# Finite size Emptiness Formation Probability for the XXZ spin chain at $\Delta = -1/2$

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# Outline

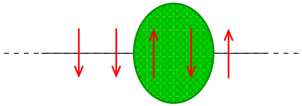
- Introduction
  - Definition of the XXZ spin chain
  - Combinatorics at $\Delta = -\frac{1}{2}$, Spin and Link-Pattern basis
  - Emptiness Formation Probability (EFP): statement of the result
- How to tackle the computation of the EFP
  - Integrability of the XXZ spin chain, inhomogenous version, exchange (qKZ) equations
  - Definition of an *inhomogeneous version* of the EFP and its characterising properties
  - Determinantal expression of the inhomogenous EFP
  - $t$-specialization and a NICE determiantal evaluation
- Conclusion and perspectives

# XXZ spin chain

Hamiltonian of the XXZ spin chain [Heisenberg '29, Bethe '31, ...]

$$H_N(\Delta) = -\frac{1}{2}\sum_{i=1}^{N} \sigma_x^i \sigma_x^{i+1} + \sigma_y^i \sigma_y^{i+1} + \Delta \sigma_z^i \sigma_z^{i+1} \qquad \Delta = -\frac{q+q^{-1}}{2}$$

This hamiltonian acts on a space $\mathcal{H}_N = (\mathbb{C}^2)^{\otimes N}$ and models a chain of spins with a nearest neighbor magnetic interaction.



The operators $\sigma_\nu^i$ acts locally on the $i$-th component of the tensor product $\mathcal{H}_N$

$$\sigma_\nu^i = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \underbrace{\sigma_\nu}_{i} \mathbf{1} \otimes \cdots \otimes \mathbf{1}$$

# XXZ spin chain

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

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Physicist are usually interested in:

- Ground state and more generally spectrum and eigenvectors;
- Observables: correlation functions at zero or finite temperature, entanglement entropy, etc.

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Thanks to the Integrability of the XXZ spin chain several quantities can be computed analytically in the thermodynamic limit $n \to \infty$.

# XXZ spin chain at $\Delta = -\frac{1}{2}$

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- $N$ odd and Periodic boundary conditions: $\sigma^{\alpha}_{N+1} = \sigma^{\alpha}_{1}$
  - Two-fold degenerate ground states $\Psi^{\pm}_{N=2n+1}$,
  - Energy: $E = -\frac{3N}{4}$,
  - Rational-valued components.

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  - Rational-valued components.
- $N$ even and Twisted periodic boundary conditions: $\sigma^{z}_{N+1} = \sigma^{z}_{1}$ and $\sigma^{\pm}_{N+1} = e^{\pm i \frac{2\pi}{3}} \sigma^{\pm}_{N+1}$ (where $\sigma^{\pm} = \sigma^{x} + \pm i \sigma^{y}$)
  - A single ground state $\Psi^{e}_{N=2n}$,
  - Energy: $E = -\frac{3N}{4}$,
  - Complex valued components.

# XXZ spin chain at $\Delta = -\frac{1}{2}$, combinatorics

Some exact result at finite size [Razumov, Stroganov; Batchelor, de Gier, Nienhius]: normalize the smallest component $\Psi_{\uparrow\cdots\uparrow\downarrow\cdot\downarrow} = 1$

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- Odd size $(N = 2n + 1)$
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- Even size ($N = 2n$)
  - Largest Component $\Psi^{+}_{\uparrow\downarrow\cdots\uparrow\downarrow}(2n) = \left(\frac{2}{\sqrt{3}}\right)^{n-1}\sqrt{A_{HT}(2n)}$
  - Sum of Components $\sum_I \Psi_I(2n) = (3)^{n/2} A_n$

Where:

- $A_n = \#$ of $n \times n$ Alternating Sign Matrices
- $A_{HT}(N) = \#$ of $N \times N$ Half-Turn Symmetric Alternating Sign Matrices

# XXZ spin chain in the Loop Basis

Defining the Temperley-Lieb generators

$$e_i = \frac{1}{2}(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z - \frac{1}{2}(q - q^{-1})(\sigma_i^z - \sigma_{i+1}^z) + \frac{1}{2})$$

The hamiltonian can be written as

$$H_N(\Delta) = -\sum_{i=1}^{N} \left( e_i + \frac{1}{2} \right)$$

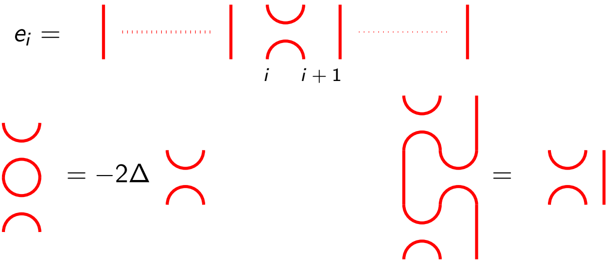
The $e_i$ satisfy the relations

$$e_i^2 = -2\Delta e_i$$

$$e_i e_{i\pm1} e_i = e_i$$

$$[e_i, e_j] = 0 \quad \text{for} \quad |i - j| \geq 2$$

The Temperley-Lieb algebra has a nice graphical representation




Each closed loop has a weight $-2\Delta$.

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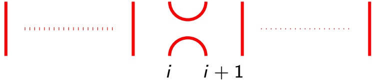
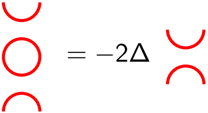
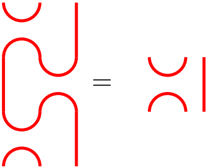
$e_i =$ 


Each closed loop has a weight $-2\Delta$.
The graphical presentation suggests a natural representation on a vector space whose basis are labeled by *link patterns*

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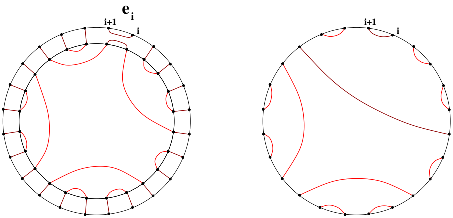

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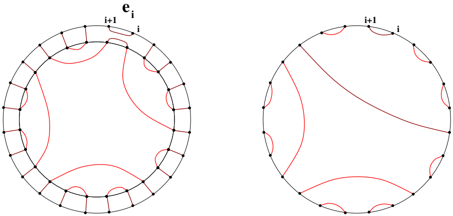



*The ground state equation becomes the equation at the core of the Razumov–Stroganov correspondence* [See Sportiello's talk]

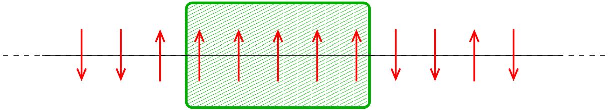
$$\sum_{i=1}^{N} (e_i - 1)\,\Psi = 0$$

**Theorem [L.C, A. Sportiello]**

The components of $\Psi$ enumerate FPL on certain domains, refined by their link pattern.

# Emptiness Formation Probability (EFP)

A quite non trivial correlation function, whose exact finite size expression has been conjectured by Razumov and Stroganov, is the Emptiness Formation Probability (EFP), namely the probability that the spins between two given position along the chain are all pointing in the up (down) direction



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- Multiple integral representation for $\Delta > -1$ [Jimbo & coll., Maillet & coll.]
- Asymptotics: Gaussian decay [Conjectured by Korepin et al.]

$$E(k) \sim k^{-\gamma} C^{-k^2}$$

Confirmed for

- $\Delta = 0$: $\gamma = \frac{1}{4}$, $C = \sqrt{2}$ [Shiroishi et al.]
- $\Delta = -\frac{1}{2}$: $\gamma = \frac{5}{36}$, $C = \frac{8}{3\sqrt{3}}$ [Kitanine et al.]

# Emptiness Formation Probability (EFP)

The formal definition of the EFP for $\mu = \{\pm, e\}$ is

$$E_N^\mu(k) = \frac{\left((\Psi_N^\mu)^*, \prod_{i=1}^k p_i^+ \cdot \Psi_N^\mu\right)}{((\Psi_N^\mu)^*, \Psi_N^\mu)}, \qquad p_i^+ = \frac{\sigma_i^z + \mathbf{1}}{2}$$

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Since in the even size case the ground state is complex valued we can define also a "Pseudo" (unphysical) EFP

$$E_{2n}^{\tilde{e}}(k) = \frac{\left(\Psi_{2n}^{e}, \prod_{i=1}^{k} p_i^+ \cdot \Psi_{2n}^{e}\right)}{(\Psi_{2n}^{e}, \Psi_{2n}^{e})}.$$

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- They have an analytical, "factorized" form!
- The Pseudo EFP is given by a ratio of enumerations of Cyclically Symmetric Self Complementary Plane Partion with a star shaped hole

# Emptiness Formation Probability (EFP)

**Theorem[L.C., conjectured by Razumov, Stroganov '00]**

$$\frac{E^{-}_{2n+1}(k-1)}{E^{-}_{2n+1}(k)} = \frac{(2k-2)!(2k-1)!(2n+k)!(n-k)!}{(k-1)!(3k-2)!(2n-k+1)!(n+k-1)!}$$

$$\frac{E^{+}_{2n+1}(k-1)}{E^{+}_{2n+1}(k)} = \frac{(2k-2)!(2k-1)!(2n+k)!(n-k+1)!}{(k-1)!(3k-2)!(2n-k+1)!(n+k)!}$$

$$\frac{E^{e}_{2n}(k-1)}{E^{e}_{2n}(k)} = \frac{(2k-2)!(2k-1)!(2n+k-1)!(n-k)!}{(k-1)!(3k-2)!(2n-k)!(n+k-1)!}$$

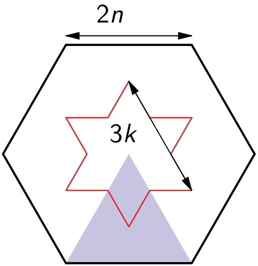
$$\frac{E^{\tilde{e}}_{2n}(k-1)}{E^{\tilde{e}}_{2n}(k)} = -q\frac{(2k-2)!(2k-1)!(2n+k-1)!(n-k)!}{(k-1)!(3k-3)!(3k-1)(2n-k)!(n+k-1)!}$$

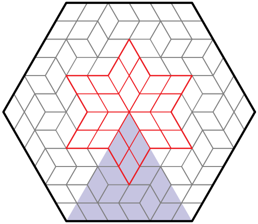
# Cyclically Symm. Self Compl. Plane Partitions

**Proposition**

$$\frac{E_{2n}^{\tilde{e}}(k-1)}{E_{2n}^{\tilde{e}}(k)} = -q\frac{CSSC(2n,k-1)}{CSSC(2n,k)}$$

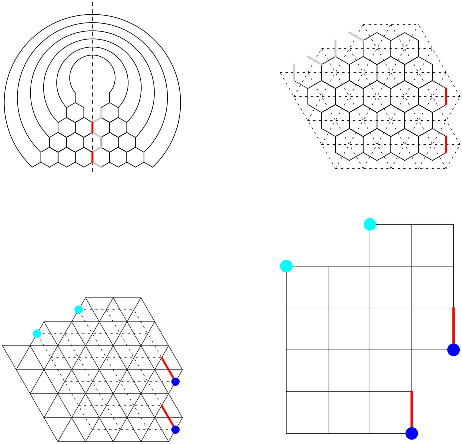
$CSSC(2n,k) = \#$ of lozange tilings of a regular hexagon of side length $2n$, which are invariant under a rotation of $\pi/3$ and have a star shaped frozen region of length $k$, $(CSSC(2n) := CSSC(2n,0) = CSSC(2n,1))$



# Cyclically Symm. Self Compl. Plane Partitions

The proof of the proposition is a simple application of a nice result of Ciucu ['97] about enumerations of dimer coverings of planar graphs with vertical symmetry.



# Cyclically Symm. Self Compl. Plane Partitions

Using the Lindström-Gessel-Viennot theorem one finds

$$CSSC(2n,k) = \det[Q_{i+k,j+k}]_{1\le i,j\le n-k}$$

with

$$Q_{i,j} = \binom{i+j-2}{2j-i-2} + \frac{1}{2}\binom{i+j-2}{2j-i-1}$$

and then open "Advanced Determinant Calculus", Theorem 40

$$CSSC(2n,k) = \prod_{j=1}^{n-k} \frac{(j-1)!(j+2k-1)!((3j+3k-2)!)^2}{(2j+k-1)!(2j+k-2)!(2j+3k-1)!(2j+3k-2)!}$$

# Some partial results

- Asymptotics $N \to \infty$ [Maillet et al. '02]

$$\lim_{N\to\infty} E_N(k) = \left(\frac{\sqrt{3}}{2}\right)^{3k^2} \prod_{j=1}^{k} \frac{\Gamma(j-1/3)\Gamma(j+1/3)}{\Gamma(j-1/2)\Gamma(j+1/2)}.$$

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- Norm [Di Francesco et al. '06]

$$E^{\pm}_{2n+1}(n) = A_{HT}(2n+1)^{-1} = \prod_{j=1}^{n} \frac{(2j-1)!^2(2j)!^2}{(j-1)!j!(3j-1)!(3j)!}$$

$$E^{\tilde{e}}_{2n}(n) = (-q)^n A_n^{-2} = (-q)^n CSSC(2n)^{-1}$$

# Integrability

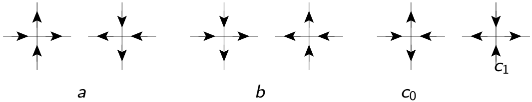
R-matrix: $\mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$

$$
\begin{array}{cc}
 & \begin{array}{cccc} \uparrow_i,\uparrow_j & \uparrow_i,\downarrow_j & \downarrow_i,\uparrow_j & \downarrow_i,\downarrow_j \end{array} \\
R_{i,j}(z) = & \begin{pmatrix} a(z) & 0 & 0 & 0 \\ 0 & b(z) & c_1(z) & 0 \\ 0 & c_0(z) & b(z) & 0 \\ 0 & 0 & 0 & a(z) \end{pmatrix} \begin{array}{c} \uparrow_i,\uparrow_j \\ \uparrow_i,\downarrow_j \\ \downarrow_i,\uparrow_j \\ \downarrow_i,\downarrow_j \end{array}
\end{array}
$$

with

$$a(z) = \frac{qz - q^{-1}}{q - q^{-1}z}, \qquad b(x) = \frac{z-1}{q - q^{-1}z}, \qquad c_i(z) = \frac{(q - q^{-1})z^i}{q - q^{-1}z}$$

The R-matrix encodes the Boltzmann weights of the configurations of the Six-Vertex model



# Integrability

Twist matrix

$$\Omega(\phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

$R$-$\Omega$ commutation

$$[R_{i,j}(x), \Omega_i(\phi) \otimes \Omega_j(\phi)] = 0$$

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Yang-Baxter equation

$$R_{i,j}(z_i/z_j)R_{i,k}(z_i/z_k)R_{j,k}(z_j/z_k) = R_{j,k}(z_j/z_k)R_{i,k}(z_i/z_k)R_{i,j}(z_i/z_j)$$

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Transfer matrix

$$T_N(y|\mathbf{z}_{\{1,\ldots,N\}}, \phi) = \mathrm{tr}_0\left[R_{0,1}(y/z_1)R_{0,2}(y/z_2)\ldots R_{0,N}(y/z_N)\Omega_0(\phi)\right]$$

# Integrability

Thanks to YBE and $R$-$\Omega$ commutation, the transfer matrices with different "$y$" commute

$$[T_N(y|\mathbf{z}_{\{1,\dots,N\}},\phi), T_N(y'|\mathbf{z}_{\{1,\dots,N\}},\phi)] = 0$$

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The hamiltonian of the XXZ spin chain is the logarithmic derivative of the transfer matrix at $z_i = 1$

$$\frac{1}{T_N(1|\mathbf{1},\phi)}\frac{dT_N(y|\mathbf{1},\phi)}{dy}\bigg|_{y=1} = -\frac{1}{q-q^{-1}}\left(H_N(\Delta) - \frac{3N}{2}\Delta\right).$$

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**Crucial observation [Razumov, Stroganov, Zinn-Justin]**

Consider the ground state with spectral parameters has a very simple eigenvalue $\lambda(y|\mathbf{z}_{\{1,\dots,N\}}) = \prod_{i=1}^{N}(a(y/z_i) + b(y/z_i))$

$$\Psi_N^\mu \rightarrow \Psi_N^\mu(\mathbf{z}), \qquad T_N(y|\mathbf{z}_{\{1,\dots,N\}},\phi)\Psi_N^\mu(\mathbf{z}) = \lambda(y)\Psi_N^\mu(\mathbf{z})$$

# q–Knizhnik–Zamolodchikov (qKZ) equations

[Razumov, Stroganov & Zinn-Justin '07]
From the previous observations and the YBE it follows that

- The ground state is polynomial in the spectral parameters.

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From the previous observations and the YBE it follows that

- The ground state is polynomial in the spectral parameters.
- The ground state satisfies a specialization $q = e^{2\pi i/3}$ of the $U_q(\hat{sl_2})$ qKZ equations

$$\check{R}_{i,i+1}(z_{i+1}/z_i)\Psi_N^\mu(\dots, z_i, z_{i+1}, \dots) = \Psi_N^\mu(\dots, z_{i+1}, z_i, \dots)$$

$$\sigma\Psi_N^\mu(z_1, z_2, \dots, z_{N-1}, z_N) = D\Psi_N^\mu(z_2, \dots, z_{N-1}, z_N, sz_1).$$

With

$$\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_{N-1} \otimes v_N) = v_2 \otimes \cdots \otimes v_{N-1} \otimes v_N \otimes v_1$$

$$\check{R}_{i,i+1}(z) = P_{i,i+1}R_{i,i+1}(z) \qquad P_{i,j}(e_i^\mu \otimes e_j^\nu) = e_i^\nu \otimes e_j^\mu,$$

$$s = q^6, \qquad D = q^{3N} q^{3(s_N^z+1)/2}.$$

# Main Idea: Inhomogenous EFP

Using the solution of the $U_q(\hat{sl_2})$ qKZ equations at level 1.

**we define an inhomogenous version of the EFP**

$$\mathcal{E}_N^{\mu}(k; \mathbf{y}_{\{1,\dots,2k\}}; \mathbf{z}_{\{1,\dots,N-k\}}) \sim$$

$$\frac{\left(\mathcal{P}_{N,k}(\mathbf{z})(\Psi_N^{\mu}(q^{-6}\mathbf{y}_{\{k+1,\dots,2k\}};\mathbf{z}))^*, \prod_{i=1}^k p_i^+ \Psi_N^{\mu}(\mathbf{y}_{\{1,\dots,k\}};\mathbf{z})\right)}{\prod_{1\le i<j\le k}(qy_i - q^{-1}y_j)(qy_{i+k}-q^{-1}y_{j+k})}$$

with $\mathcal{P}_{N,k}(\mathbf{z}) = \prod_{i=k+1}^N (z_i p_i^+ + p_i^-)$.

**and of the Pseudo EFP**

$$\mathcal{E}_N^{\tilde{e}}(k; \mathbf{y}_{\{1,\dots,2k\}}; \mathbf{z}_{\{1,\dots,N-k\}}) \sim$$

$$\frac{\left(\Psi_N^{\mu}(k;q^{-6}\mathbf{y}^{-1}_{\{k+1,\dots,2k\}};\mathbf{z}^{-1}), \prod_{i=1}^k p_i^+ \Psi_N^{\mu}(k;\mathbf{y}_{\{1,\dots,k\}};\mathbf{z})\right)}{\prod_{1\le i<j\le k}(qy_i - q^{-1}y_j)(qy_{i+k}-q^{-1}y_{j+k})}$$

# Main Idea: Inhomogenous EFP

Using the solution of the $U_q(\hat{sl}_2)$ qKZ equations at level 1.

**we define an inhomogenous version of the EFP**

$$\mathcal{E}_N^{\mu}(k;\mathbf{y}_{\{1,\dots,2k\}};\mathbf{z}_{\{1,\dots,N-k\}}) \sim$$

$$\frac{\left(\mathcal{P}_{N,k}(\mathbf{z})(\Psi_N^{\mu}(q^{-6}\mathbf{y}_{\{k+1,...,2k\}};\mathbf{z}))^*,\prod_{i=1}^k p_i^+\Psi_N^{\mu}(\mathbf{y}_{\{1,...,k\}};\mathbf{z})\right)}{\prod_{1\le i<j\le k}(qy_i-q^{-1}y_j)(qy_{i+k}-q^{-1}y_{j+k})}$$

with $\mathcal{P}_{N,k}(\mathbf{z}) = \prod_{i=k+1}^{N}(z_i p_i^+ + p_i^-)$.

**and of the Pseudo EFP**

$$\mathcal{E}_N^{\tilde{e}}(k;\mathbf{y}_{\{1,\dots,2k\}};\mathbf{z}_{\{1,\dots,N-k\}}) \sim$$

$$\frac{\left(\Psi_N^{\mu}(k;q^{-6}\mathbf{y}^{-1}_{\{k+1,...,2k\}};\mathbf{z}^{-1}),\prod_{i=1}^k p_i^+\Psi_N^{\mu}(k;\mathbf{y}_{\{1,...,k\}};\mathbf{z})\right)}{\prod_{1\le i<j\le k}(qy_i-q^{-1}y_j)(qy_{i+k}-q^{-1}y_{j+k})}$$

We shall compute these and then take the specialization $z_i = y_j = 1$.

# Properties of $\mathcal{E}^{\mu}_{N}(k; \mathbf{y}; \mathbf{z})$ and of $\mathcal{E}^{\tilde{e}}_{2n}(k; \mathbf{y}; \mathbf{z})$

1 Symmetries under $z_i \leftrightarrow z_j$, $y_i \leftrightarrow y_j$.

# Properties of $\mathcal{E}_N^{\mu}(k;\mathbf{y};\mathbf{z})$ and of $\mathcal{E}_{2n}^{\tilde{e}}(k;\mathbf{y};\mathbf{z})$

1. Symmetries under $z_i \leftrightarrow z_j$, $y_i \leftrightarrow y_j$.
2. Specialization $z_i = 0$ or $z_i \to \infty$

$$\lim_{z_{2n+1}\to\infty} z_{2n+1}^{-2n}\mathcal{E}_{2n+1}^{-}(k;\mathbf{y};\mathbf{z}) \sim \mathcal{E}_{2n}^{e}(k;\mathbf{y};\mathbf{z}\setminus z_{2n+1})$$

$$\mathcal{E}_{2n+2}^{e}(k;\mathbf{y};\mathbf{z})|_{z_{2n+2}=0} \sim \mathcal{E}_{2n+1}^{+}(k;\mathbf{y};\mathbf{z}\setminus z_{2n+2})$$

# Properties of $\mathcal{E}_N^{\mu}(k;\mathbf{y};\mathbf{z})$ and of $\mathcal{E}_{2n}^{\tilde{e}}(k;\mathbf{y};\mathbf{z})$

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3. Recursion relation for $z_j = q^{\pm 2} z_i$

$$\frac{\mathcal{E}_N^{\mu}(k;\mathbf{y};\mathbf{z})|_{z_j=q^2 z_i}}{\mathcal{E}_{N-2}^{\mu}(k;\mathbf{y};\mathbf{z}\setminus\{z_i,z_j\})} \sim$$

$$z_i \prod_{\ell=1}^{2k} \frac{q y_\ell - q^{-1} z_i}{q - q^{-1}} \prod_{\substack{1\le \ell \le N-k \\ \ell \ne i,j}} \frac{(q z_\ell - q^{-1} z_i)(q^3 z_i - q^{-1} z_\ell)}{-(q-q^{-1})^2}$$

# Properties of $\mathcal{E}^{\mu}_{N}(k;\mathbf{y};\mathbf{z})$ and of $\mathcal{E}^{\tilde{e}}_{2n}(k;\mathbf{y};\mathbf{z})$

1 Symmetries under $z_i \leftrightarrow z_j$, $y_i \leftrightarrow y_j$.

2 Specialization $z_i = 0$ or $z_i \to \infty$

$$\lim_{z_{2n+1}\to\infty} z_{2n+1}^{-2n}\mathcal{E}^{-}_{2n+1}(k;\mathbf{y};\mathbf{z}) \sim \mathcal{E}^{e}_{2n}(k;\mathbf{y};\mathbf{z}\setminus z_{2n+1})$$

$$\mathcal{E}^{e}_{2n+2}(k;\mathbf{y};\mathbf{z})|_{z_{2n+2}=0} \sim \mathcal{E}^{+}_{2n+1}(k;\mathbf{y};\mathbf{z}\setminus z_{2n+2})$$

3 Recursion relation for $z_j = q^{\pm 2} z_i$

$$\frac{\mathcal{E}^{\tilde{e}}_{2n}(k;\mathbf{y};\mathbf{z})|_{z_j=q^2 z_i}}{\mathcal{E}^{\tilde{e}}_{2n-2}(k;\mathbf{y};\mathbf{z}\setminus\{z_i,z_j\})} \sim$$

$$\prod_{\ell=1}^{2k}\frac{qy_\ell - q^{-1}z_i}{q-q^{-1}}\prod_{\substack{1\le\ell\le 2n-k\\ \ell\neq i,j}}\frac{(qz_\ell - q^{-1}z_i)(q^3 z_i - q^{-1}z_\ell)}{-(q-q^{-1})^2}$$

# Properties of $\mathcal{E}_N^{\mu}(k;\mathbf{y};\mathbf{z})$ and of $\mathcal{E}_N^{\tilde{e}}(k;\mathbf{y};\mathbf{z})$

4 Degree in $z_i$ = less than $2n \pm 1$.

# Properties of $\mathcal{E}_N^{\mu}(k;\mathbf{y};\mathbf{z})$ and of $\mathcal{E}_N^{\tilde{e}}(k;\mathbf{y};\mathbf{z})$

4. Degree in $z_i$ = less than $2n \pm 1$.
5. Factorized cases

$$\mathcal{E}_{2k}^{e/\tilde{e}}(k;\mathbf{y};\mathbf{z}) = \prod_{1\le i<j\le k} \frac{(qz_i - q^{-1}z_j)(qz_j - q^{-1}z_i)}{(q-q^{-1})^2}$$

$$\mathcal{E}_{2k+1}^{+}(k+1;\mathbf{y};\mathbf{z}) = \prod_{1\le i<j\le k} \frac{(qz_i - q^{-1}z_j)(qz_j - q^{-1}z_i)}{(q-q^{-1})^2} \prod_{i=1}^{k} z_i^2$$

$$\mathcal{E}_{2k+1}^{-}(k;\mathbf{y};\mathbf{z}) = \prod_{1\le i<j\le k+1} \frac{(qz_i - q^{-1}z_j)(qz_j - q^{-1}z_i)}{(q-q^{-1})^2}.$$

These properties completely determine $\mathcal{E}_N^{\mu}(k;\mathbf{y};\mathbf{z})$ and $\mathcal{E}_{2n}^{\tilde{e}}(k;\mathbf{y};\mathbf{z})$!

# Combinatorial point $q = e^{2\pi i/3}$, $k = 0$

Let us concentrate on $\mathcal{E}_N^{\tilde{e}}(k; \mathbf{y}; \mathbf{z})$. For $q = e^{2\pi i/3}$ the recursion becomes

$$\frac{\mathcal{E}_{2n}^{\tilde{e}}(k; \mathbf{y}; \mathbf{z})|_{z_j = q^2 z_i}}{\mathcal{E}_{2n-2}^{\tilde{e}}(k; \mathbf{y}; \mathbf{z} \setminus \{z_i, z_j\})} \sim \left( \prod_{\substack{1 \le \ell \le 2n-k \\ \ell \neq i,j}} \frac{q z_\ell - q^{-1} z_i}{q - q^{-1}} \right)^2$$

This is the square of the recursion relation satisfied by the Schur polynomial $S_{\lambda(2n,0)}(\mathbf{z})$ corresponding to the Young Tableau

$$\lambda(m, r) = \{\lfloor \frac{r}{2} \rfloor, \lfloor \frac{r+1}{2} \rfloor, \dots, \lfloor \frac{r+i-1}{2} \rfloor, \dots, \lfloor \frac{r+m-1}{2} \rfloor\}$$

# Combinatorial point $q = e^{2\pi i/3}$, generic $k$

For generic $k$ the product $S_{\lambda(2n,0)}(\mathbf{z}, \mathbf{y}_I) S_{\lambda(2n,0)}(\mathbf{z}, \mathbf{y}_{I^c})$ satisfies the recursion relation but: it has the "wrong" initial conditons and moreover the degree and symmetries in $y$ are wrong.

# Combinatorial point $q = e^{2\pi i/3}$, generic $k$

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Is it possible to take linear combination of $S_{\lambda(2n,0)}(\mathbf{z}, \mathbf{y}_I) S_{\lambda(2n,0)}(\mathbf{z}, \mathbf{y}_{I^c})$ for different $I$ such that one has the right initial conditions?

# Combinatorial point $q = e^{2\pi i/3}$, generic $k$

Let $\tilde\rho, \tilde\sigma$ be strictly increasing infinite sequences of nonnegative integers. Introduce the following matrices

$$
\mathcal{M}^{(\tilde\rho,\tilde\sigma)}(r,s;\mathbf{y};\mathbf{z}) =
\begin{pmatrix}
z_1^{\tilde\rho_1} & z_1^{\tilde\rho_2} & \dots & z_1^{\tilde\rho_{r+s}} & 0 & 0 & \dots & 0 \\
z_2^{\tilde\rho_1} & z_2^{\tilde\rho_2} & \dots & z_2^{\tilde\rho_{r+s}} & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
z_r^{\tilde\rho_1} & z_r^{\tilde\rho_2} & \dots & z_r^{\tilde\rho_{r+s}} & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & z_1^{\tilde\sigma_1} & z_1^{\tilde\sigma_2} & \dots & z_1^{\tilde\sigma_{r+s}} \\
0 & 0 & \dots & 0 & z_2^{\tilde\sigma_1} & z_2^{\tilde\sigma_2} & \dots & z_2^{\tilde\sigma_{r+s}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & z_r^{\tilde\sigma_1} & z_r^{\tilde\sigma_2} & \dots & z_r^{\sigma_{r+s}} \\
y_1^{\tilde\rho_1} & y_1^{\tilde\rho_2} & \dots & y_1^{\tilde\rho_{r+s}} & y_1^{\tilde\sigma_1} & y_1^{\tilde\sigma_2} & \dots & y_1^{\tilde\sigma_{r+s}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
y_{2s}^{\tilde\rho_1} & y_{2s}^{\tilde\rho_2} & \dots & y_{2s}^{\tilde\rho_{r+s}} & y_{2s}^{\tilde\sigma_1} & y_{2s}^{\tilde\sigma_2} & \dots & y_{2s}^{\tilde\sigma_{r+s}}
\end{pmatrix}
$$

# Combinatorial point $q = e^{2\pi i/3}$

Define the following polynomials

$$\mathcal{S}^{(\tilde\rho,\tilde\sigma)}(r,s;\mathbf{y};\mathbf{z}) = \frac{\det \mathcal{M}^{(\tilde\rho,\tilde\sigma)}(r,s;\mathbf{y};\mathbf{z})}{\prod_{1\le i<j\le r}(z_i-z_j)^2 \prod_{1\le i<j\le 2s}(y_i-y_j) \prod_{\substack{1\le i\le r\\ 1\le j\le 2s}}(z_i-y_j)}.$$

Using the Laplace expansion along the first $r+s$ columns we can write $\mathcal{S}^{(\tilde\rho,\tilde\sigma)}(r,s;\mathbf{y};\mathbf{z})$ as a bilinear in Schur polynomials

$$\mathcal{S}^{(\tilde\rho,\tilde\sigma)}(r,s;\mathbf{y};\mathbf{z}) =$$

$$\sum_{\substack{I\subset\{1,\dots 2s\}\\ |I|=s}} (-1)^{\epsilon(I)} \frac{\prod_{i<j\in I}(y_i-y_j)\prod_{i<j\in I^c}(y_i-y_j)}{\prod_{1\le i<j\le 2s}(y_i-y_j)} S_{\rho(r+s)}(\mathbf{z},\mathbf{y}_I) S_{\sigma(r+s)}(\mathbf{z},\mathbf{y}_{I^c}),$$

# Combinatorial point $q = e^{2\pi i/3}$

Now let us introduce the following family of integer sequences

$$\tilde{\lambda}_i(r) = \lfloor \frac{3i - 3 + r}{2} \rfloor, \qquad \begin{array}{c} \tilde{\lambda}(0) = \{0, 1, 3, 4, 6, 7, \dots\} \\ \tilde{\lambda}(1) = \{0, 2, 3, 5, 6, 8, \dots\} \\ \tilde{\lambda}(2) = \{1, 2, 4, 5, 7, 8, \dots\} \\ \dots \end{array}$$

## Theorem [L.C.]

$$\mathcal{E}^{-}_{2n+1}(k; \mathbf{y}; \mathbf{z}) \sim \mathcal{S}^{(\tilde{\lambda}(0),\tilde{\lambda}(1))}(2n + 1 - k, k; \mathbf{y}; \mathbf{z})$$

$$\mathcal{E}^{+}_{2n+1}(k; \mathbf{y}; \mathbf{z}) \sim \mathcal{S}^{(\tilde{\lambda}(1),\tilde{\lambda}(2))}(2n + 1 - k, k; \mathbf{y}; \mathbf{z})$$

$$\mathcal{E}^{e}_{2n}(k; \mathbf{y}; \mathbf{z}) \sim \mathcal{S}^{(\tilde{\lambda}(0),\tilde{\lambda}(1))}(2n - k, k; \mathbf{y}; \mathbf{z})$$

$$\mathcal{E}^{\tilde{e}}_{2n}(k; \mathbf{y}; \mathbf{z}) \sim \mathcal{S}^{(\tilde{\lambda}(0),\tilde{\lambda}(2))}(2n - k, k; \mathbf{y}; \mathbf{z})$$

# $t-$ specialization

Setting

$$\mathbf{z} = \{z_1 = t^0, z_2 = t^1, \dots, z_{N-k} = t^{N-k-1}\}$$

and

$$\mathbf{y} = \{y_1 = t^{N-k}, \dots, y_{2k} = t^{N+k-1}\}$$

the functions $\mathcal{E}_N^{\mu}(k; \mathbf{y}; \mathbf{z})$ and $\mathcal{E}_{2n}^{\tilde{e}}(k; \mathbf{y}; \mathbf{z})$, as rational functions of $t$, have simple factors.

Let us look at an example

# $t-$ specialization

$$\mathcal{M}^{(\tilde\lambda(0),\tilde\lambda(1))}(3,2;\mathbf{z}(t);t^3\mathbf{y}(t)) =$$

$$\begin{pmatrix}
t^{0\cdot 0} & t^{1\cdot 0} & t^{3\cdot 0} & t^{4\cdot 0} & t^{6\cdot 0} & 0 & 0 & 0 & 0 & 0 \\
t^{0\cdot 1} & t^{1\cdot 1} & t^{3\cdot 1} & t^{4\cdot 1} & t^{6\cdot 1} & 0 & 0 & 0 & 0 & 0 \\
t^{0\cdot 2} & t^{1\cdot 2} & t^{3\cdot 2} & t^{4\cdot 2} & t^{6\cdot 2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t^{0\cdot 0} & t^{2\cdot 0} & t^{3\cdot 0} & t^{5\cdot 0} & t^{6\cdot 0} \\
0 & 0 & 0 & 0 & 0 & t^{0\cdot 1} & t^{2\cdot 1} & t^{3\cdot 1} & t^{5\cdot 1} & t^{6\cdot 1} \\
0 & 0 & 0 & 0 & 0 & t^{0\cdot 2} & t^{2\cdot 2} & t^{3\cdot 2} & t^{5\cdot 2} & t^{6\cdot 2} \\
t^{0\cdot 3} & t^{1\cdot 3} & t^{3\cdot 3} & t^{4\cdot 3} & t^{6\cdot 3} & t^{0\cdot 3} & t^{2\cdot 3} & t^{3\cdot 3} & t^{5\cdot 3} & t^{6\cdot 3} \\
t^{0\cdot 4} & t^{1\cdot 4} & t^{3\cdot 4} & t^{4\cdot 4} & t^{6\cdot 4} & t^{0\cdot 4} & t^{2\cdot 4} & t^{3\cdot 4} & t^{5\cdot 4} & t^{6\cdot 4} \\
t^{0\cdot 5} & t^{1\cdot 5} & t^{3\cdot 5} & t^{4\cdot 5} & t^{6\cdot 5} & t^{0\cdot 5} & t^{2\cdot 5} & t^{3\cdot 5} & t^{5\cdot 5} & t^{6\cdot 5} \\
t^{0\cdot 6} & t^{1\cdot 6} & t^{3\cdot 6} & t^{4\cdot 6} & t^{6\cdot 6} & t^{0\cdot 6} & t^{2\cdot 6} & t^{3\cdot 6} & t^{5\cdot 6} & t^{6\cdot 6}
\end{pmatrix}$$

# $t-$ specialization

$$\mathcal{M}^{(\tilde\lambda(0),\tilde\lambda(1))}(3,2;\mathbf{z}(t);t^3\mathbf{y}(t)) =$$

$$\begin{pmatrix}
v_1^0 & t^{1\cdot0} & v_2^0 & t^{4\cdot0} & v_3^0 & 0 & 0 & 0 & 0 & 0 \\
v_1^1 & t^{1\cdot1} & v_2^1 & t^{4\cdot1} & v_3^1 & 0 & 0 & 0 & 0 & 0 \\
v_1^2 & t^{1\cdot2} & v_2^2 & t^{4\cdot2} & v_3^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & v_1^0 & t^{2\cdot0} & v_2^0 & t^{5\cdot0} & v_3^0 \\
0 & 0 & 0 & 0 & 0 & v_1^1 & t^{2\cdot1} & v_2^1 & t^{5\cdot1} & v_3^1 \\
0 & 0 & 0 & 0 & 0 & v_1^2 & t^{2\cdot2} & v_2^2 & t^{5\cdot2} & v_3^2 \\
v_1^3 & t^{1\cdot3} & v_2^3 & t^{4\cdot3} & v_3^3 & v_1^3 & t^{2\cdot3} & v_2^3 & t^{5\cdot3} & v_3^3 \\
v_1^4 & t^{1\cdot4} & v_2^4 & t^{4\cdot4} & v_3^4 & v_1^4 & t^{2\cdot4} & v_2^4 & t^{5\cdot4} & v_3^4 \\
v_1^5 & t^{1\cdot5} & v_2^5 & t^{4\cdot5} & v_3^5 & v_1^5 & t^{2\cdot5} & v_2^5 & t^{5\cdot5} & v_3^5 \\
v_1^6 & t^{1\cdot6} & v_2^6 & t^{4\cdot6} & v_3^6 & v_1^6 & t^{2\cdot6} & v_2^6 & t^{5\cdot6} & v_3^6
\end{pmatrix}$$

$v_1 = 1$, $v_2 = t^3$, $v_3 = t^3$.

# $t-$ specialization

$$\mathcal{M}^{(\tilde\lambda(0),\tilde\lambda(1))}(3,2;\mathbf{z}(t);t^3\mathbf{y}(t)) =$$

$$\begin{pmatrix}
t^{0\cdot 0} & {a_1}^0 & t^{3\cdot 0} & (\lambda a_1)^0 & t^{6\cdot 0} & 0 & 0 & 0 & 0 & 0 \\
t^{0\cdot 1} & {a_1}^1 & t^{3\cdot 1} & (\lambda a_1)^1 & t^{6\cdot 1} & 0 & 0 & 0 & 0 & 0 \\
t^{0\cdot 2} & {a_1}^2 & t^{3\cdot 2} & (\lambda a_1)^2 & t^{6\cdot 2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t^{0\cdot 0} & {a_2}^0 & t^{3\cdot 0} & (\lambda a_2)^0 & t^{6\cdot 0} \\
0 & 0 & 0 & 0 & 0 & t^{0\cdot 1} & {a_2}^1 & t^{3\cdot 1} & (\lambda a_2)^1 & t^{6\cdot 1} \\
0 & 0 & 0 & 0 & 0 & t^{0\cdot 2} & {a_2}^2 & t^{3\cdot 2} & (\lambda a_2)^2 & t^{6\cdot 2} \\
t^{0\cdot 3} & {a_1}^3 & t^{3\cdot 3} & (\lambda a_1)^3 & t^{6\cdot 3} & t^{0\cdot 3} & {a_2}^3 & t^{3\cdot 3} & (\lambda a_2)^3 & t^{6\cdot 3} \\
t^{0\cdot 4} & {a_1}^4 & t^{3\cdot 4} & (\lambda a_1)^4 & t^{6\cdot 4} & t^{0\cdot 4} & {a_2}^4 & t^{3\cdot 4} & (\lambda a_2)^4 & t^{6\cdot 4} \\
t^{0\cdot 5} & {a_1}^5 & t^{3\cdot 5} & (\lambda a_1)^5 & t^{6\cdot 5} & t^{0\cdot 5} & {a_2}^5 & t^{3\cdot 5} & (\lambda a_2)^5 & t^{6\cdot 5} \\
t^{0\cdot 6} & {a_1}^6 & t^{3\cdot 6} & (\lambda a_1)^6 & t^{6\cdot 6} & t^{0\cdot 6} & {a_2}^6 & t^{3\cdot 6} & (\lambda a_2)^6 & t^{6\cdot 6}
\end{pmatrix}$$

$\lambda = t^3, a_1 = t, a_2 = t^2.$

# A NICE determinantal evaluation

We are lead to consider the determinaint of a matrix of the form

$$\mathcal{G}^{(\ell,r,s)}(\mathbf{v};\lambda,a_1,a_2)=\begin{pmatrix} D^{(0)}_{\ell+r;\ell}(\mathbf{v}) & D^{(0)}_{\ell+r;r+s}(a_1\vec{\lambda}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D^{(0)}_{\ell+r;\ell}(\mathbf{v}) & D^{(0)}_{\ell+r;r+s}(a_2\vec{\lambda}) \\ D^{(\ell+r)}_{2s;\ell}(\mathbf{v}) & D^{(\ell+r)}_{2s;r+s}(a_1\vec{\lambda}) & D^{(\ell+r)}_{2s;\ell}(\mathbf{v}) & D^{(\ell+r)}_{2s;r+s}(a_2\vec{\lambda}) \end{pmatrix},$$

where $\mathbf{v}=\{v_1,\dots,v_\ell\}$, $a_i\vec{\lambda}=\{a_i,a_i\lambda,\dots,a_i\lambda^{r+s-1}\}$ and the blocks consist of the following rectangular matrices

$$D^{(j)}_{m;\ell}(\mathbf{v})=\begin{pmatrix} v_1^j & v_2^j & \dots & v_\ell^j \\ v_1^{j+1} & v_2^{j+1} & \dots & v_\ell^{j+1} \\ \vdots & \vdots & \ddots & \vdots \\ v_1^{j+m-1} & v_2^{j+m-1} & \dots & v_\ell^{j+m-1} \end{pmatrix}.$$

# A NICE determinantal evaluation

We have the following determinantal evaluation

## Proposition [L.C.]

$$\det \mathcal{G}^{(\ell,r,s)}(\mathbf{v};\lambda,a_1,a_2) = \prod_{1\le i,j\le \ell}(v_i - v_j)^2 \prod_{\alpha=1,2}\prod_{\substack{1\le i\le \ell\\ 1\le j\le r+s}}(v_i - \lambda^{j-1}a_\alpha)\det \mathcal{G}^{(0,r,s)}(\lambda,a_1,a_2),$$

with

$$\det \mathcal{G}^{(0,r,s)}(\lambda;a_1;a_2) = (a_1a_2)^{\binom{r+s}{2}} \prod_{1\le i,j\le s}(\lambda^{j-1}a_1 - \lambda^{i-1}a_2)\ \mathcal{D}^{(r,s)}(\lambda),$$

and

$$\mathcal{D}^{(r,s)}(\lambda) = \frac{(-1)^{s(r+s)}\lambda^{s\left(\binom{r}{2}-\binom{r+s}{2}\right)}\prod_{1\le i<j\le r}(\lambda^{j-1}-\lambda^{i-1})\prod_{1\le i<j\le r+2s}(\lambda^{j-1}-\lambda^{i-1})}{\prod_{1\le i,j\le s}(\lambda^{j+s-1}-\lambda^{i-1})}.$$

# $t-$ specialization

$$\begin{array}{rl} \mathcal{E}^{e}_{2n}(k;t) \to & \mathcal{M}^{(\tilde\lambda(0),\tilde\lambda(1))}(2n-k,k;\mathbf{z}(t);\mathbf{y}(t)) = \mathcal{G}^{(n,n-k,k)}(\{t^{3i-3}\};t^3,t,t^2) \\ \mathcal{E}^{\tilde e}_{2n}(k;t) \to & \mathcal{M}^{(\tilde\lambda(0),\tilde\lambda(2))}(2n-k,k;\mathbf{z}(t);\mathbf{y}(t)) = \mathcal{G}^{(n,n-k,k)}(\{t^{3i-2}\};t^3,1,t^2) \\ \mathcal{E}^{-}_{2n+1}(k;t) \to & \mathcal{M}^{(\tilde\lambda(0),\tilde\lambda(1))}(2n+1-k,k;\mathbf{z}(t);\mathbf{y}(t)) = \mathcal{G}^{(n+1,n-k,k)}(\{t^{3i-3}\};t^3,t,t^2) \\ \mathcal{E}^{+}_{2n+1}(k;t) \to & \mathcal{M}^{(\tilde\lambda(1),\tilde\lambda(2))}(2n+1-k,k;\mathbf{z}(t);\mathbf{y}(t)) = \mathcal{G}^{(n,n-k+1,k)}(\{t^{3i-2}\};t^3,1,t) \end{array}.$$

Using the above proposition we can write the $t$-evaluation of the inhomogenous EFP in terms of the usual $t$-numbers and $t$-factorial

$$[i]_t = \frac{t^i-1}{t-1} \quad \text{and} \quad [n]_t! = \prod_{i=1}^{n}[i]_t.$$

# $t-$ specialization: result

$$\frac{\mathcal{E}^{e}_{2n}(k-1;t)}{\mathcal{E}^{e}_{2n}(k;t)} \sim \frac{[2n+k-1]_t![n-k]_{t^3}![2k-1]_{t^3}![2k-2]_{t^3}!}{[2n-k]_t![n+k-1]_{t^3}![k-1]_{t^3}![3k-2]_t!}$$

$$\frac{\mathcal{E}^{\tilde{e}}_{2n}(k-1;t)}{(-q)\mathcal{E}^{\tilde{e}}_{2n}(k;t)} \sim \frac{[2n+k-1]_t![n-k]_{t^3}![2k-1]_{t^3}![2k-2]_{t^3}!}{[2n-k]_t![n+k-1]_{t^3}![k-1]_{t^3}![3k-3]_t![3k-1]_t}$$

$$\frac{\mathcal{E}^{-}_{2n+1}(k-1;t)}{\mathcal{E}^{-}_{2n+1}(k;t)} \sim \frac{[2n+k]_t![n-k]_{t^3}![2k-1]_{t^3}![2k-2]_{t^3}!}{[2n-k+1]_t![n+k-1]_{t^3}![k-1]_{t^3}![3k-2]_t!}$$

$$\frac{\mathcal{E}^{+}_{2n+1}(k-1;t)}{\mathcal{E}^{+}_{2n+1}(k;t)} \sim \frac{[2n+k]_t![n-k+1]_{t^3}![2k-1]_{t^3}![2k-2]_{t^3}!}{[2n-k+1]_t![n+k]_{t^3}![k-1]_{t^3}![3k-2]_t!}$$

The coefficients in front are of the form $t^{\beta}\left(\frac{[3]_t}{3}\right)^{k-1} \xrightarrow[t\to 1]{} 1.$

# What if $q$ is generic?

We have found a multiresidua formula

$$\mathcal{E}^{\mu}_{2n-\epsilon(\mu)}(k;\mathbf{y};\mathbf{z}) \sim$$

$$\oint_{\{qz_i\}} \prod_{\ell=1}^{n-k} \frac{dw_\ell w^{\alpha(\mu)} \prod_{j=1}^{2k}(w_\ell - y_j)}{2\pi i \prod_{i=1}^{2n-k-\epsilon(\mu)}(w_\ell - qz_i)(w_\ell - q^{-1}z_i)} K_{n-k}(\mathbf{w})$$

where $\epsilon(\pm) = \pm 1, \epsilon(e) = \epsilon(\tilde{e}) = 0$,

$$\alpha(+) = \alpha(e) = 0, \quad \alpha(\tilde{e}) = 1, \quad \alpha(-) = 2$$

and

$$K_N(\mathbf{w}) = \prod_{1\leq i<j\leq N} (w_i - w_j)^2 (qw_i - q^{-1}w_j)(qw_j - q^{-1}w_i).$$

# Perspectives

- Consider other correlation functions.
- Other boundary conditions (e.g. open chain with diagonal $K-$ matrices)
- Higher spins or also $U_q(sl_N)$ spin chain.

Thank you!