Combinatorics of the ASEP (asymmetric simple exclusion process)

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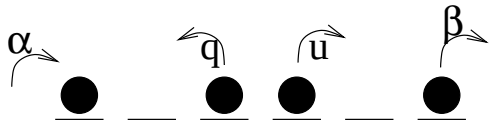
Based on joint works with Corteel, as well as Sasamoto, Stanley, and Stanton.

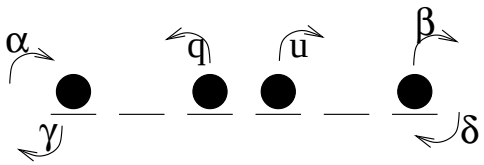
Program

I. Definition of the asymmetric exclusion process (ASEP)

II. The totally non-negative part of the Grassmannian

III. 

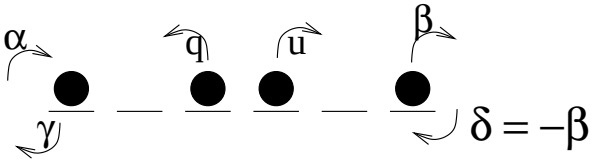
, permutation tableaux, permutations

IV. 

and staircase tableaux

V. Enumeration of staircase tableaux

VI. A physical interpretation of a nonsensical specialization:



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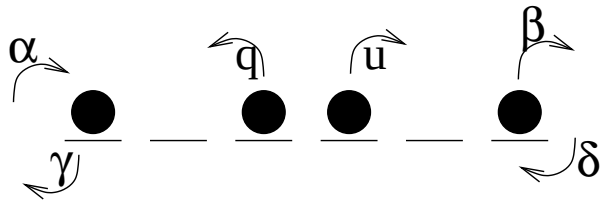
Part I: The Asymmetric Simple Exclusion Process

Fix a 1D lattice of n sites. Choose parameters $\alpha, \beta, \gamma, \delta, q$ in $[0, 1]$.

- Each site i is either occupied by a particle $\sigma_i = 1$ or empty $\sigma_i = 0$.



Let B_n be the set of all 2^n words of length n on letters $\{\circ, \bullet\}$.

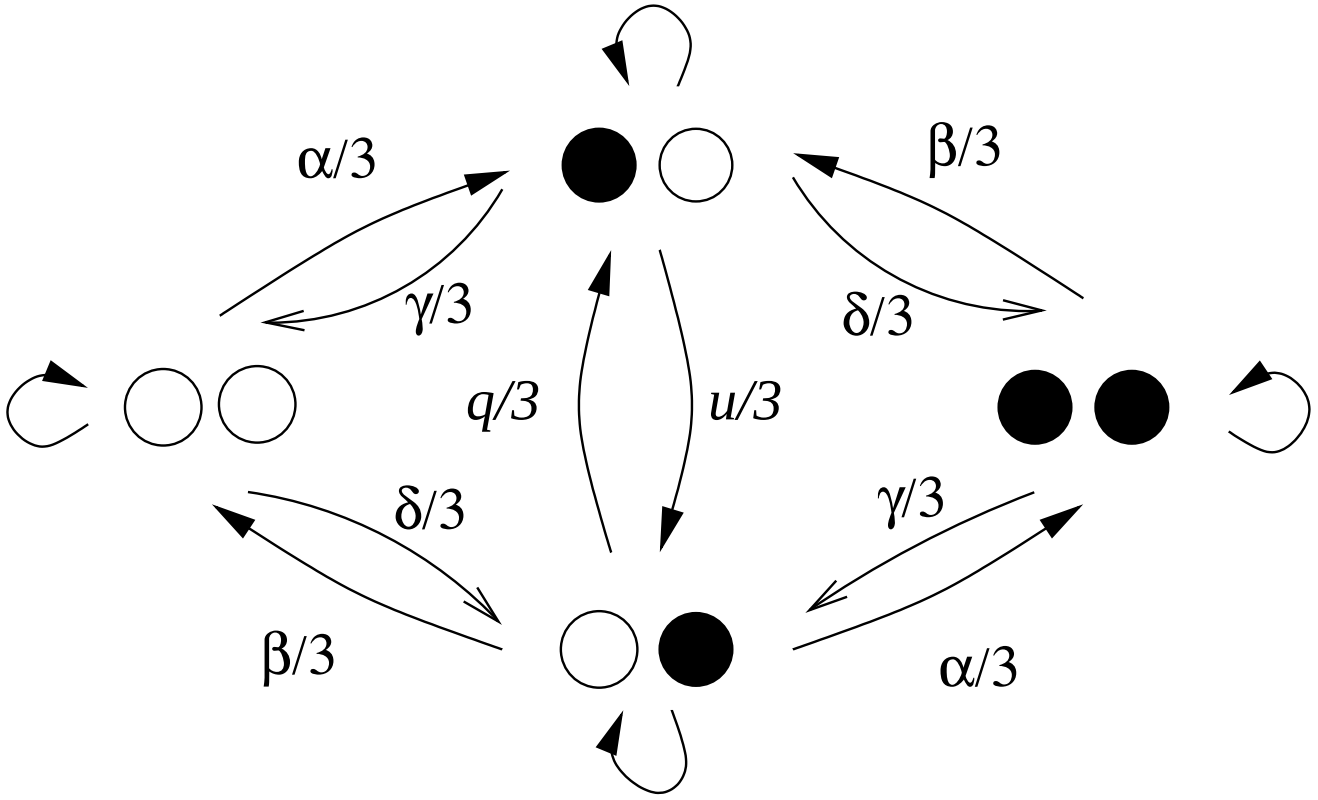
The ASEP is the Markov chain on B_n with transition probabilities:

- If $X = A\bullet\circ B$ and $Y = A\circ\bullet B$ then $P_{X,Y} = \frac{u}{n+1}$ and $P_{Y,X} = \frac{q}{n+1}$.
- If $X = \circ B$ and $Y = \bullet B$ then $P_{X,Y} = \frac{\alpha}{n+1}$ and $P_{Y,X} = \frac{\gamma}{n+1}$.
- If $X = B\bullet$ and $Y = B\circ$ then $P_{X,Y} = \frac{\beta}{n+1}$ and $P_{Y,X} = \frac{\delta}{n+1}$.
- Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.

(Explain the name ASEP)

The ASEP

The state diagram of the ASEP for $n = 2$.



The ASEP (finite lattice, open boundaries)

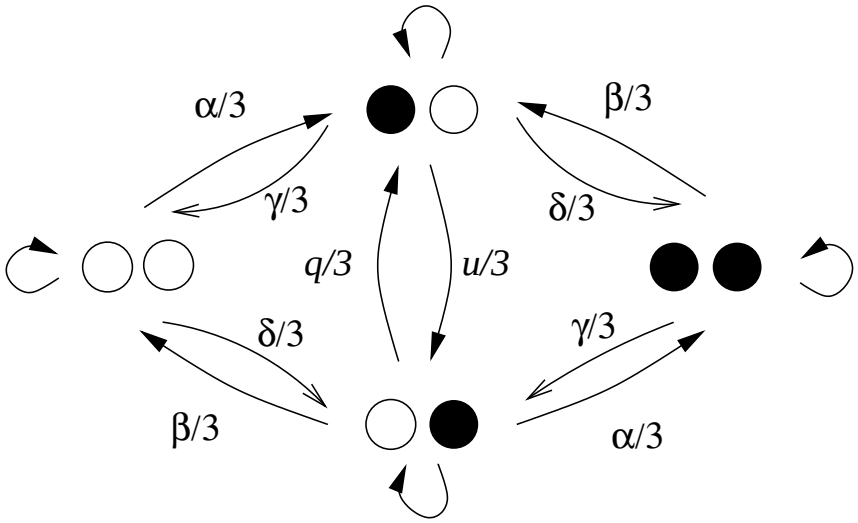
• This flavor of the ASEP was introduced by biologists (MacDonald, Gibbs, Pipkin, 1968); and independently by a mathematician (Spitzer, 1970). By now, > 400 papers on the arXiv!

Connections and applications:

- Biology: is a model for translation in protein synthesis
- Integrable systems: is closely connected to the XXZ model
- Statistical physics: is an example of a nonequilibrium system; exhibits phase transitions
- Special functions: connected to Askey-Wilson polynomials
- Combinatorics and probability: stationary distribution is connected to many combinatorial objects

Stationary Distribution of the ASEP

The ASEP has a unique stationary distribution – that is, it has a unique left eigenvector of the transition matrix associated with eigenvalue 1. This is called the steady state.



(Solve for prob.'s, say when $u = 1$, $\alpha = \beta = 1$, and $\gamma = \delta = 0$.)

Stationary distribution of the ASEP: comments & questions

The stationary distribution of the ASEP on a finite lattice with open boundaries is non-trivial:

- Unlike the ASEP on the infinite lattice $\mathbb{Z}$, the stationary distribution is in general not Bernoulli

A combinatorialist's perspective:

- Can one give a combinatorial formula for the stationary distribution?
- What are the appropriate combinatorial objects?

Analogies with the Razumov-Stroganov world

	ASEP	Razumov-Stroganov
Markov chain	on $\{\circ, \bullet\}^n$ (2^n)	on link patterns (C_n)
Symmetries	Particle-hole symmetry, left-right symmetry	Dihedral symmetry
Partition function	$(n+1)!$ if $\gamma = \delta = 0$; $4^n n!$ in general case	$\|ASM\|'s \leftrightarrow \|FPL\|$'s
Combinatorics	Permutation tableaux; Staircase tableaux	ASM's $\leftrightarrow$ FPL's
Lattice model	Connection to XXZ	Connection to XXZ
Geometry	Totally non-negative part of Grassmannian	Orbital varieties; Zinn-Justin, Knutson

Part II: The totally non-negative part of the Grassmannian

The totally non-negative part of the Grassmannian Gr_{kn}^{+} is the subset of the real Grassmannian with all Plucker coordinates non-negative.

Theory of total positivity for flag varieties (including Grassmannians) was developed by Lusztig; motivated by connections to his canonical bases.

Postnikov studied Gr_{kn}^{+} and found it had a nice cell decomposition. He showed that cells were in bijection with certain tableaux he called ⅃-diagrams.

The totally non-negative part of the Grassmannian

Definition: A JL-*diagram* is a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ (where $\lambda_i \geq 0$) together with a filling with 0's and +'s such that:

• There is no 0 which has a + above it in the same column *and* a + to its left in the same row.

+	+	**0**	**0**	+
0	**0**	+	**0**	
+	+	+	+	
0	**0**	+		

• JL-diagrams contained in $k \times (n-k)$ rectangle are in bijection with cells of Gr^+_{kn}. (Number of +'s $\leftrightarrow$ dimension of cell.)

• A JL-diagram $\mathcal{T}$ is called a *permutation tableaux* if each column contains at least one +. (These are in bijection with permutations.) Define $\mathrm{wt}(\mathcal{T}) = number(+'s) - number(columns)$.

Enumeration: q-Eulerian numbers

Theorem (W.) Enumerated cells in Gr^+_{kn}, by counting J-diagrams contained in a $k \times (n-k)$ rectangle, according to number of +'s.

What is the polynomial $\hat{E}_{k,n}(q)$ enumerating the perm. tableaux $\mathcal{T}$ contained in a $k \times (n-k)$ rectangle, according to $\mathrm{wt}(\mathcal{T})$?

Theorem (W.)

$$\hat{E}_{k,n}(q) = q^{k-k^2} \sum_{i=0}^{k-1} (-1)^i [k-i]_q{}^n q^{ki-k} \left(\binom{n}{i} q^{k-i} + \binom{n}{i-1} \right).$$

Here, $[i]_q := 1 + q + q^2 + \cdots + q^{i-1}$.

Remark. $\hat{E}_{k,n}(q)$ specializes at $q = -1, 0, 1$ to binomial numbers, Narayana numbers, and Eulerian numbers. It counts perms with k weak excedances in S_n according to crossings.

A first connection to ASEP

Theorem (Corteel): Let $\alpha = \beta = 1, \gamma = \delta = 0$. In the steady state, the probability that the ASEP with n sites is in a configuration with precisely k particles is:

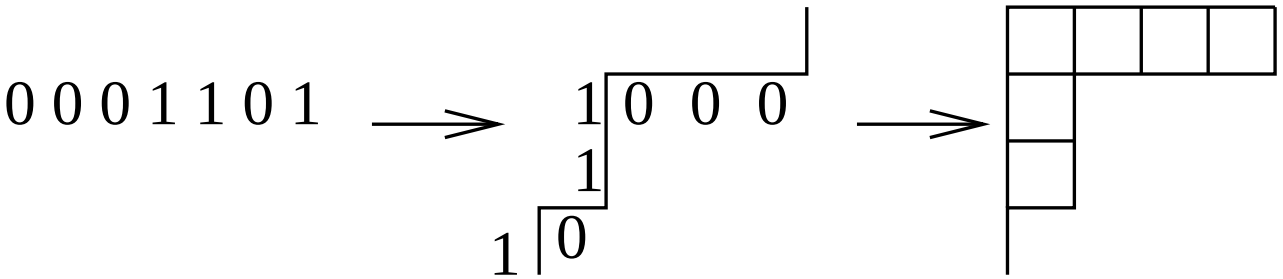
$$\frac{\hat{E}_{k+1,n+1}(q)}{Z_n}$$

Here, Z_n is the *partition function* for the ASEP – the sum of the probabilities of all possible states. So $Z_n = \sum_{k=0}^{n} \hat{E}_{k+1,n+1}(q)$.

Question: Corteel's result doesn't say anything about the *location* of the particles. How can we refine this result?

Part III: Permutation tableaux and the ASEP ($\gamma = \delta = 0$)

There is a easy bijection between words σ in $\{0,1\}^n$ and partitions of semiperimeter $n+1$ (where each column has length at least one):



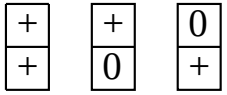
This associates the partition $\lambda(\sigma)$ to σ.

Theorem (Corteel, W). Let $\alpha = \beta = 1, \gamma = \delta = 0$. In the steady state, the probability that the ASEP is in configuration σ is:

$$\frac{\sum_{\mathcal{T}} q^{\mathrm{wt}(\mathcal{T})}}{Z_n}$$

where the sum is over all permutation tableaux of shape $\lambda(\sigma)$.

Example

State	Partition	Perm Tableaux	Weight	Probability
● ●			$q^0=1$	1/(q+5)
○ ●			$q^0=1$	1/(q+5)
○ ○			$q^0=1$	1/(q+5)
● ○			q +2	(q+2)/(q+5)

How to prove this: the Matrix Ansatz

Let $P_n(\sigma_1, \ldots, \sigma_n)$ be the prob. of configuration σ in stationarity.

Theorem: (Derrida, Evans, Hakim, Pasquier '93) Consider ASEP (with $\gamma = \delta = 0$). Suppose D and E are matrices, $|V\rangle$ a column vector, $\langle W|$ a row vector, such that:

$$DE - qED = D + E \tag{1}$$

$$\beta D|V\rangle = |V\rangle \tag{2}$$

$$\alpha\langle W|E = \langle W| \tag{3}$$

Then

$$P_n(\sigma_1, \ldots, \sigma_n) = \frac{\langle W|(\prod_{i=1}^{n}(\sigma_i D + (1-\sigma_i)E))|V\rangle}{\langle W|(D+E)^n|V\rangle}.$$

Ex. $P_n(1,1,0,0,1) \propto \langle W|DDEED|V\rangle$.

Proof that enumerating perm tableaux according to shape yields steady state probabilities

(Here $\alpha = \beta = 1, \gamma = \delta = 0$.)

Idea: Construct D, E, V, and W which satisfy the Matrix Ansatz, *and* enumerate perm tableaux in the following sense:

- D and E are "transfer matrices" for permutation tableaux.
- E.g. $\langle W|DDEED|V\rangle$ is the generating function for perm tableaux of "shape $DDEED$."

Recall: $[i] = 1 + q + \cdots + q^i$. Let $[i]^{(k)}$ denote the kth derivative of $[i]$ with respect to q.

When $\alpha = \beta = 1$, $\gamma = \delta = 0$, D is the (infinite) upper triangular matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

E is the (infinite) lower triangular matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & [2] & 0 & 0 & 0 & \dots \\ 1 & [3]' & [3] & 0 & 0 & \dots \\ 1 & \frac{[4]''}{2} & [4]' & [4] & 0 & \dots \\ 1 & \frac{[5]'''}{6} & \frac{[5]''}{2} & [5]' & [5] & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

$\langle W|$ is the row vector $(1, 0, 0, \dots)$, and
$|V\rangle$ is the column vector $(1, 1, 1, \dots)^t$.

Then $DE - qED = D + E, \quad D|V\rangle = |V\rangle, \quad \langle W|E = \langle W|.$

Furthermore, $\langle W|(\prod_{i=1}^{n}(\tau_i D + (1-\tau_i)E))|V\rangle$ enumerates permutation tableaux of shape $\lambda(\tau)$.

So the Matrix Ansatz implies:

the steady state prob. that the ASEP is in configuration τ

$\propto$ the generating function for perm-tableaux of shape $\lambda(\tau)$.

Result and proof generalize to the ASEP with α, β, q general, $\gamma = \delta = 0$. Just introduce two stats on perm-tableaux.

Recap of the result

Theorem (Corteel, W). Let $\gamma = \delta = 0$. In the steady state, the prob. that the ASEP on a lattice of n sites is in configuration σ is:

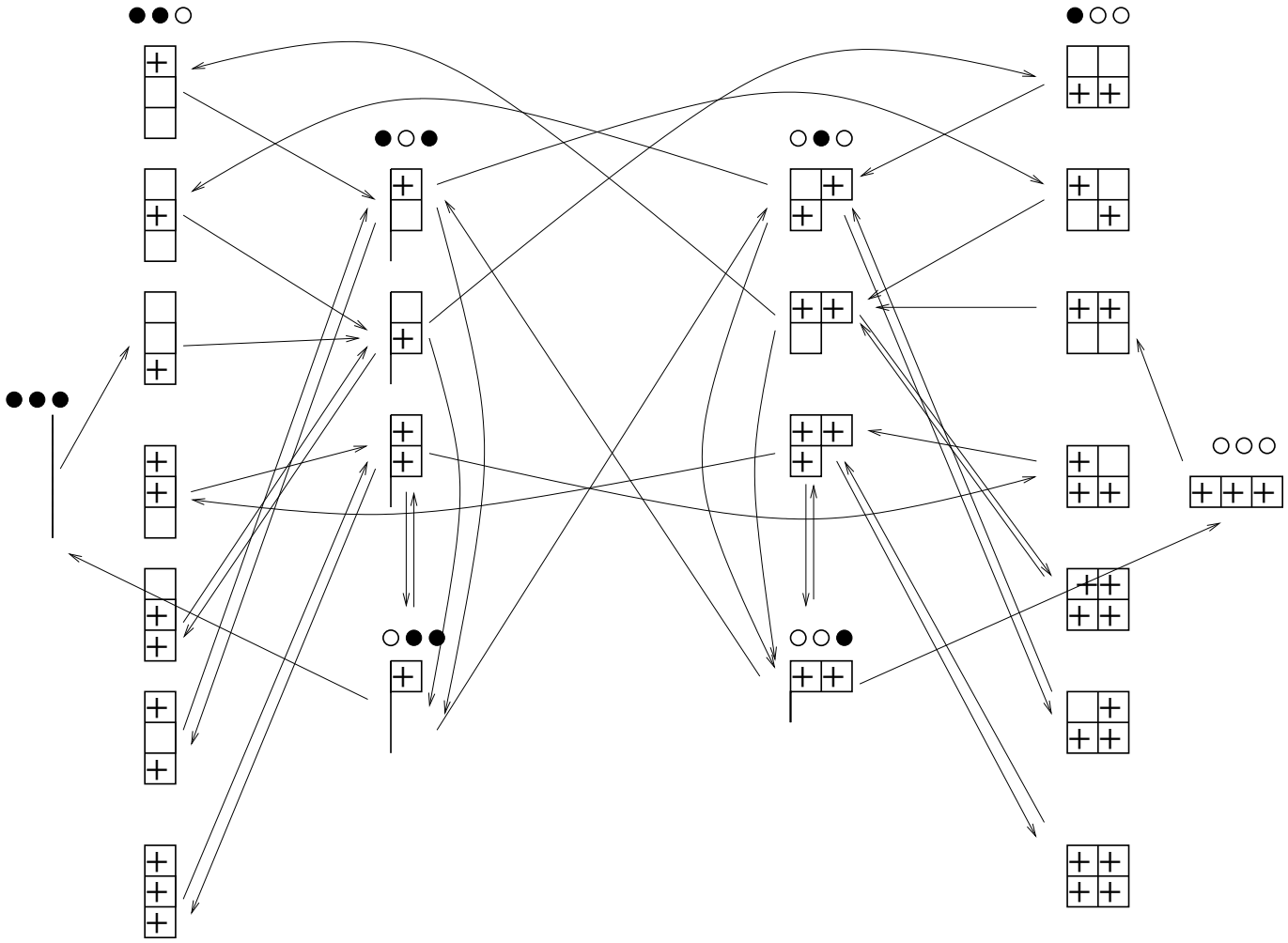
$$\frac{\sum_{\mathcal{T}} q^{\mathrm{wt}(\mathcal{T})}}{Z_n}$$

where the sum is over all permutation tableaux of shape $\lambda(\sigma)$.

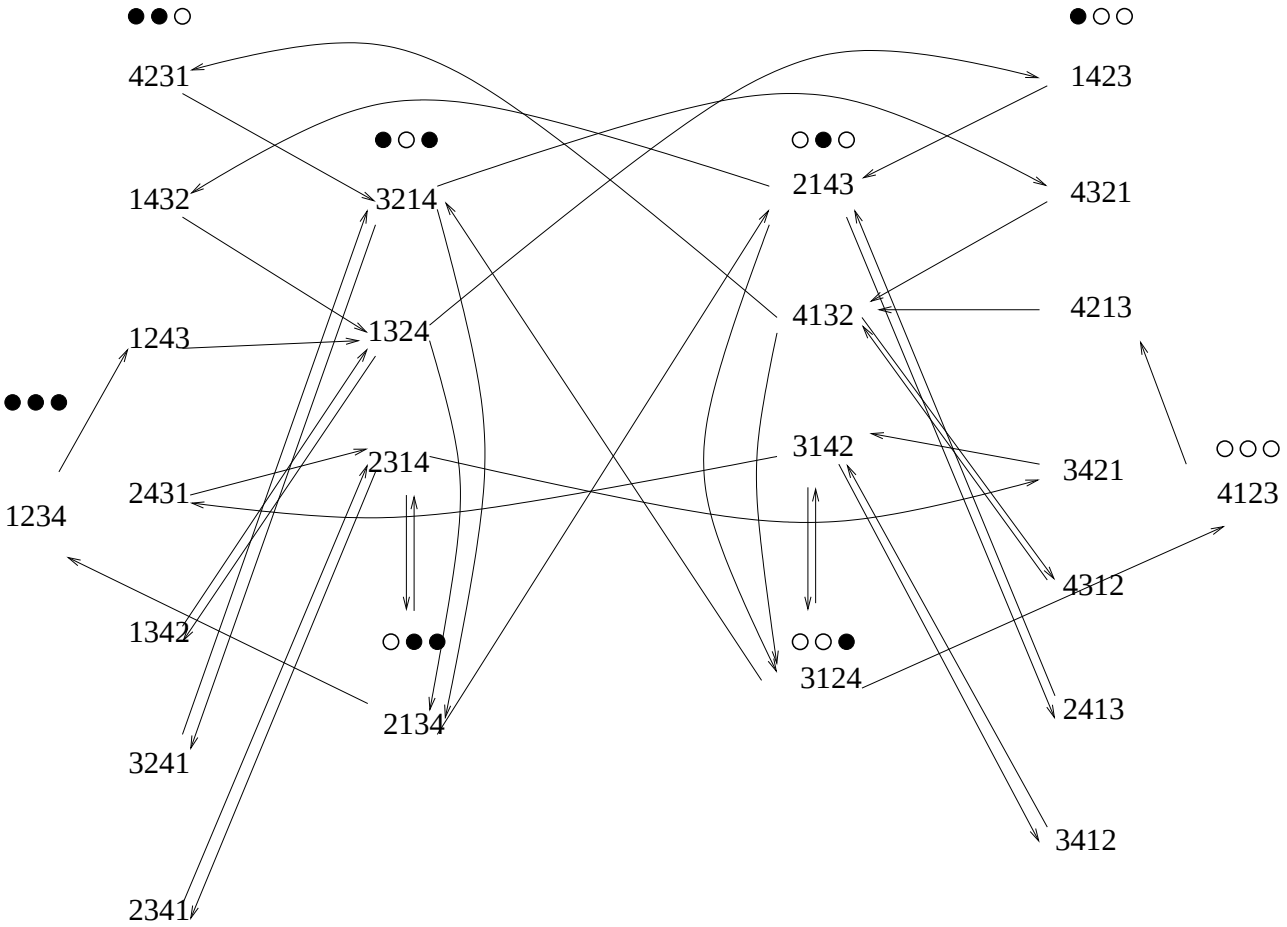
• Recall that permutation tableaux correspond to cells in Gr^+_{kn}. So this gives an indirect link between the Grassmannian and the ASEP. Is there a more direct connection?

• Is there a more intuitive way to understand why the stationary distribution of the ASEP has to do with permutation tableaux? (or permutations)? Is there a way to lift the dynamics of the ASEP to the tableaux themselves?

Lifting the ASEP dynamics to the perm-tableaux

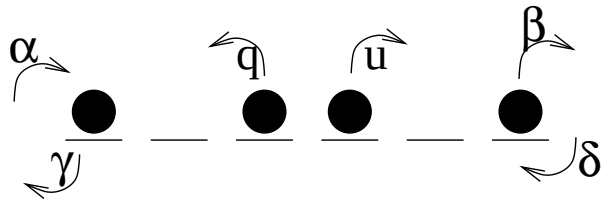


The corresponding Markov chain on permutations



Part IV: On the ASEP with $\alpha, \beta, \gamma, \delta, q$ general

How to compute the stationary distribution in the general case?



Matrix Ansatz still works (Derrida et al, '93): if D, E, W, V satisfy

$$DE - qED = D + E \tag{4}$$

$$(\beta D - \delta E)|V\rangle = |V\rangle \tag{5}$$

$$\langle W|(\alpha E - \gamma D) = \langle W| \tag{6}$$

then

$$P_n(\sigma_1, \ldots, \sigma_n) = \frac{\langle W|(\prod_{i=1}^{n}(\sigma_i D + (1-\sigma_i)E))|V\rangle}{\langle W|(D+E)^n|V\rangle}.$$

No solution was found to this Ansatz until 2003!

Stationary Distribution of the ASEP, general case

First computed by Uchiyama-Sasamoto-Wadati, 2003, by finding relationship with Askey-Wilson polynomials and using the *Matrix Ansatz*. Formula involves products of matrices d and e, where

$$d = \begin{bmatrix} d_0^{\natural} & d_0^{\sharp} & 0 & \cdots \\ d_0^{\flat} & d_1^{\natural} & d_1^{\sharp} & \\ & d_1^{\flat} & d_2^{\natural} & \ddots \\ \vdots & & \ddots & \ddots \end{bmatrix}, \qquad e = \begin{bmatrix} e_0^{\natural} & e_0^{\sharp} & 0 & \cdots \\ e_0^{\flat} & e_1^{\natural} & e_1^{\sharp} & \\ & e_1^{\flat} & e_2^{\natural} & \ddots \\ \vdots & & \ddots & \ddots \end{bmatrix},$$

$$\begin{aligned} d_n^{\natural} &= \frac{q^{n-1}}{(1-q^{2n-2}abcd)(1-q^{2n}abcd)} \\ &\times [bd(a+c)+(b+d)q-abcd(b+d)q^{n-1}-\{bd(a+c)+abcd(b+d)\}q^n \\ &-bd(a+c)q^{n+1}+ab^2cd^2(a+c)q^{2n-1}+abcd(b+d)q^{2n}], \\ e_n^{\natural} &= \frac{q^{n-1}}{(1-q^{2n-2}abcd)(1-q^{2n}abcd)} \\ &\times [ac(b+d)+(a+c)q-abcd(a+c)q^{n-1}-\{ac(b+d)+abcd(a+c)\}q^n \\ &-ac(b+d)q^{n+1}+a^2bc^2d(b+d)q^{2n-1}+abcd(a+c)q^{2n}], \end{aligned}$$

$$d_n^{\sharp} = \frac{1}{1-q^n ac}\mathcal{A}_n, \qquad e_n^{\sharp} = -\frac{q^n ac}{1-q^n ac}\mathcal{A}_n, \qquad d_n^{\flat} = -\frac{q^n bd}{1-q^n bd}\mathcal{A}_n, \qquad e_n^{\flat} = \frac{1}{1-q^n bd}\mathcal{A}_n,$$

$$\mathcal{A}_n = \left[\frac{(1-q^{n-1}abcd)(1-q^{n+1})(1-q^n ab)(1-q^n ac)(1-q^n ad)(1-q^n bc)(1-q^n bd)(1-q^n cd)}{(1-q^{2n-1}abcd)(1-q^{2n}abcd)^2(1-q^{2n+1}abcd)} \right]^{1/2},$$

$$a = \frac{1-q-\alpha+\gamma+\sqrt{(1-q-\alpha+\gamma)^2+4\alpha\gamma}}{2\alpha}, \qquad b = \frac{1-q-\beta+\delta+\sqrt{(1-q-\beta+\delta)^2+4\beta\delta}}{2\beta},$$

$$c = \frac{1-q-\alpha+\gamma-\sqrt{(1-q-\alpha+\gamma)^2+4\alpha\gamma}}{2\alpha}, \qquad d = \frac{1-q-\beta+\delta-\sqrt{(1-q-\beta+\delta)^2+4\beta\delta}}{2\beta}.$$

Remark. The Uchiyama-Sasamoto-Wadati (USW) formula is amazing, but quite complicated ...

Question: Can one hope for a *simple combinatorial* formula for the stationary distribution?

What combinatorial objects will play the role of permutation tableaux??

To answer this question, it is helpful to first transform permutation tableaux into some objects which are more symmetric.

Permutation tableaux ↔ Alternative tableaux

A *topmost* + is a + which is topmost in its column, but NOT in the first row. A *rightmost restricted* 0 is a 0 which lies below a +, and is the rightmost in its row with this property.

+	+	**0**	**0**
0	**0**	+	**0**
+	+	+	+
0	**0**	+	

⇓

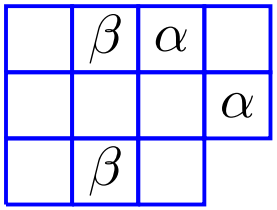
	β	α	
			α
	β		

Cut off top row
Topmost $+ \to \alpha$
Rightmost restricted $0 \to \beta$

Note: each box left of β is empty
Note: each box above α is empty

Definition: (Viennot '08) an *alternative tableau* is a Young diagram where boxes are empty, or contain α or β, such that each box above α is empty, and each box left of β is empty.

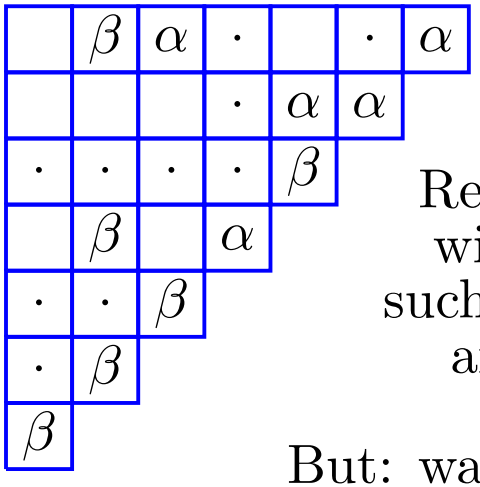
2nd transformation:
put alternative tableaux in staircase shape



Encode the southeast border of Young diagram by diagonal boxes in staircase tableau

Require boxes above α to be empty
Require boxes left of β to be empty

$\Downarrow$



Resulting tableau is in a staircase, with all diagonal boxes occupied, such that each box above α is empty, and each box left of β is empty.

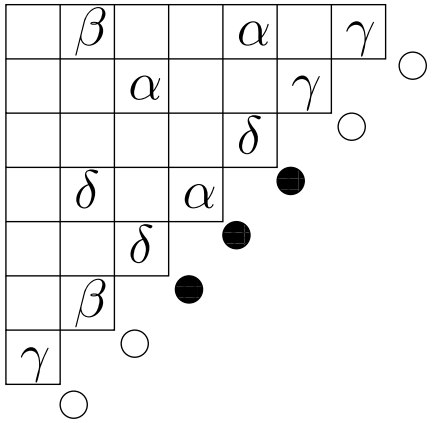
But: want to get γ's and δ's into the picture!

New combinatorial objects: staircase tableaux

Definition. (Corteel–W.) A *staircase tableau* of size n is a Young diagram of shape $(n, n-1, \ldots, 2, 1)$, whose boxes are either empty, or filled with $\alpha, \beta, \gamma, \delta$, such that:

- all boxes above an α or γ are empty.
- all boxes left of a β or δ are empty.
- all boxes on the southeast border are nonempty.

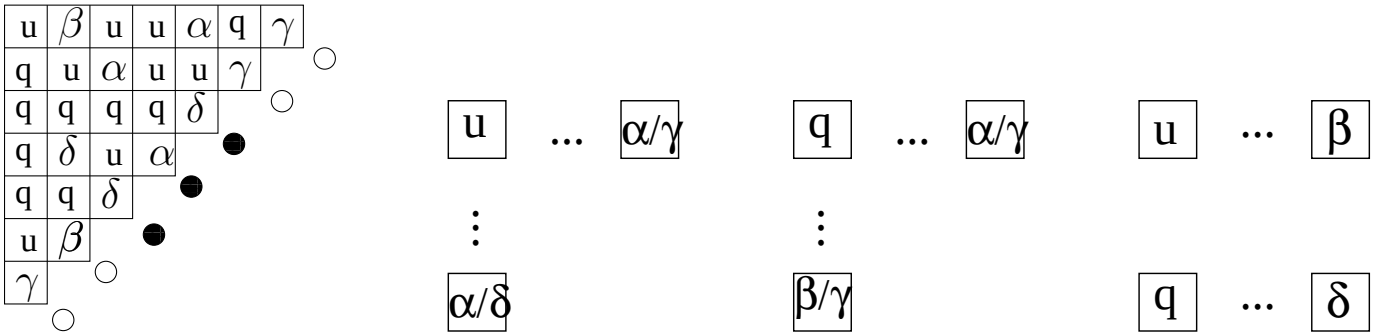
(remark on symmetries of ASEP)



Define its *type* to be the word in $\{\bullet, \circ\}^n$ obtained by reading the southeast border and assigning a $\bullet$ to an α or δ and a $\circ$ to a β or γ.

Staircase tableaux and the ASEP

Assign q or u to each blank box, according to *(asymmetric!)* RULE.
Define *weight* $\mathrm{wt}(\mathcal{T})$ of $\mathcal{T}$ as product of all boxes.



Let $Z_n = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})$, summing over all tableaux of size n.

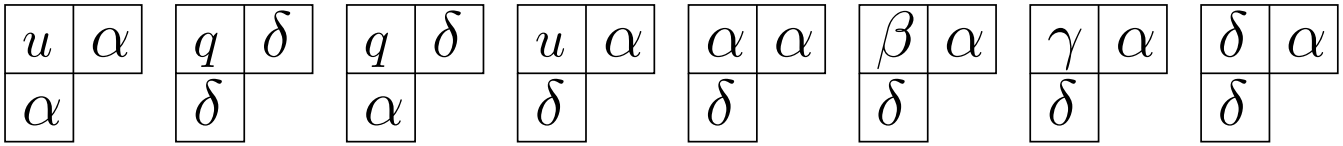
Theorem. (Corteel–W.) Consider the ASEP with parameters $\alpha, \beta, \gamma, \delta, q, u$ general. The steady state probability that the ASEP is in configuration σ is

$$\frac{\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})}{Z_n},$$

summing over all tableaux $\mathcal{T}$ of type σ.

Example

Fix $n = 2$. What is the steady state probability of the state $\bullet\bullet$?



Answer:

$$\frac{\alpha^2 u + \delta^2 q + \alpha\delta q + \alpha\delta u + \alpha^2\delta + \alpha\beta\delta + \alpha\gamma\delta + \alpha\delta^2}{Z_2}.$$

Some open problems and comments

- Can one lift the dynamics of the ASEP (with all parameters general) to the staircase tableaux?

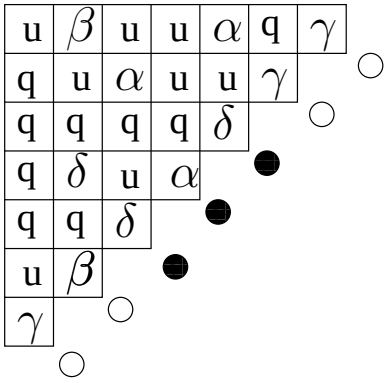
- Recall that in the case $\gamma = \delta = 0$, the stationary distribution of the ASEP was connected to permutation tableaux, which in turn are connected to total positivity on the Grassmannian. Is there a geometric interpretation of staircase tableaux?

- There is a notion of *double Grassmannian* and a theory of total positivity for it. This may be the right place to look ...

Part V: Enumeration of staircase tableaux

Recall: the *weight* $\mathrm{wt}(\mathcal{T})$ of $\mathcal{T}$ is product of all boxes.

WLOG set $u = 1$.



Def: Let $Z_n(y; \alpha, \beta, \gamma, \delta; q) = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T}) y^{black(\mathcal{T})}$, where:

- the sum is over all tableaux of size n;
- $black(\mathcal{T})$ is the number of black particles in the type of $\mathcal{T}$.

Call this the *enhanced* partition function. Setting $y = 1$ gives the partition function of the ASEP, ie the weighted sum over all staircase tableaux of size n.

Explicit formula for (enhanced) partition function

Theorem. (Corteel-Stanley-Stanton-W.) The enhanced partition function $Z_n(y;\alpha,\beta,\gamma,\delta;q)$ equals

$$Z_n(y;\alpha,\beta,\gamma,\delta;q) = (abcd;q)_n \left(\frac{\alpha\beta}{1-q}\right)^n \sum_{k=0}^{n} \frac{(ab,ac/y,ad;q)_k}{(abcd;q)_k} q^k$$

$$\times \sum_{j=0}^{k} q^{-(k-j)^2} (a^2/y)^{j-k} \frac{(1+y+q^{k-j}a+q^{j-k}y/a)^n}{(q,q^{2j-2k+1}y/a^2;q)_{k-j}(q,a^2q^{1-2j+2k}/y;q)_j},$$

where

$$a = \frac{1-q-\alpha+\gamma+\sqrt{(1-q-\alpha+\gamma)^2+4\alpha\gamma}}{2\alpha}, \quad b = \frac{1-q-\beta+\delta+\sqrt{(1-q-\beta+\delta)^2+4\beta\delta}}{2\beta},$$

$$c = \frac{1-q-\alpha+\gamma-\sqrt{(1-q-\alpha+\gamma)^2+4\alpha\gamma}}{2\alpha}, \quad d = \frac{1-q-\beta+\delta-\sqrt{(1-q-\beta+\delta)^2+4\beta\delta}}{2\beta}.$$

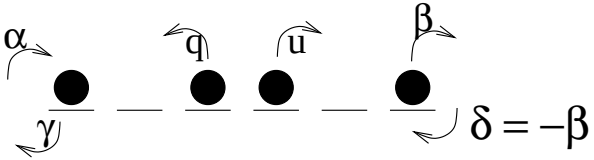
Staircase tableaux generalize many combinatorial objects!

(table in CSSW paper)

α	β	γ	δ	q	y	$Z_n(y;\alpha,\beta,\gamma,\delta;q)$
1	1	1	1	1	1	$4^n n! = 4n!!!!$
1	1	1	0	1	1	$(2n+1)!!$
1	1	0	0	1	1	$(n+1)!$
1	1	0	0	0	1	$C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$
1	1	1	1	0	0	$2F_{2n}$ (Fibonacci)

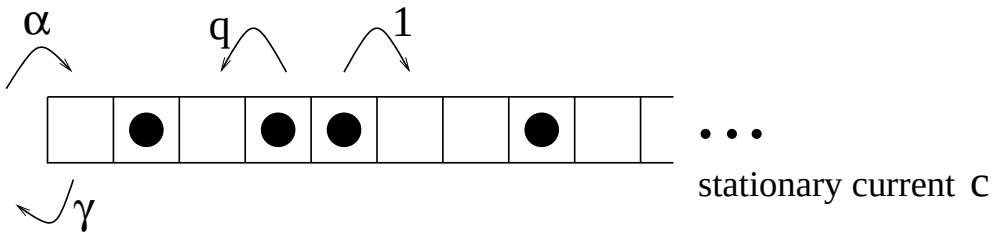
α	β	γ	δ	q	y	$Z_n(y;\alpha,\beta,\gamma,\delta;q)$
α	β	γ	δ	1	1	$\prod_{j=0}^{n-1}(\alpha+\beta+\gamma+\delta+j(\alpha+\gamma)(\beta+\delta))$
α	β	γ	$-\beta$	q	1	$\prod_{j=0}^{n-1}(\alpha+q^j\gamma)$
α	β	γ	β	q	-1	$(-1)^n\prod_{j=0}^{n-1}(\alpha-q^j\gamma)$
α	0	γ	0	q	y	$\prod_{j=0}^{n-1}(y\alpha+q^j\gamma)$
0	β	γ	0	q	y	$\prod_{j=0}^{n-1}(\beta+\beta\gamma[j]_q+\gamma q^j)$

What is the meaning of the nonphysical specialization $\delta = -\beta$? It doesn't make sense to let a probability be negative!



Part VI: ASEP on semi-infinite lattice

Grosskinsky (thesis) studied the ASEP on a semi-infinite lattice.
Depends on parameters α, γ, q.
(Grosskinsky considered the case $\gamma = q = 0$.)



This Markov chain has infinitely many states – so the probability of a given state is 0. Also, there are many stationary measures (parameterized by c).

Still, one may look at the leftmost n sites and ask: for a given stationary measure, what is the steady state probability of seeing a particular configuration?

ASEP on semi-infinite lattice

Sasamoto asked me if combinatorial methods would work here ...

Recall: the *weight* $\mathrm{wt}(\mathcal{T})$ of tableau $\mathcal{T}$ is product of all boxes.

Let Ψ denote the specialization $\beta \to c$ and $\delta \to -c$.

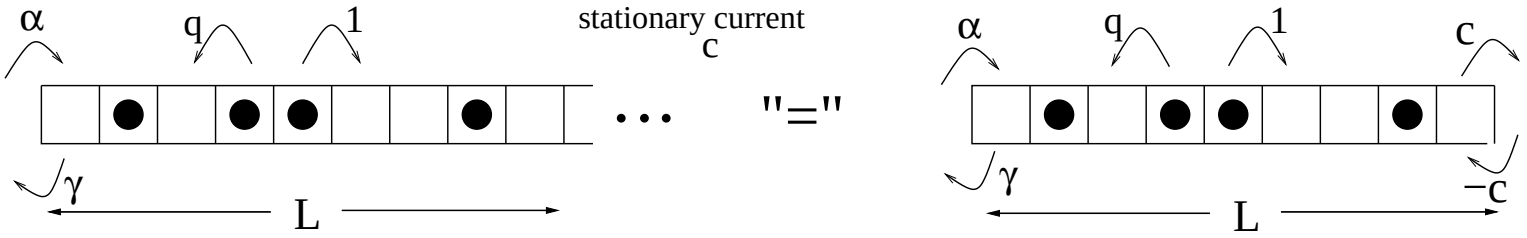
Theorem (Sasamoto-W.): Consider the ASEP on a semi-infinite lattice with parameters α, β, q, and c. Let $\sigma \in \{\bullet, \circ\}^n$. The steady state probability that the leftmost n sites of the ASEP are in configuration σ is

$$\Psi\left(\frac{\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})}{Z_n}\right),$$

where the sum is over all staircase tableaux $\mathcal{T}$ of type σ.

ASEP on semi-infinite lattice

"Theorem":



This provides a physical interpretation of a nonsensical specialization of the ASEP on a finite lattice!

Remarks on negative specializations of Markov chains

- Consider a Markov chain whose transition matrix is written in terms of one or more parameters. Typically this "makes sense" only if those parameters are positive.
- Recall: the stationary distribution of a Markov chain is the (unique) left eigenvector of the transition matrix associated with eigenvalue 1.
- One can may choose a specialization of the parameters and consider the corresponding specialization of the left eigenvector.
- If one makes one or more parameters negative, when can we still give a physical meaning to the corresponding "stationary distribution"?

Open problems for the ASEP

• Find a combinatorial proof of the formula for the stationary distribution in terms of staircase tableaux, by:
lifting the dynamics of the ASEP to a Markov chain on staircase tableaux. (We have done this for $\gamma = \delta = 0$.)

• Define a more refined version of the ASEP which keeps track of "birth certificates" for particles: whether each particle and hole enters from the left or right of the lattice. Prove a formula for the stationary distribution, in terms of staircase tableaux.

Open problems for the ASEP

- Are there physical interpretations of some of the other specializations of the enhanced partition function?

- Is there a connection between staircase tableaux and the combinatorics of the double Grassmannian?

- For Razumov-Stroganov aficionados: can techniques from that world be used for ASEP? can one encode staircase tableaux as vertex configurations?

Thank you! (plus references)

Tableaux combinatorics for the asymmetric exclusion process (with Sylvie Corteel), *Advances in Applied Math*, 2007.

A Markov chain on permutations which projects to the PASEP (with Sylvie Corteel), *IMRN*, 2007.

Staircase tableaux, the ASEP, and Askey-Wilson polynomials (with S. Corteel), *Proc. Nat. Acad. Sci.* March, 2010.

Tableaux combinatorics for the ASEP and Askey-Wilson polynomials (with S. Corteel), *Duke Math. J.*, September, 2011.

Formulae for Askey-Wilson moments and enumeration of staircase tableaux (with Corteel, Stanley, Stanton), to appear in *Transactions of the AMS*.

Work in progress with T. Sasamoto.

Some ideas of the proof: generalize the Matrix Ansatz

Recall the Derrida et al Matrix Ansatz:

Theorem: Consider ASEP on lattice of n sites, with parameters $\alpha, \beta, \gamma, \delta, q$. If D, E, W, V satisfy

$$DE - qED = D + E \tag{7}$$

$$(\beta D - \delta E)|V\rangle = |V\rangle \tag{8}$$

$$\langle W|(\alpha E - \gamma D) = \langle W| \tag{9}$$

then the steady state probability of being in state $(\sigma_1, \ldots, \sigma_n)$ is

$$P_n(\sigma_1, \ldots, \sigma_n) = \frac{\langle W|(\prod_{i=1}^{n}(\sigma_i D + (1-\sigma_i)E))|V\rangle}{\langle W|(D+E)^n|V\rangle}.$$

Some ideas of the proof: generalize the Matrix Ansatz

Theorem. Suppose D and E are matrices, $\langle W|$ and $|V\rangle$ vectors, such that: *for all words X and Y in D and E*, we have

$$\langle W|X(DE - qED)Y|V\rangle = \langle W|X(D+E)Y|V\rangle, \tag{10}$$

$$\langle W|X(\beta D - \delta E)|V\rangle = \langle W|X|V\rangle, \tag{11}$$

$$\langle W|(\alpha E - \gamma D)Y|V\rangle = \langle W|Y|V\rangle. \tag{12}$$

$$\text{Then } P_n(\sigma_1, \ldots, \sigma_n) = \frac{\langle W|(\prod_{i=1}^{n}(\sigma_i D + (1-\sigma_i)E))|V\rangle}{\langle W|(D+E)^n|V\rangle}$$

Remark. Compare the three relations above with those from the usual Matrix Ansatz:

$$DE - qED = D + E, \quad (\beta D - \delta E)|V\rangle = |V\rangle, \quad \langle W|(\alpha E - \gamma D) = \langle W|$$

Generalize the Matrix Ansatz again

Theorem. Suppose D and E are matrices, $\langle W|$ and $|V\rangle$ vectors, and λ_n is a family of constants, such that:
for all words X and Y in D and E, we have

$$\langle W|X(DE - qED)Y|V\rangle = \lambda_{|X|+|Y|+2}\langle W|X(D+E)Y|V\rangle, \quad (13)$$

$$\langle W|X(\beta D - \delta E)|V\rangle = \lambda_{|X|+1}\langle W|X|V\rangle, \quad (14)$$

$$\langle W|(\alpha E - \gamma D)Y|V\rangle = \lambda_{|Y|+1}\langle W|Y|V\rangle. \quad (15)$$

$$\text{Then } P_n(\sigma_1, \ldots, \sigma_n) = \frac{\langle W|(\prod_{i=1}^{n}(\sigma_i D + (1-\sigma_i)E))|V\rangle}{\langle W|(D+E)^n|V\rangle}$$

Remark. One may adapt the Derrida-Evans-Hakim-Pasquier proof to prove the above theorem.

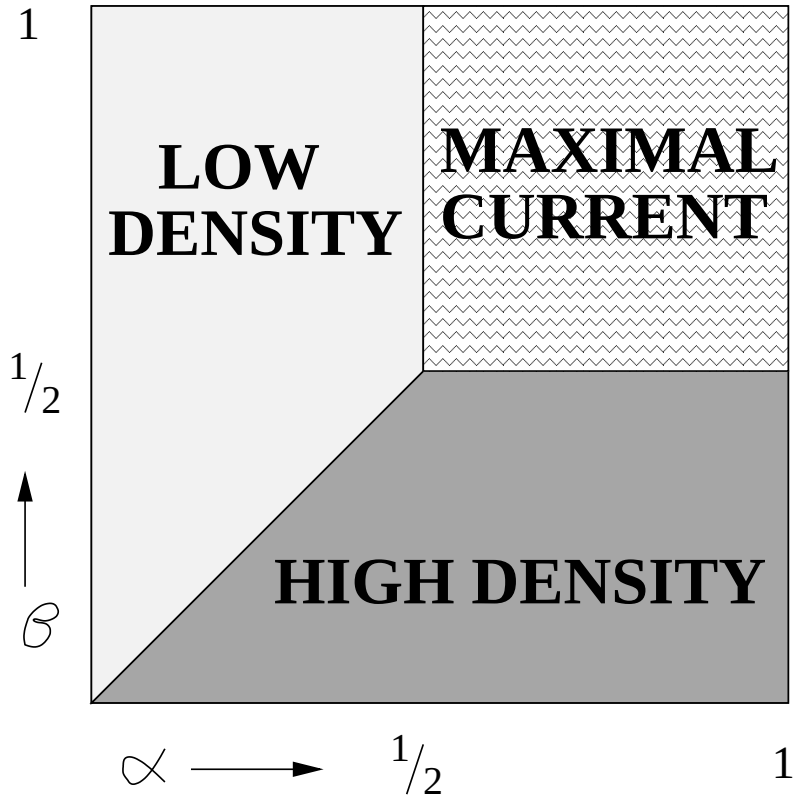
Strategy of proof of combinatorial formula for stationary distribution of ASEP (with $\alpha, \beta, \gamma, \delta, q$ general)

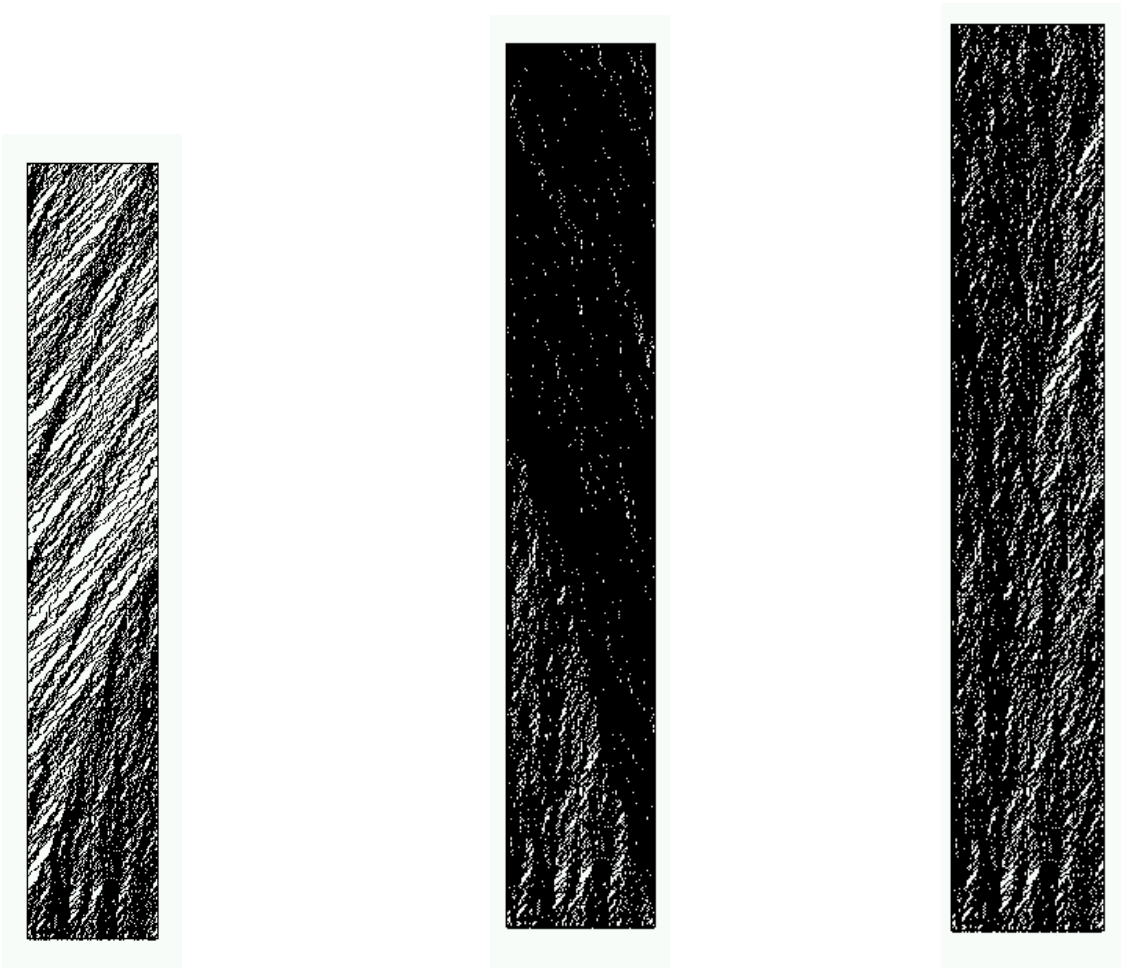
- Guess combinatorial formula – staircase tableaux. (*tricky – we were looking for more symmetric objects*)
- Find matrices D, E and vectors $\langle W|, |V\rangle$ such that products of them enumerate staircase tableaux of a given type; D and E are "transfer matrices" for the tableaux. (*straightforward*)
- Prove that $D, E, \langle W|, |V\rangle$ satisfy the three infinite families of relations (previous slide), with $\lambda_n = \alpha\beta - q^{n-1}\gamma\delta$. (*hard!*)
- Generalized Matrix Ansatz (*straightforward*) then implies that the generating function for staircase tableaux of type σ is the probability that the ASEP is in state σ.

Some features of the ASEP *with open boundaries*

The ASEP with open boundaries exhibits boundary-induced *phase transitions.*

(Here, $\gamma = \delta = q = 0$.)





(a) $\alpha = 0.2, \beta = 1$ (b) $\alpha = 1, \beta = 0.2$ (c) $\alpha = \beta = 1$

http://front.math.ucdavis.edu/9910.0270 (Sasamoto)

Application of staircase tableaux to orthogonal polynomials.

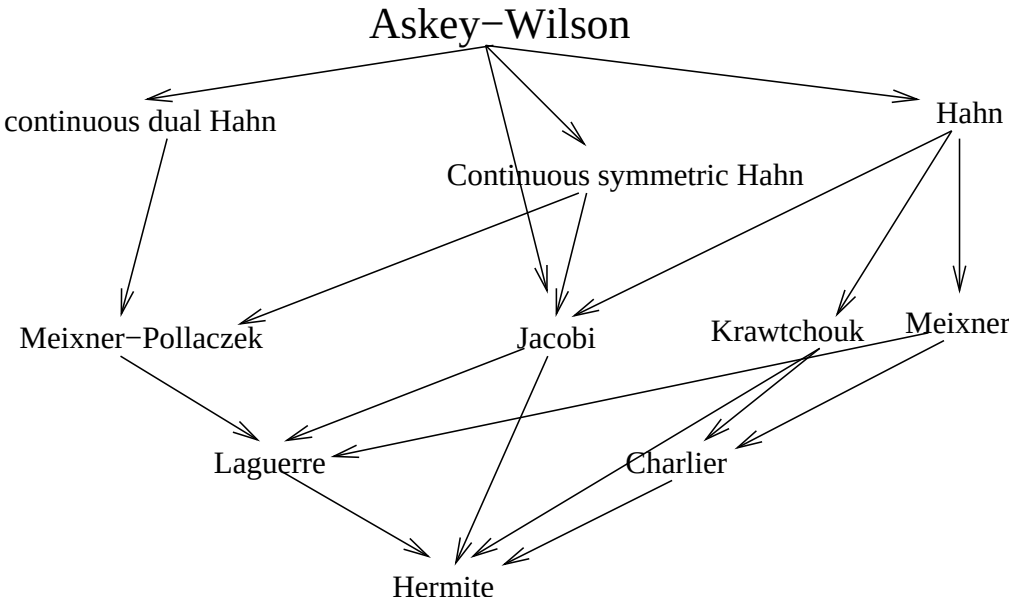
• We say $\{P_k(x)\}_{k\geq 0}$ is a family of *orthogonal polynomials* if there exists a linear function $f : K[x] \to K$ such that:

- $\deg(P_k) = k$ for $k \geq 0$,
- $f(P_k P_\ell) = 0$ if $k \neq \ell$,
- $f(P_k^2) \neq 0$ for $k \geq 0$.

Then $f(x^n) = \mu_n$ is called the nth *moment*.

• Equivalent to the usual definition: choose a probability measure μ on $\mathbb{R}$. Then $\{P_k(x)\}_{k\geq 0}$ are orthogonal with respect to μ if $\int_{\mathbb{R}} P_n(x)P_k(x)d\mu(x) = 0$ for $n \neq k$.

The hierarchy of classical orthogonal polynomials



• Much work giving a combinatorial interpretations to moments: Ismail, Stanton, Viennot, Kim, Zeng, Zeilberger, Foata, Strehl, Garsia, Remmel, Jackson, Simion .... But there had been nothing for Askey-Wilson moments.

• However, work of USW provided a close link between the ASEP and the Askey-Wilson moments.

A combinatorial formula for Askey-Wilson moments

Askey-Wilson polynomials $P_n(x, a, b, c, d|q)$ are q-orthogonal polynomials in four free parameters a, b, c, d besides q.

Recall $Z_n = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})$, summing over all tableaux of size n.

Define $\alpha, \beta, \gamma, \delta$ by

$$\alpha = \frac{1-q}{1+ac+a+c}, \qquad \beta = \frac{1-q}{1+bd+b+d},$$
$$\gamma = \frac{-(1-q)ac}{1+ac+a+c}, \qquad \delta = \frac{-(1-q)bd}{1+bd+b+d}.$$

Theorem (Corteel–W.) The kth moment of the Askey-Wilson polynomials is given by

$$\mu_k = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \left(\frac{1-q}{2}\right)^{\ell} \frac{Z_\ell}{\prod_{i=0}^{\ell-1}(\alpha\beta - \gamma\delta q^i)}.$$

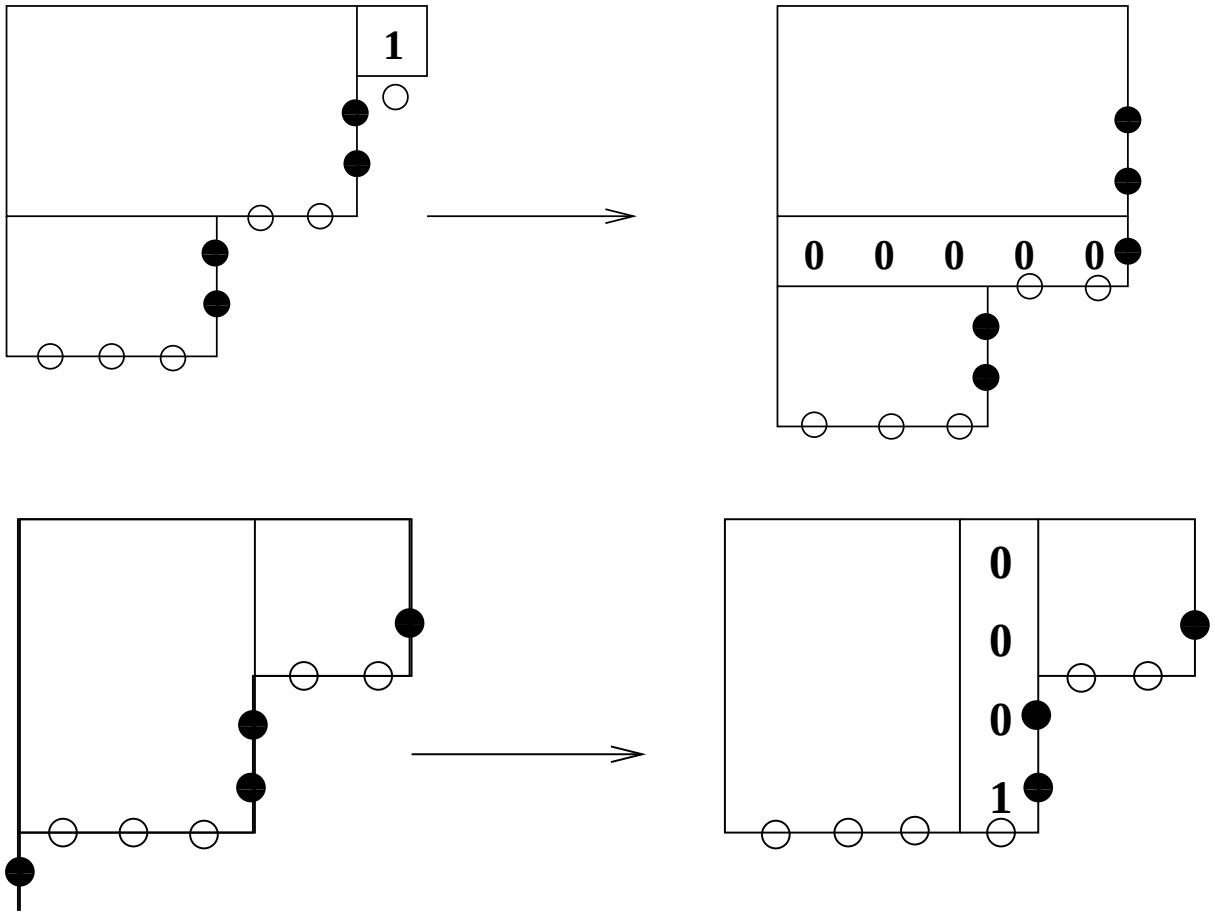
Explicit formula for Askey-Wilson moments

Using technology developed by Ismail-Stanton, we can also prove:

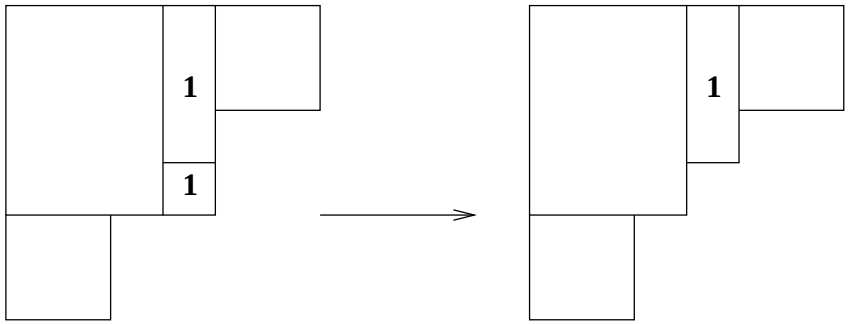
Theorem. (Corteel-Stanley-Stanton-W.) The Askey-Wilson moments $\mu_n(a,b,c,d|q)$ are

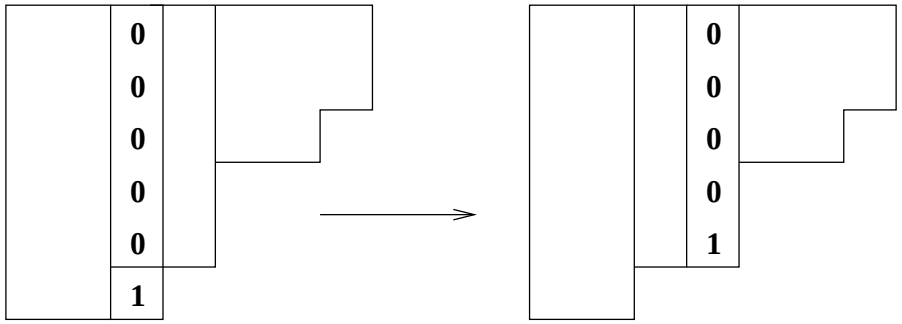
$$\frac{1}{2^n}\sum_{k=0}^{n}\frac{(ab,ac,ad;q)_k}{(abcd;q)_k}q^k\sum_{j=0}^{k}\frac{q^{-(k-j)^2}a^{2j-2k}(q^{k-j}a+q^{j-k}/a)^n}{(q,q^{2j-2k+1}/a^2;q)_{k-j}(q,q^{2k-2j+1}a^2;q)_j}.$$

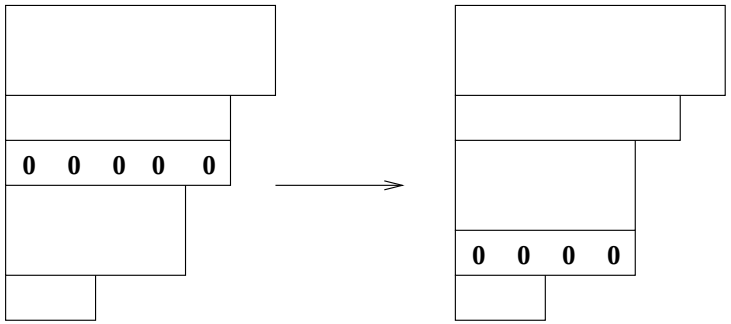
Transitions in the "PT chain": enter and exit



Transitions in the "PT chain": hop right







Transitions in the "PT chain": hop left