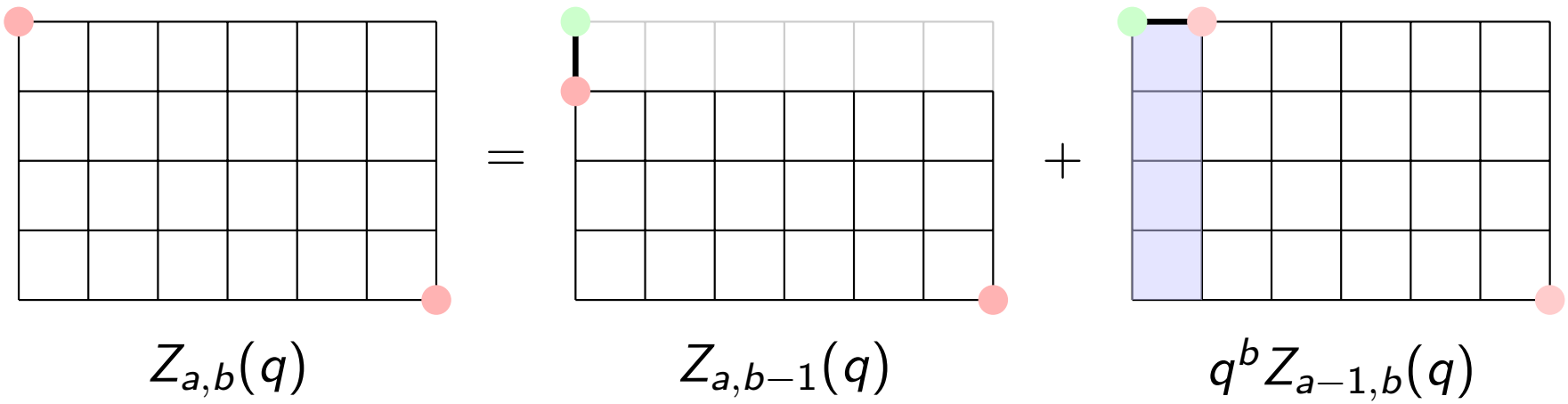
Proof.



A key ingredient: the q-Stirling formula

Lemma (Stirling)

$$\ell! = \sqrt{2\pi\ell}\exp\left(\int_0^\ell \log u \mathrm{d}u\right)(1+o(1))$$

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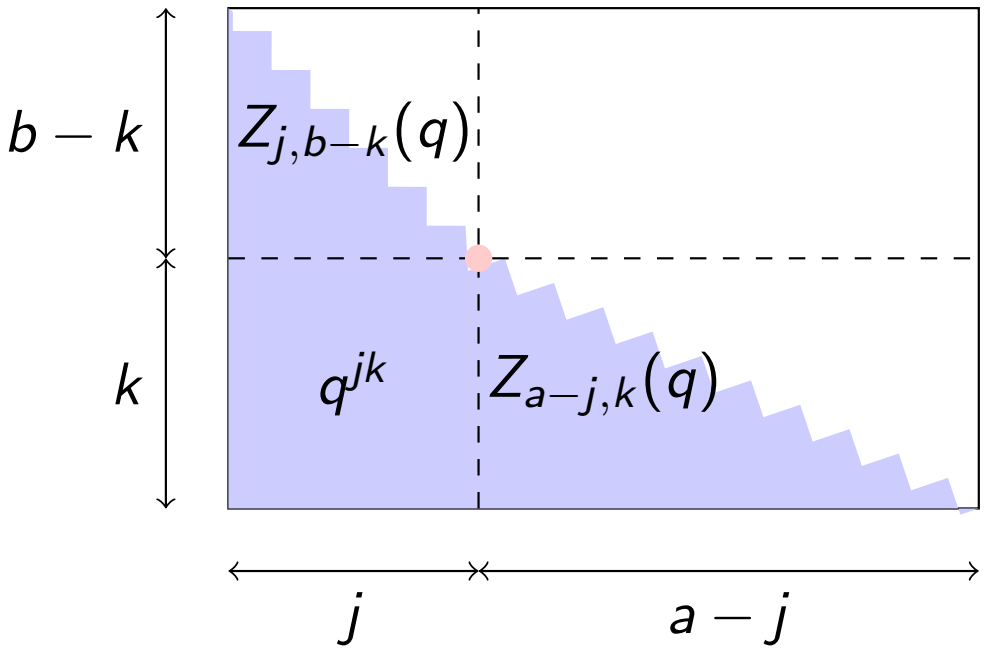
Position of the curve: maximum likelihood



$$\mathbf{P}(Y_j = k) = \frac{q^{kj} Z_{j,b-k} Z_{a-j,k}}{Z_{a,b}}$$

unimodality of the distribution:

$$\frac{\mathbf{P}(Y_j = k)}{\mathbf{P}(Y_j = k+1)} \geq 1 \Leftrightarrow k \geq L^\sharp(j)$$

Limit of $L^\sharp(j)$ should give the limit shape if it exists.

Convergence to the limit shape

- Rotate by 45 degrees
- (X_t) boundary of the random Young diagram
- $a_n + b_n = 2n$, $\frac{a_n}{2n} \to \rho \in (0,1)$, $q = e^{-c/n}$

Theorem

As n goes to ∞, $(X_{2nt}/2n)_{t\in[0,1]}$ converges in probability in $\mathcal{C}([0,1])$ to the deterministic curve

$$L_{\rho,c} = \frac{1}{2} - \rho + \frac{1}{c} \log \frac{\sinh(ct) + e^{c(\rho-\frac{1}{2})} \sinh c(\frac{1}{2} - t)}{\sinh \frac{c}{2}}$$

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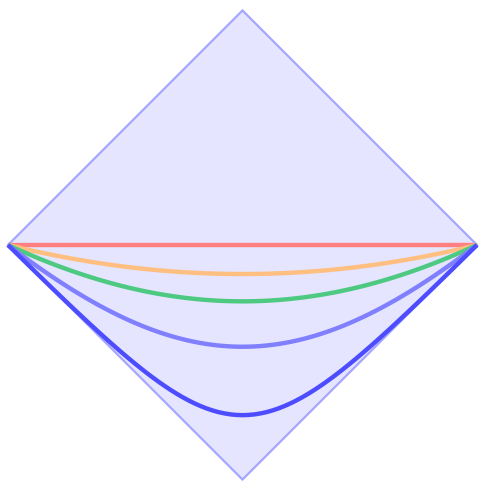
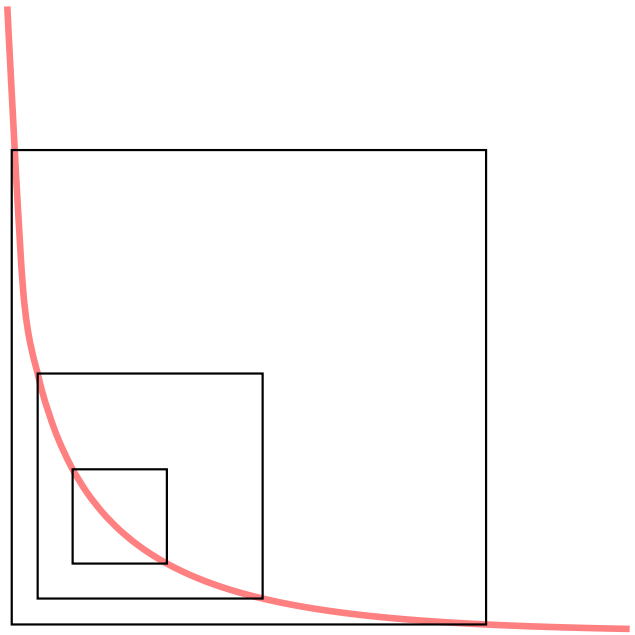


Link with Vershik's curve

for a square $\rho = \frac{1}{2}$, $L_{1/2,c} = \frac{1}{c} \log \frac{\cosh(c(t-1/2))}{\cosh c/2}$



Everything was contained in Vershik's curve...
Already noticed by Petrov (microcanonical ensemble: n fixed)
What about fluctuations around the limit shape?

Convergence of fluctuations

- $\frac{a_n}{a_n+b_n} = \rho + o\left(\frac{1}{\sqrt{n}}\right)$
- $\tilde{X}_t = \frac{\frac{1}{2}X_{2nt} - nL_{\rho,c}(t)}{\sqrt{n}}$
- $f(t) = 2\cosh c(x - \frac{1}{2})$ (for $\rho = \frac{1}{2}$)

Theorem

As n goes to ∞, the distribution of $f(t)\tilde{X}_t$ converges in $\mathcal{D}([0,1])$ to the Ornstein–Uhlenbeck bridge with parameter c.

Ornstein-Uhlenbeck bridge

Definition

Let $c > 0$. The Ornstein-Uhlenbeck process with parameter c is the centered Gaussian process $(Z_s, s \in [0,1])$ with covariance

$$\mathbf{E}(Z_s Z_t) = \frac{\sinh(cs)\sinh(c(1-t))}{c\sinh(c)}$$

- Ornstein-Uhlenbeck process conditionned to come back to 0 at time 1
- Recover the Brownian bridge when $c \to 0$

Ideas for the proof of the fluctuations

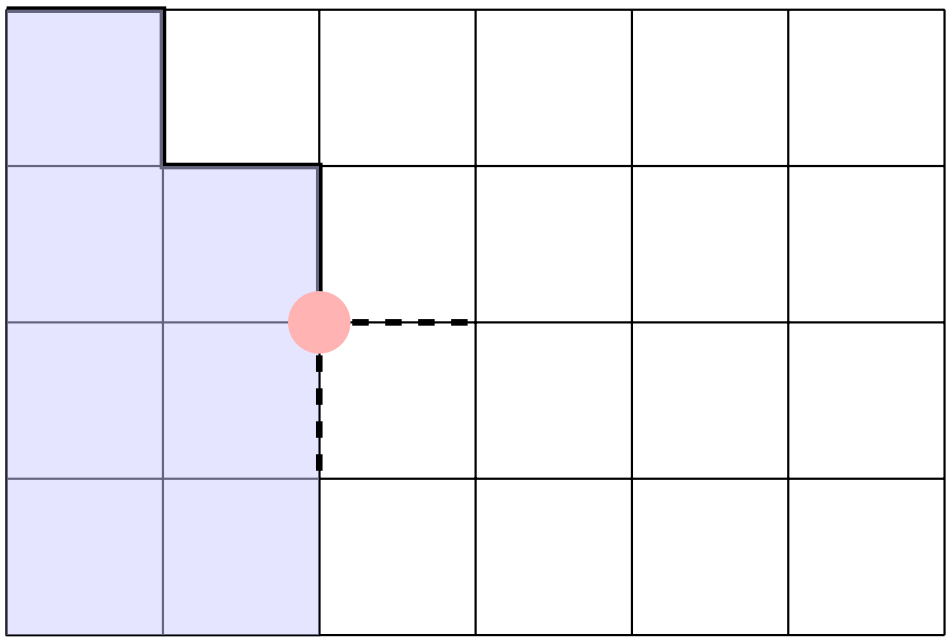
Proof in two steps

- Convergence of finite dimensional distributions
- Tightness

Key ingredients in the proof

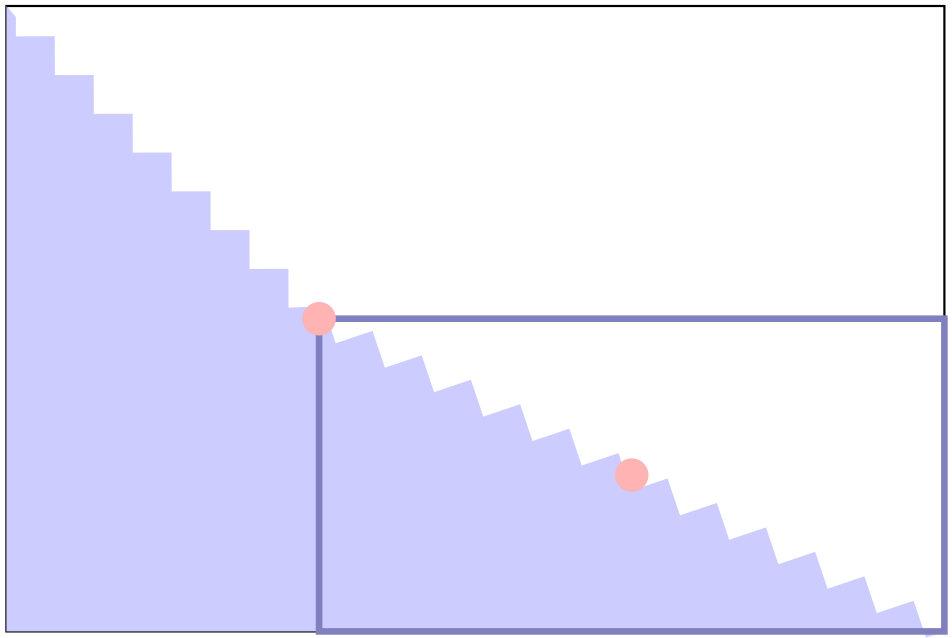
1. q-Stirling formula (again!)
2. Markov property
3. Scaling argument

Markov property



Knowing the beginning of the curve, the probability of going S or E depends only on the position of the tip.

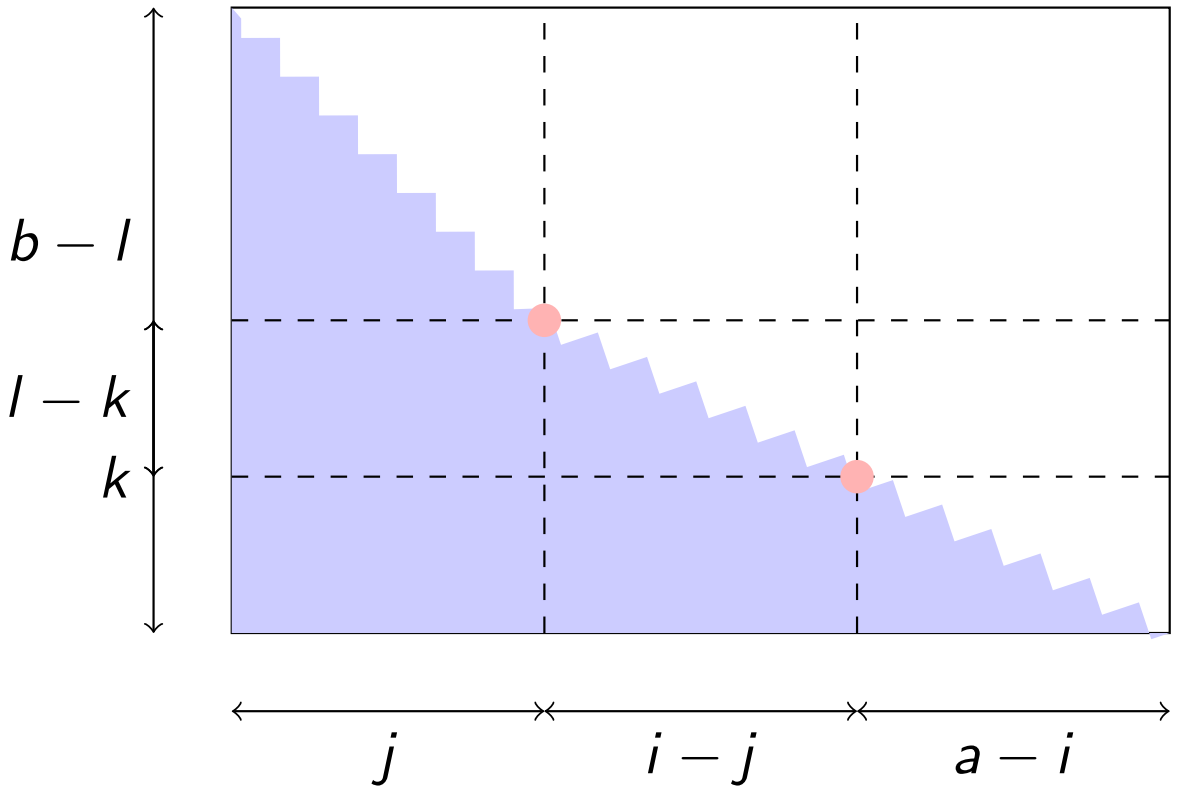
Scaling argument



Conditioning on the first point, the position of the second is reduced to the location of the interface in a smaller box.

2-point correlation function

$$\mathbf{P}(Y_j = k, Y_i = l) = ?$$



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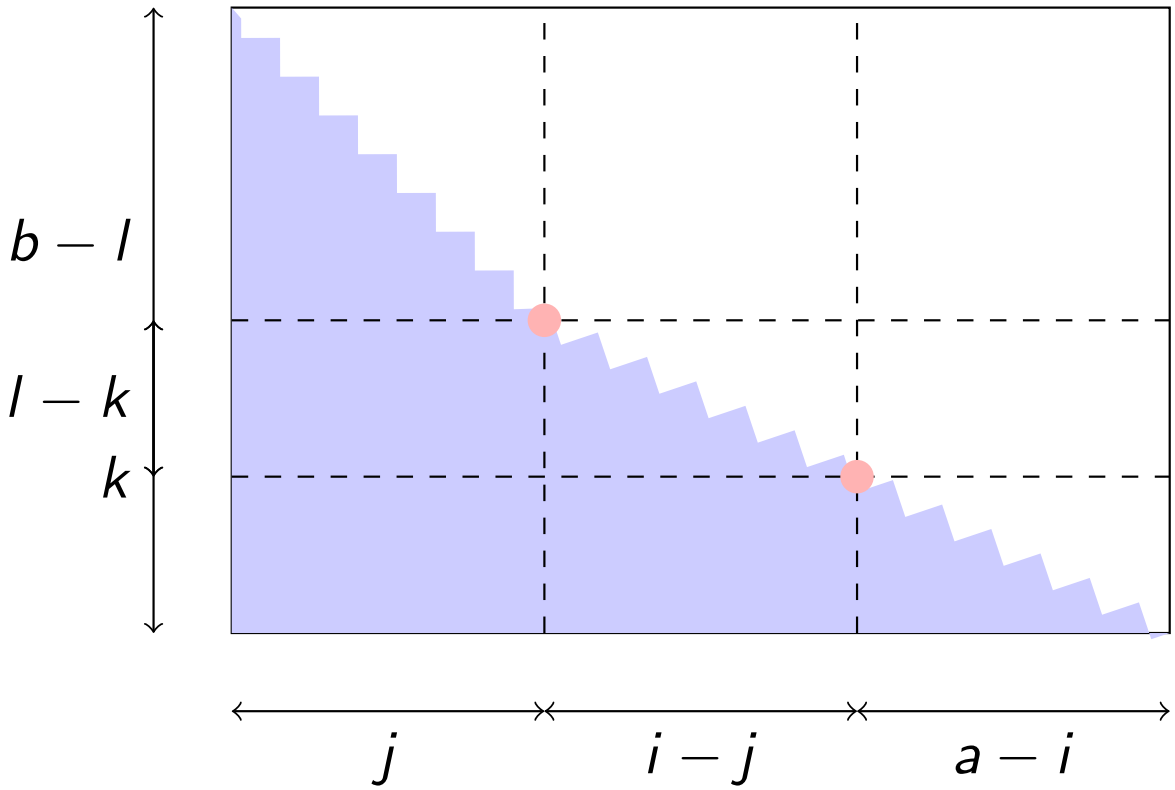


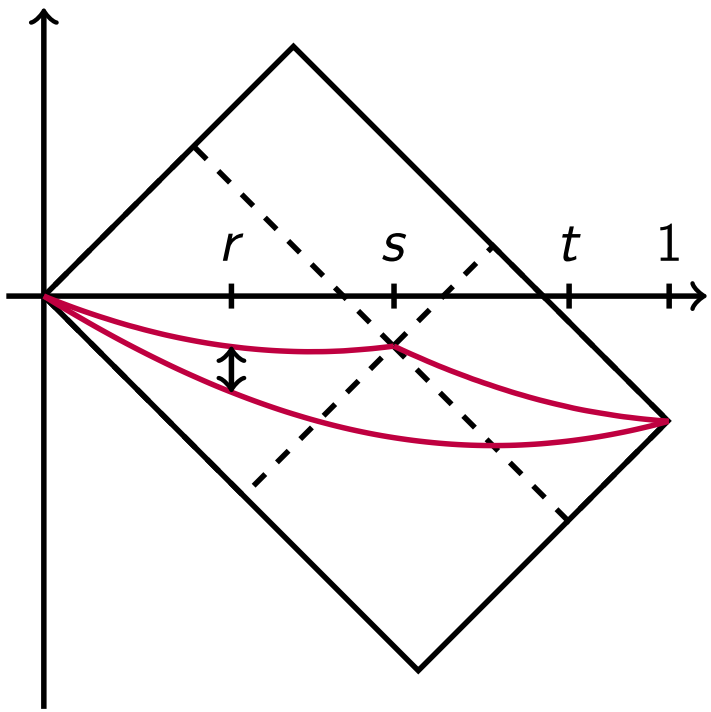
Markov property + rescaling: automatically all fi.di. marginals.

Tightness

Billingsley criterion: there exists $C > 0$ s.t for all $0 \leq r \leq s \leq t \leq 1$, all $\alpha > 0$, and all n

$$\mathbf{P}(|\tilde{X}_s - \tilde{X}_r| \geq \alpha; |\tilde{X}_t - \tilde{X}_s| \geq \alpha) \leq \frac{C(t-r)^2}{\alpha^4}$$

control of the "sticking condition"



a little bit further

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  - clue: conditionning the unconstrained fluctuation process, get Pittel's result