Asymptotics of dimers on tori and cylinders

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Outline

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Dimers

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Finite undirected graph $G = (V, E)$

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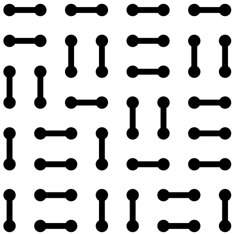
A **dimer configuration** or **perfect matching** on G is a subset $\mathfrak{m} \subseteq E$ such that every $v \in V$ is covered by exactly one $e \in \mathfrak{m}$

Dominoes

Dimer configuration on region in $\mathbb{Z}^2$



Dominoes

Dimer configuration on region in $\mathbb{Z}^2 \longleftrightarrow$ domino tiling

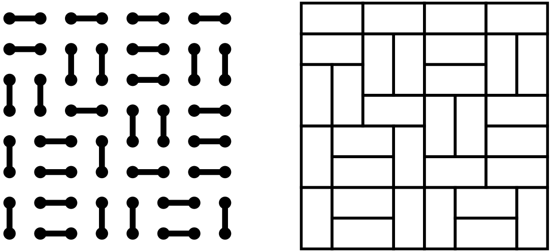
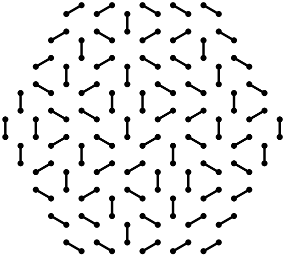
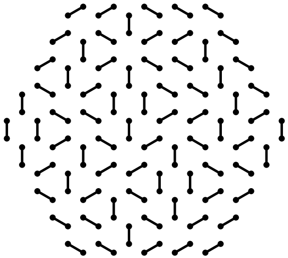


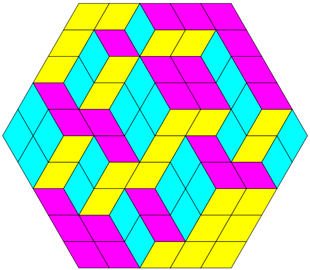
Figure: Kenyon PCMI '07

Dimer configuration on region in hexagonal lattice



Dimer configuration on region in hexagonal lattice
$\longleftrightarrow$ lozenge tiling





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[If $\nu(e) \equiv 1$ (unweighted), Z counts perfect matchings of G]

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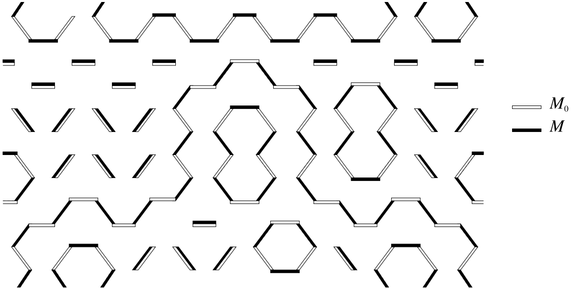

Figure: Boutillier–de Tilière AOP '09

Dimers on bipartite lattices: height function

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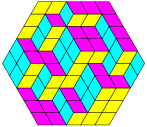

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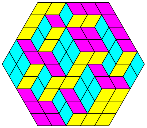

Defined on general bipartite graph using representation of $\mathfrak{m}$ as a black-to-white flow

[Levitov PRL '90, Zheng and Sachdev PRB '89, Blöte and Hilhorst JPA '82, Thurston AMM '90]

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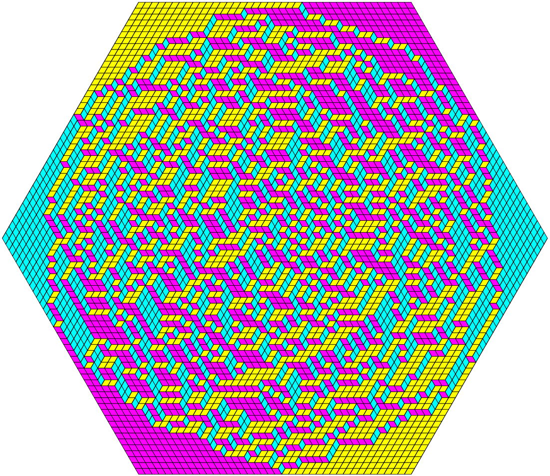

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- Conformal invariance of double-dimer loops: Kenyon '11

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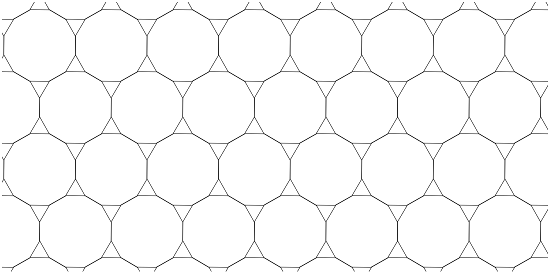

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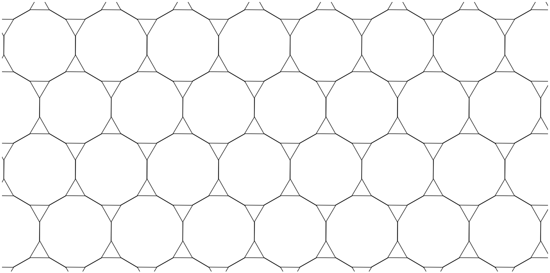

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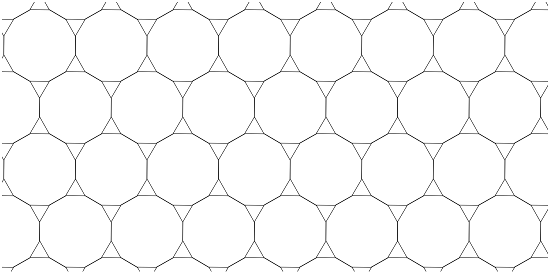
edges within small triangles: weight 1
remaining edges: weight a (symmetric)



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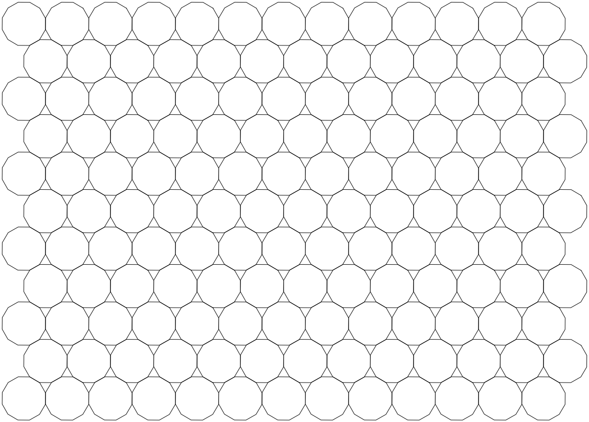

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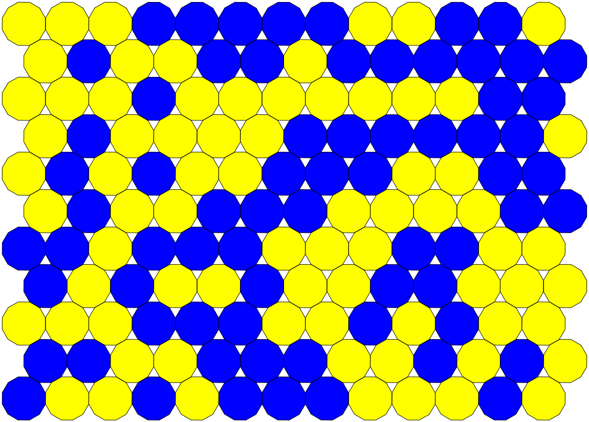

Ising spins on triangular lattice

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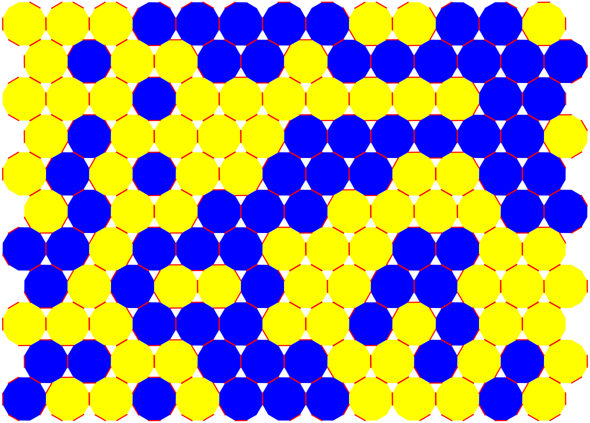

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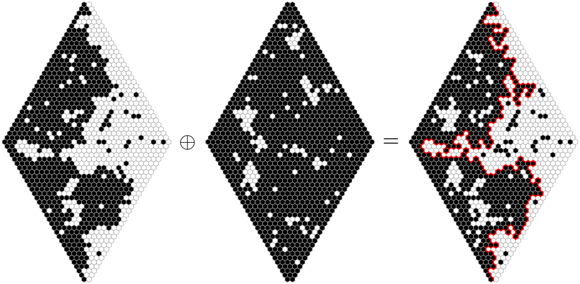

Figure: Wilson '11

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- It is an open problem to develop a mathematical understanding of this phenomenon

Outline

1. Introduction: the dimer model
2. The dimer partition function
3. Some open questions

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— including phase transitions and critical phenomena

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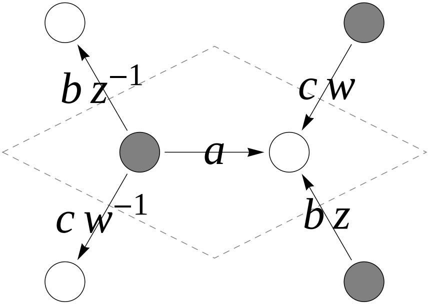
Definition:
$P(z, w) \equiv \det K(z, w)$, **where**
$K(z, w) \equiv$ **Fourier transform of** K

Dimer characteristic polynomial: example



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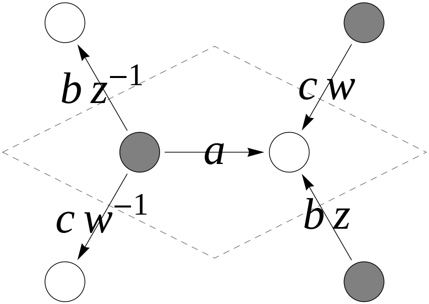

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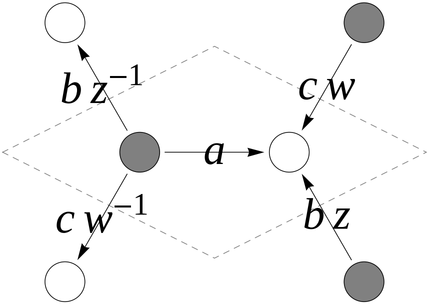

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$$P(z,w) = \det K(z,w) = |a + bz + cw|^2$$

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Critical dimer models exhibit long-range correlations
(correlations with polynomial rather than exponential decay)

[see e.g. Kenyon–Okounkov–Sheffield Annals '06]

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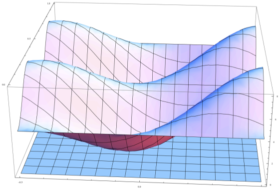

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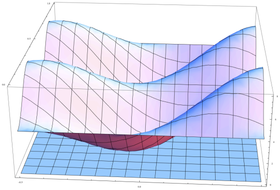

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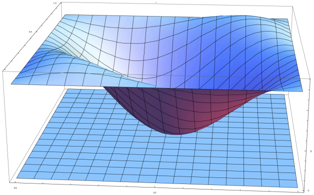

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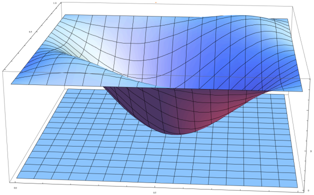

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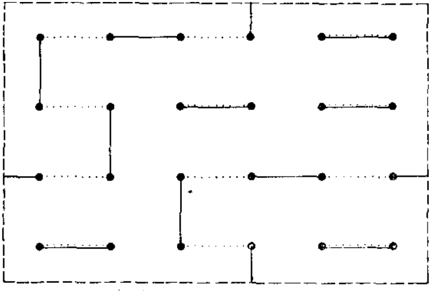

Figure: Kasteleyn Physica '61

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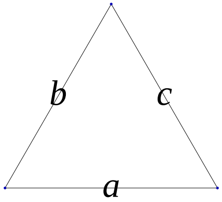
Triangular lattice with weights a, b, c (off-critical)



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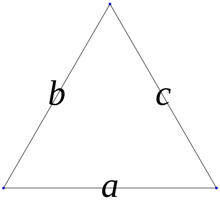

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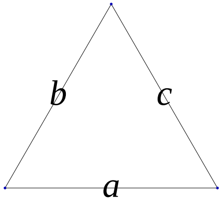

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$$\mathbb{P}^{\mathsf{dd}}(\mathbf{L}\text{ even}) - 1/2 \sim C_a m^2 \left|\frac{a^2-1}{2}\right|^{2n}$$

If $a = \sqrt{3}$ (critical), $\mathbb{P}^{\mathsf{dd}}(\mathbf{L}\text{ even}) - \mathbb{P}^{\mathsf{dd}}(\mathbf{L}\text{ odd})$ converges to

Cylinder loop statistics: Fisher lattice

Application 2:
Symmetric Fisher lattice on cylinder:

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$$\mathbb{P}^{\text{dd}}(\mathbf{L}\text{ even}) - 1/2 \sim C_a m^2 \left|\frac{a^2-1}{2}\right|^{2n}$$

If $a = \sqrt{3}$ (critical), $\mathbb{P}^{\text{dd}}(\mathbf{L}\text{ even}) - \mathbb{P}^{\text{dd}}(\mathbf{L}\text{ odd})$ converges to

$$\frac{\sum_{j\in 2\mathbb{Z}} Q^{j^2} - \sum_{j\in 2\mathbb{Z}+1} Q^{j^2}}{\sum_{j\in\mathbb{Z}} Q^{j^2}}, \quad Q \equiv (e^{-\pi/\sqrt{3}})^{m/n}$$

Outline

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- On the cylinder when $\mathbb{P}^{\mathrm{dd}}(\mathbf{L} \text{ even}) \to 1$, does $\mathbf{L} \to 0$ in probability?

Thank you!