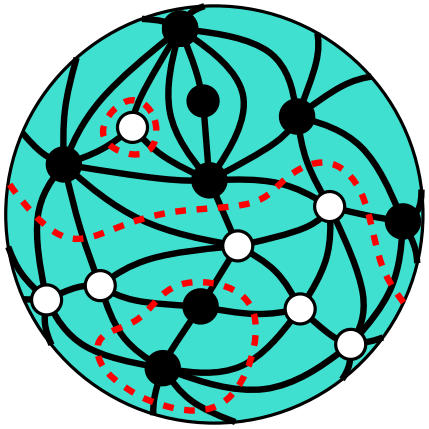
Percolation on random triangulations

Olivier Bernardi (MIT)

Joint work with

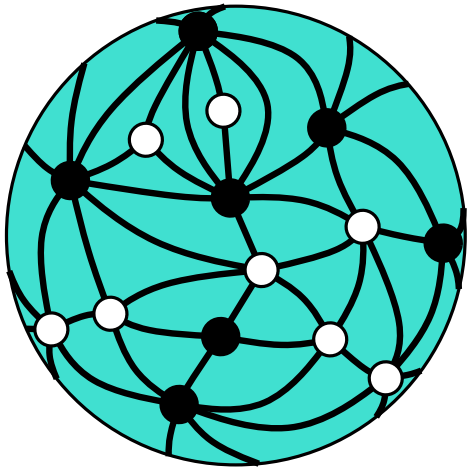
Grégory Miermont (Université Paris-Sud)

Nicolas Curien (École Normale Supérieure)



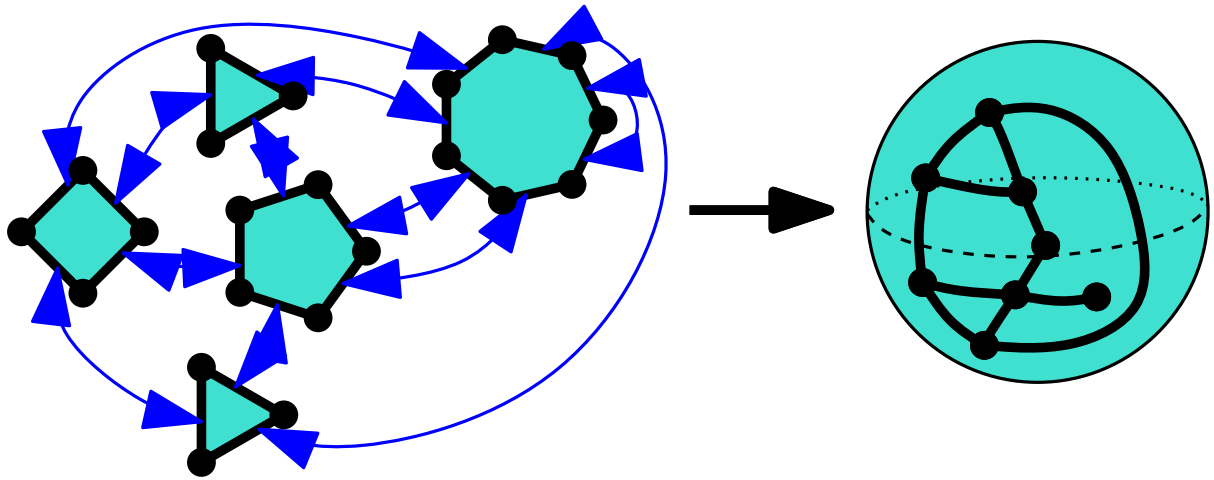
MSRI, January 2012

Model and motivations



Planar maps

Def. A **planar map** is a way of forming the sphere by gluing polygons.



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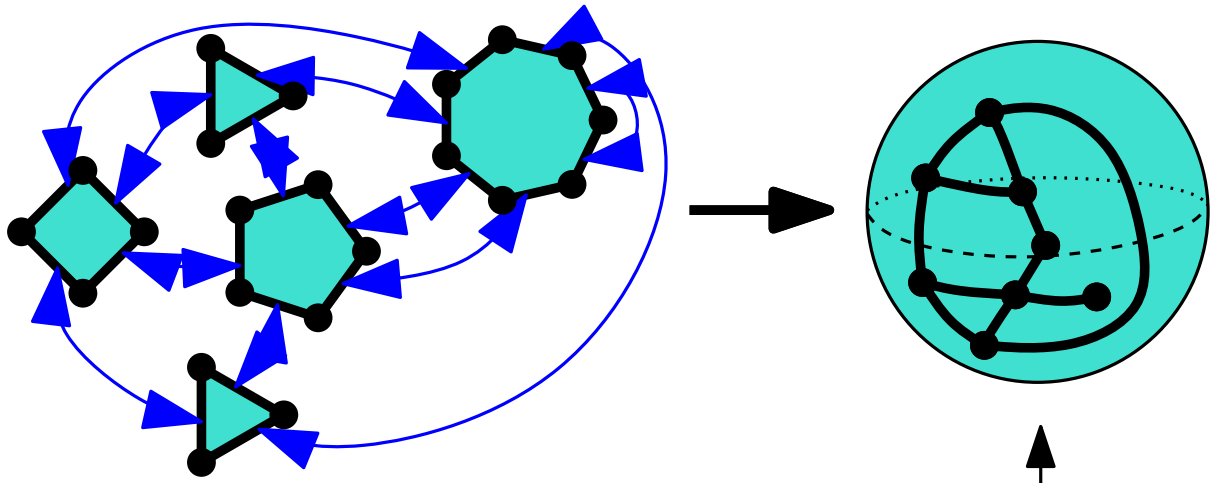

Embedded connected planar graph considered up to homeomorphism.

Planar maps

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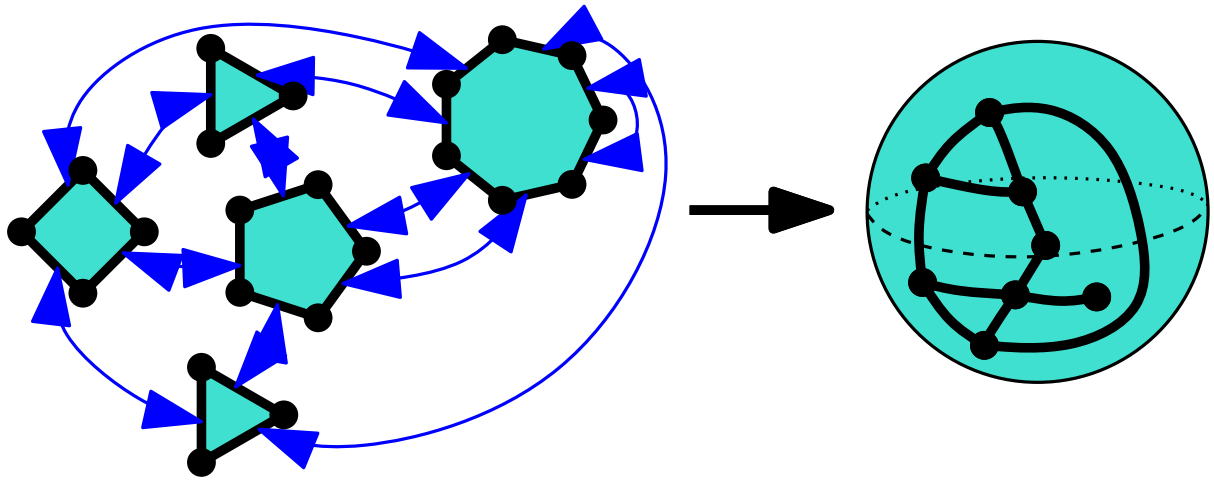
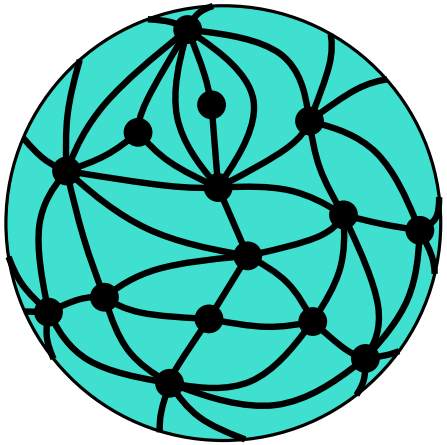


Def. A **triangulation** is a planar map made of triangles.



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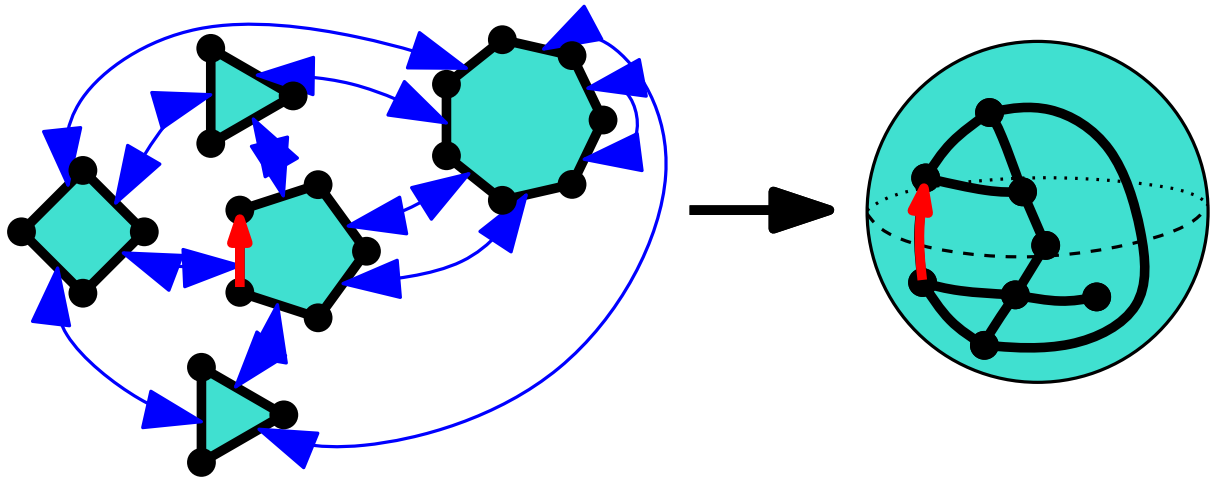

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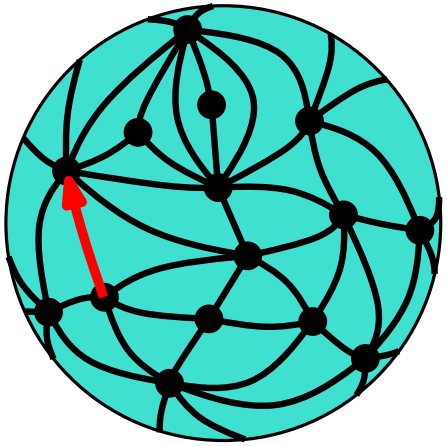
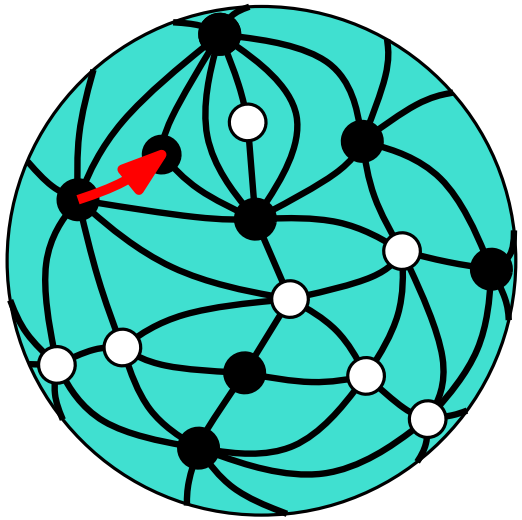


Def. A map is **rooted** if an edge is chosen and oriented.

Percolation

Site percolation: vertices are black with probability p,
white with probability $1 - p$.



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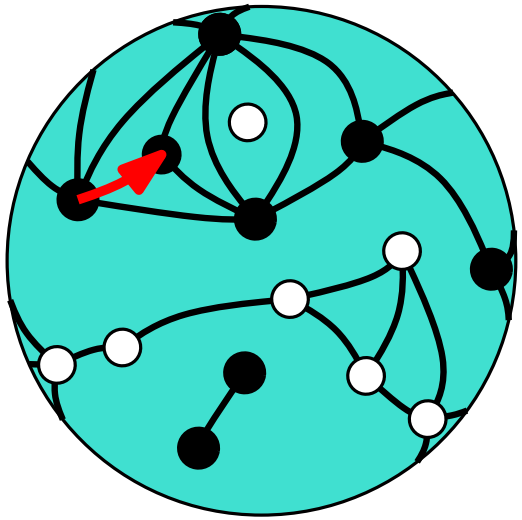
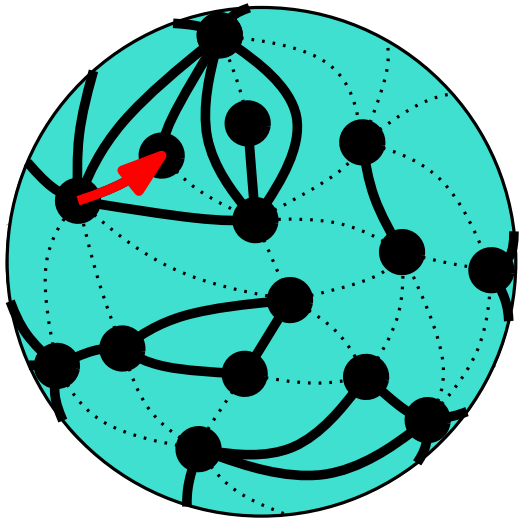


The **clusters** are the connected components.

The **gasket** is the black cluster containing the root-vertex.

Percolation

Bond percolation: edges are open with probability p,
closed with probability $1-p$.



The **clusters** are the monochromatic connected components.

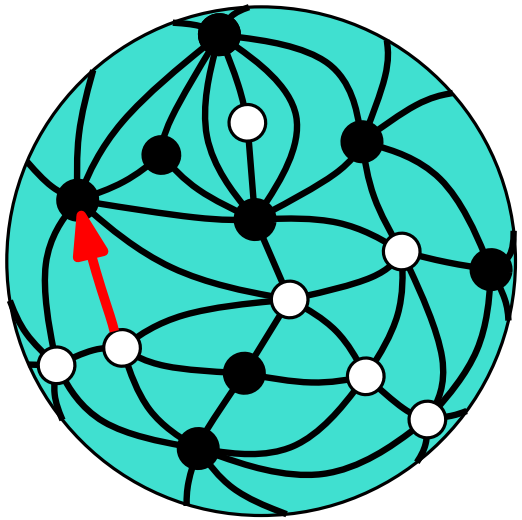
The **gasket** is the open cluster containing the root-vertex.

Boltzmann model

Model: pick a rooted triangulation + site percolation with probability proportional to

$$z^{\#\text{edges}} p^{\#\text{black vertices}} (1-p)^{\#\text{white vertices}}.$$

edge activity **percolation probability**



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Thm [Tutte 62]: The number T_n of rooted triangulations with n edges satisfies

$$T_n \sim c\, n^{-5/2} z_0^{-n}, \quad \text{where } z_0 = 432^{-1/6},$$

hence the model is **admissible** for edge activity z in $[0, z_0]$.

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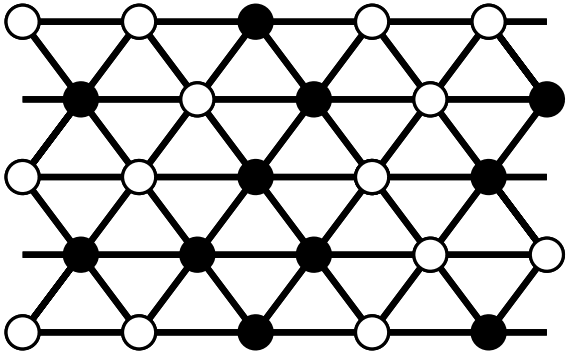
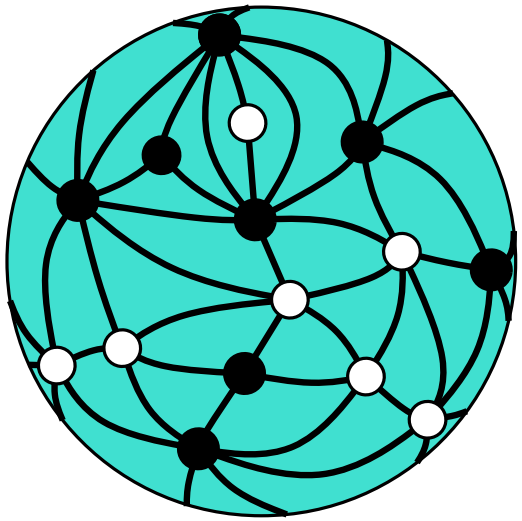
Remark: The distribution of the size of the random triangulation T has **light tail** (exponential decay) if $z < z_0$,
heavy tail (polynomial decay) if $z = z_0$.

From now on, we fix the edge activity to be z_0.

Motivations

Phase transition?





regular triangulation

random triangulation

Critical probability:

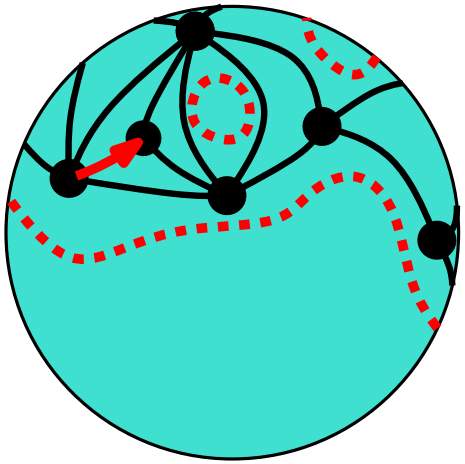
$p_c = \inf\{p,\ \mathbb{P}_p(\text{infinite black cluster}) > 0\}$

Critical probability ?

Motivations

Motivation 1. Study the phase transition:

- Transition for size of gasket: heavy-tailed / light-tailed at $p = p_c$.



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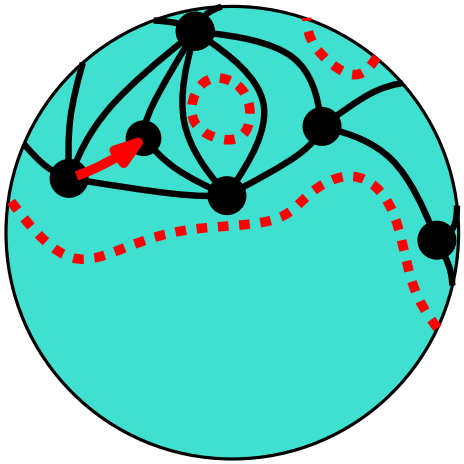
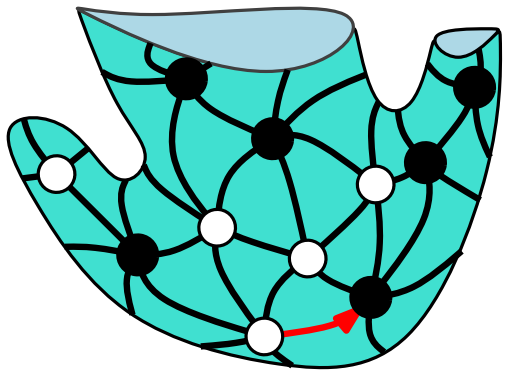


Related results. Percolation has been studied on (loopless) infinite triangulations (UIPT) [Angel,Schramm 04, Angel 03, Angel,Curien 12].

In that context, $p_c = 1/2$ for site perco, and $p_c = 1/4$ for bond perco.



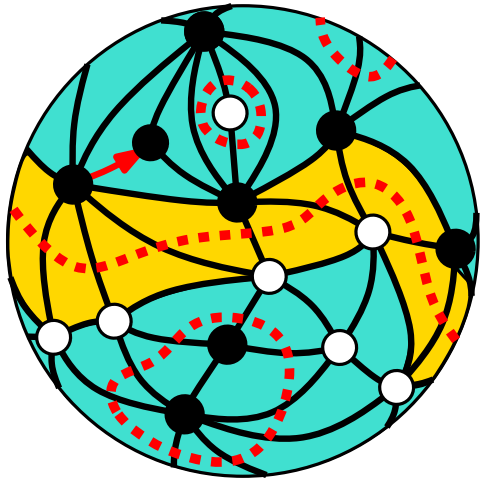
random infinite triangulation (UIPT)

Motivations

Motivation 1. Study the phase transition:

- Transition for size of gasket: heavy-tailed / light-tailed at $p = p_c$.
- Show that interfaces are long (heavy-tailed) only at $p = p_c$.

Study the degree of random faces of the gasket.



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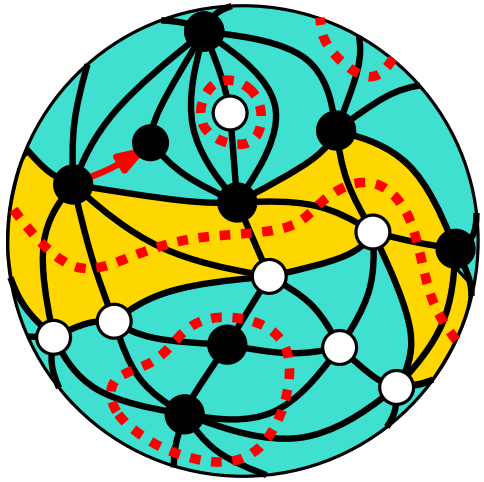

Related results.

- The tail of the distribution of the degree of faces has been predicted using the KPZ conjecture.
- The tail of the distribution of the degree of faces for the gasket of "a rigid" O(n) model on quadrangulations has been identified in [Borot, Bouttier, Guitter 12].

Motivations

Motivation 1. Study the phase transition:

- Transition for size of gasket: heavy-tailed / light-tailed at $p = p_c$.
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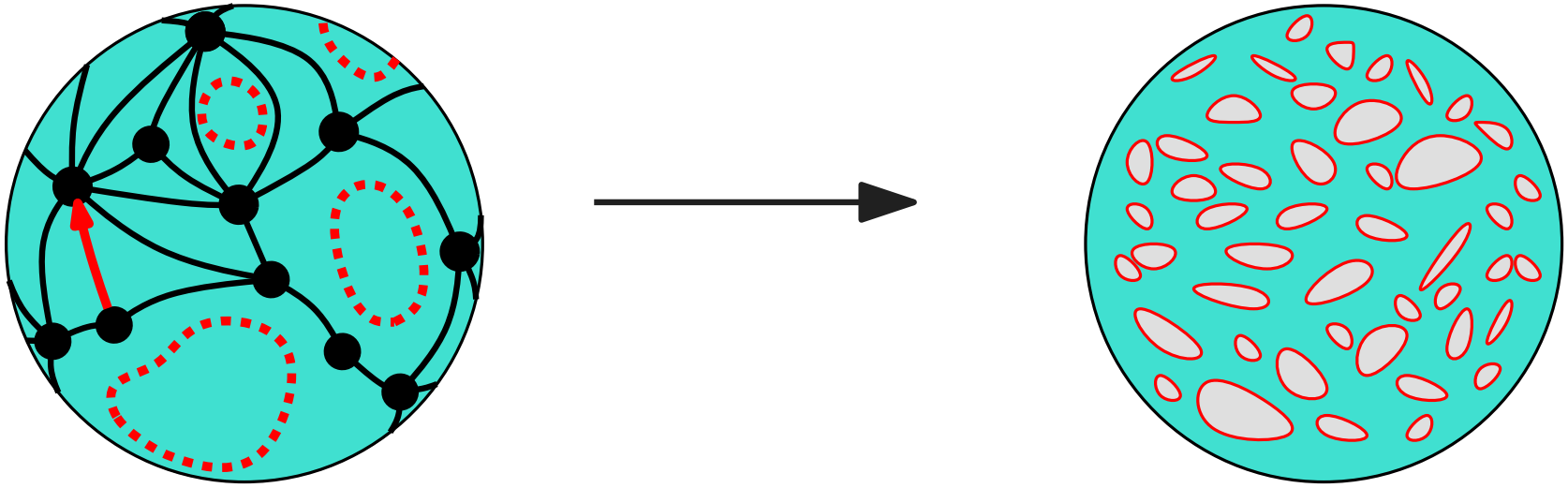
Study the degree of random faces of the gasket.

Motivation 2. Show that the gasket of critical percolation satisfy the asumptions of [Le Gall, Miermont 11] for convergence (as random metric space) toward the **stable map**.

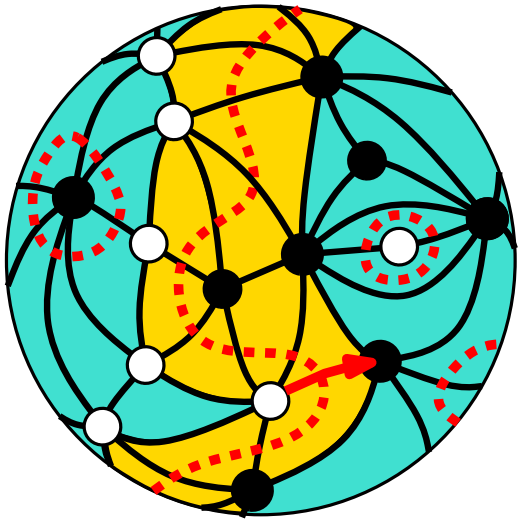


Analytic combinatorics

Counting problem.

Tail of the distribution of the length of an interface?

$\Longrightarrow$ Study the total weight of maps rooted on interface of size n.



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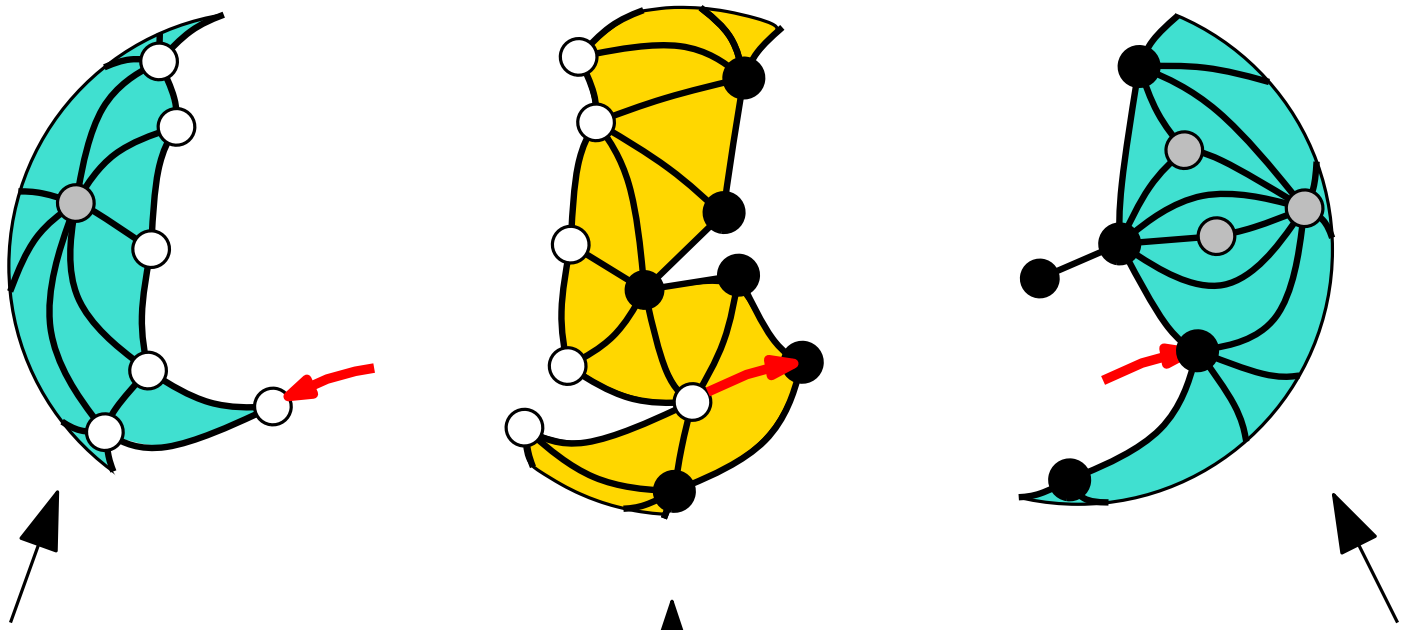

Triangulation of outer degree i

Weight $z_0^{\#\text{edges}}(1-p)^{\#\text{outer vertices}}$

Necklace: $\binom{i+j}{i}$

Triangulation of outer degree j

Weight $z_0^{\#\text{edges}}p^{\#\text{outer vertices}}$

Results.

Theorem:

• The total weight W_n of triangulations with outer degree n with black outer vertices satisfies:

$$W_n \sim \begin{cases} c(p)\, n^{-3/2}\, K(p)^n & \text{if } p < 1/2, \\ c(p)\, n^{-5/3}\, K(p)^n & \text{if } p = 1/2, \\ c(p)\, n^{-5/2}\, K(p)^n & \text{if } p > 1/2. \end{cases}$$

where $1/K(p)$ is the largest root of
$(18px^3 - 9x^3 + 5x^3\sqrt{3} + 6x^2\sqrt{3} - 48\sqrt{3}x + 32\sqrt{3})(2px - x + \sqrt{3}x - 2\sqrt{3})$.

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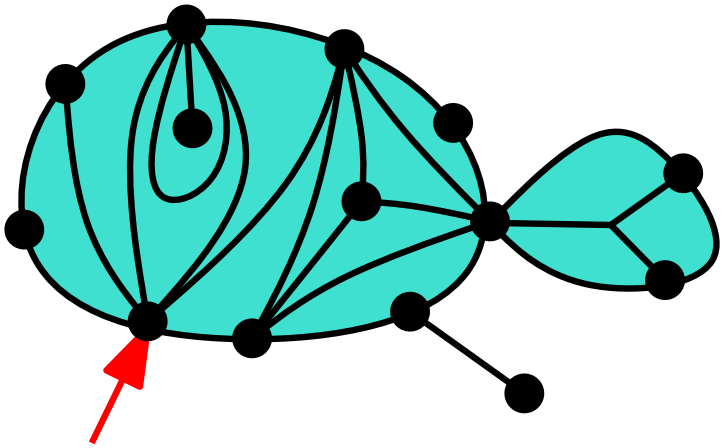
• The probability that the black-white interface started at the root-edge of a random percolated triangulation has length n decreases **polynomially** in $n^{-10/3}$ if $p = 1/2$, and **exponentially** if $p \neq 1/2$.

Recursive decomposition of triangulations

Generating function: $G(x,z) = \sum_T x^{\#\text{outer degree}}\, z^{\#\text{edges}}$



$=$ $?$

Triangulation with boundary

Recursive decomposition of triangulations

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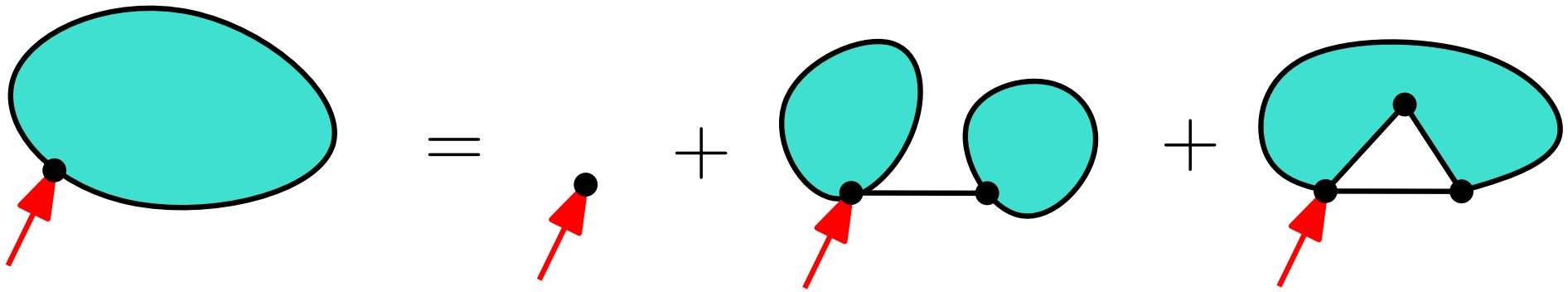

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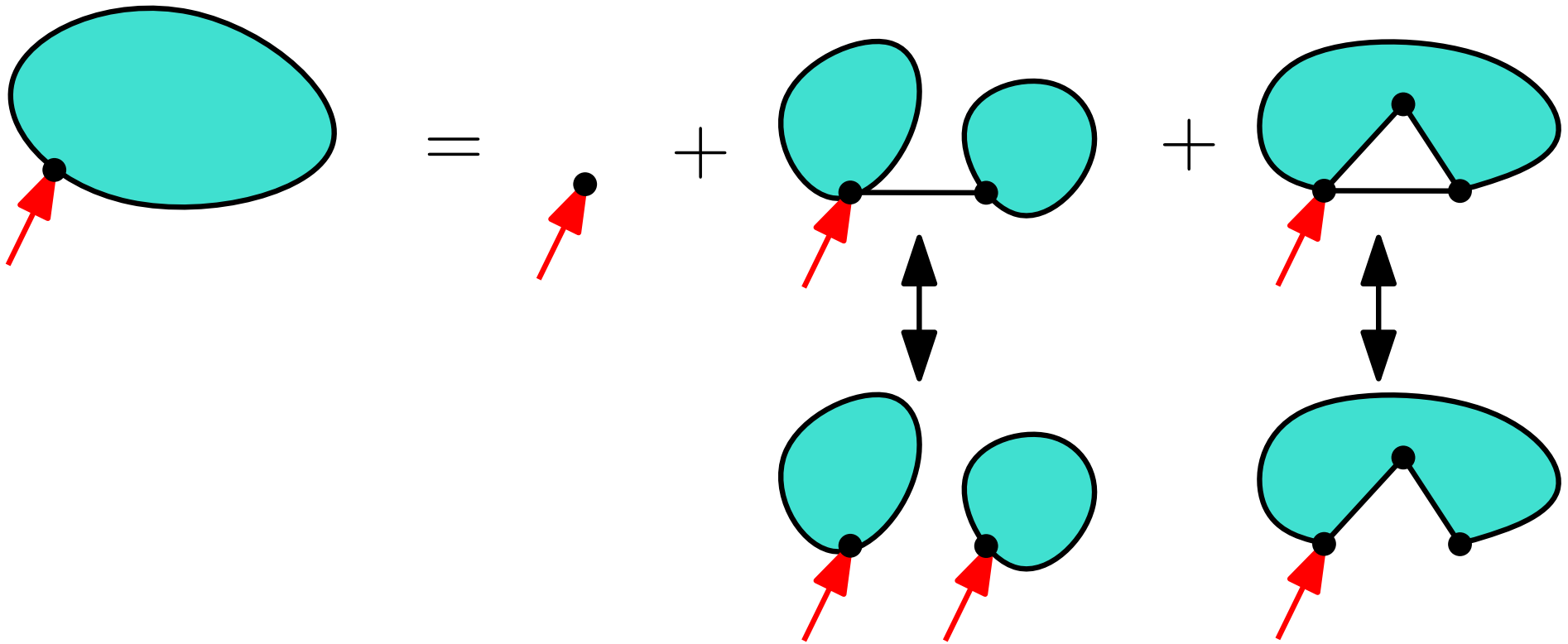

$$G(x,z) = 1 + x^2 z\, G(x,z) \times G(x,z) \;+\; \frac{z}{x}(G(x,z) - 1 - x[x^1]G(x,z))$$

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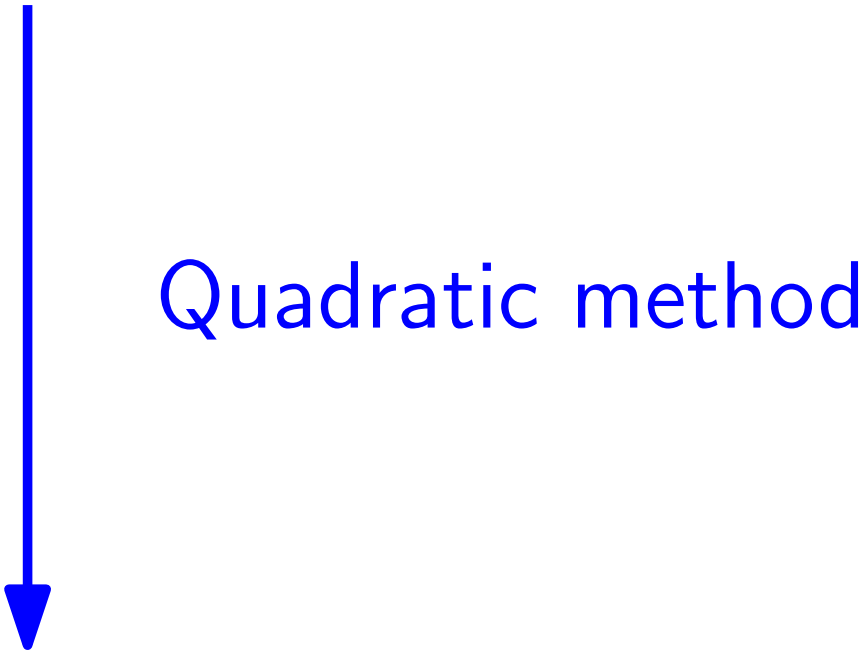
Problem: this kind of **functional equation** does not directly give counting results.

Solution: use **quadratic method** to obtain an **algebraic equation**.

Quadratic method - version Bousquet-Mélou, Jehanne 06.

Input: **Functional equation** $P(G(x,z), G_1(z), x, z) = 0$.



Output: **Algebraic equation** $Q(G_1(z), z) = 0$

Quadratic method - version Bousquet-Mélou, Jehanne 06.

Input: **Functional equation** $P(G(x,z), G_1(z), x, z) = 0$.

Theorem: Under mild hypotheseses, there exists a series $X \equiv X(z)$ such that $P_1'(G(X(z),z), G_1(z), X(z), z) = 0$. Thus,

$$\begin{cases} P(G(X(z),z), G_1(z), X(z), z) & = 0 \\ P_1'(G(X(z),z), G_1(z), X(z), z) & = 0 \\ P_3'(G(X(z),z), G_1(z), X(z), z) & = 0. \end{cases}$$

Hence polynomial elimination of $X(z)$ and $G(X(z),z)$ gives:

Output: **Algebraic equation** $Q(G_1(z), z) = 0$

Counting triangulations

Generating function: $G(x) \equiv G(x,z) = \sum_T x^{\#\text{outer degree}}\, z^{\#\text{edges}}$

- Recursive decomposition -

Functional equation: $G(x) = 1 + x^2 z\, G(x)^2 + \dfrac{z}{x}\,(G(x) - 1 - xG_1)$

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- Quadratic method -

Algebraic equation:

$$64z^5{G_1}^3 - 96z^4{G_1}^2 + z{G_1}^2 + 30z^3G_1 - G_1 - 27z^5 + z^2 = 0$$

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- Analytic combinatorics -

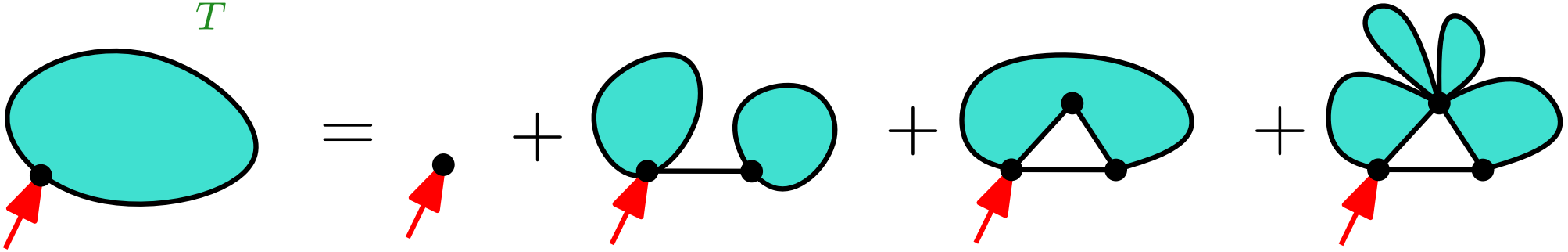
Asymptotic result:

The number $T_n = [z^{n+2}]G_1(z)$ of triangulations with n edges satisfies:

$$T_n \;\sim\; c\, n^{-5/2}\, z_0^{-n}, \qquad \text{where } z_0 = 432^{-1/6}.$$

Counting triangulations with weight p per external vertex

$$G(x) = \sum_T p^{\#\text{outer vertices}} x^{\#\text{outer degree}} z^{\#\text{edges}}$$



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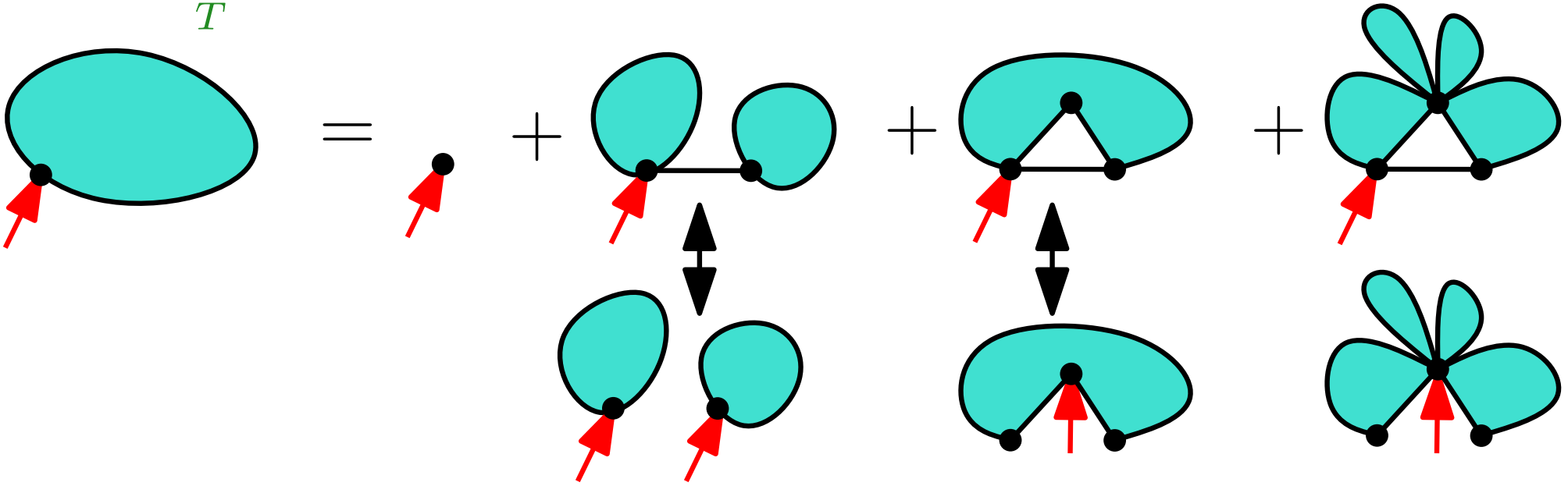

$$G(x) = p + x^2 z\, G(x)^2 + \frac{z}{px}(\tilde{G}(x) - xp\, G_1) + \frac{z}{x}(G(x) - \tilde{G}(x))$$

where $G_1 = [x]G(x)/p$ and $G(x) = \dfrac{p}{1 - \tilde{G}(x)/p}$.

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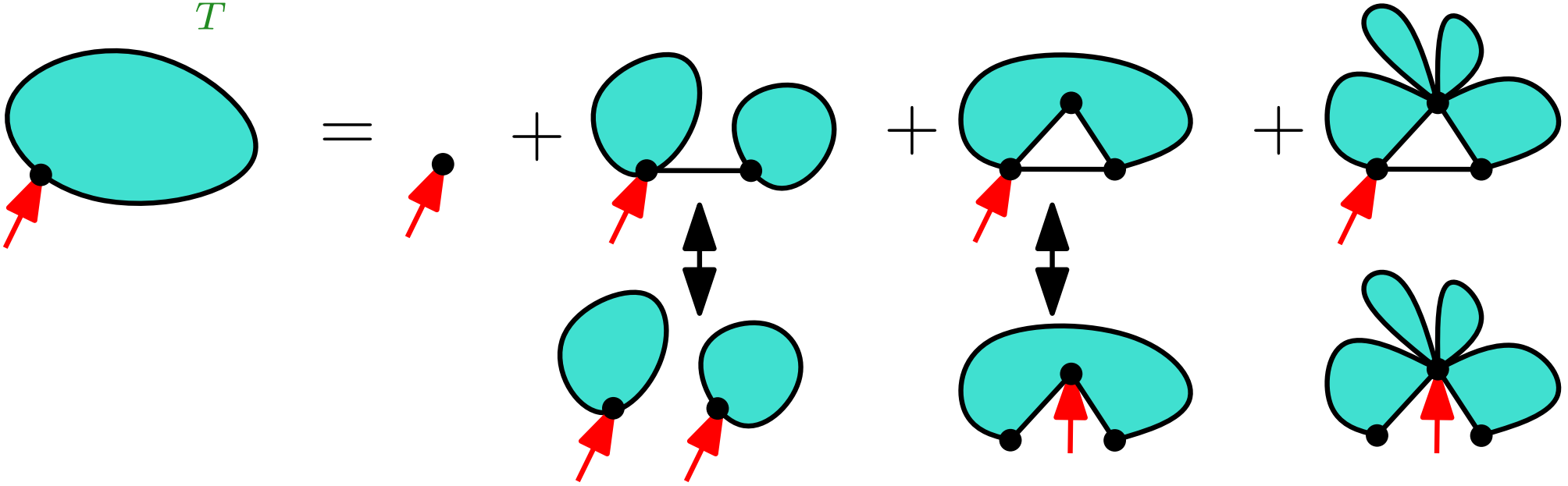

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Moreover,

$$64z^5 {G_1}^3 - 96z^4 {G_1}^2 + z{G_1}^2 + 30z^3 G_1 - G_1 - 27z^5 + z^2 = 0$$

$\Rightarrow$ **Algebraic equation**: $P(G(x), p, x, z) = 0$.

Counting triangulations with weight p per external vertex

$$H(p,x) = G(p,x,z_0) = \sum_T p^{\#\text{outer vertices}} x^{\#\text{outer degree}} z_0^{\#\text{edges}}$$

Algebraic equation: $P_0(H(p,x),p,x) = 0$.

- Analytic combinatorics -

Theorem: The total weight $W_n = [x^n]H(p,x)$ of triangulations with outer degree n with black outer vertices satisfies:

$$W_n \sim \begin{cases} c(p)\, n^{-3/2}\, K(p)^n & \text{if } p < 1/2, \\ c(p)\, n^{-5/3}\, K(p)^n & \text{if } p = 1/2, \\ c(p)\, n^{-5/2}\, K(p)^n & \text{if } p > 1/2. \end{cases}$$

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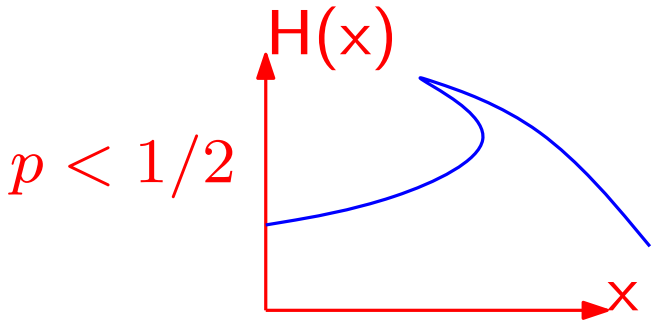
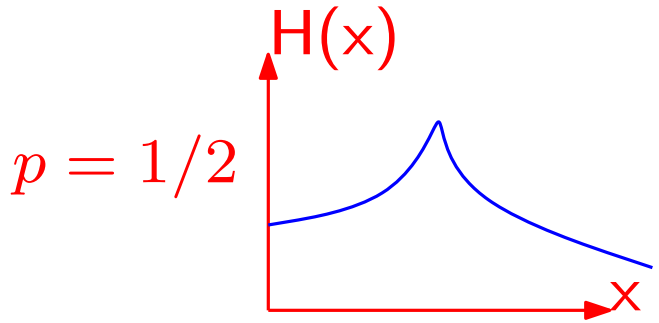
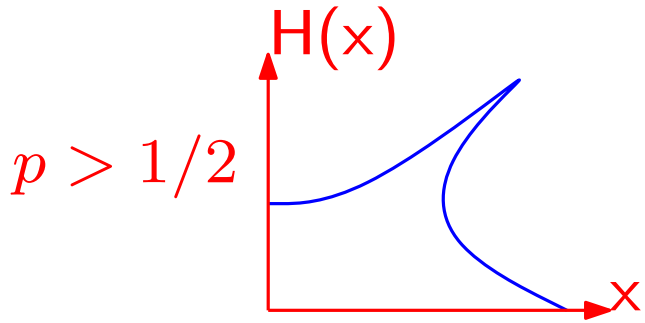


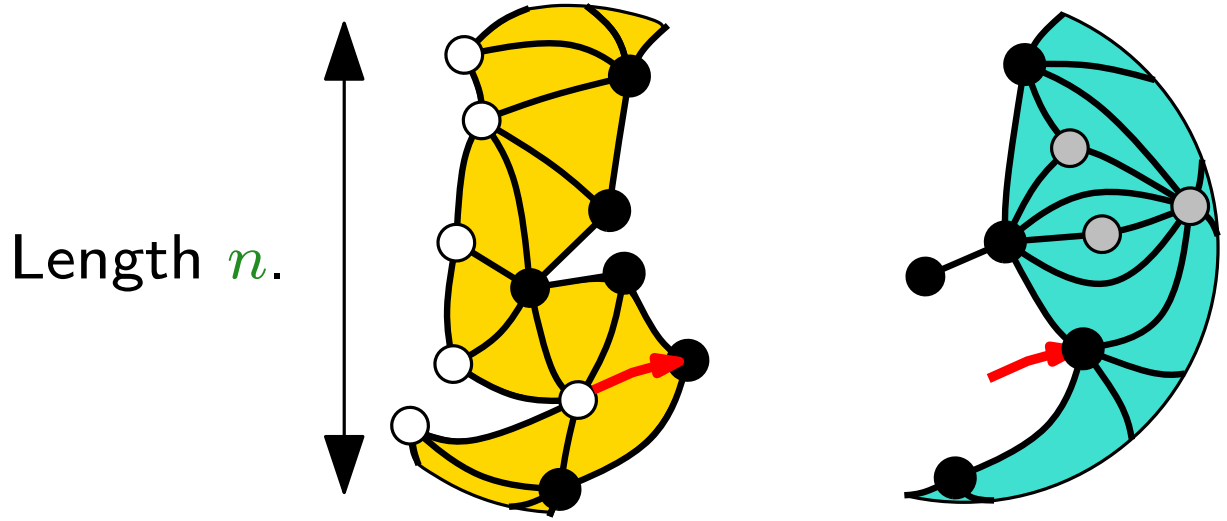




Integrating the Necklace

Lemma. Total weight of necklace + black component is $Q_n := \sum_{k \geq 0} \binom{n}{k} W_k$.

Moreover if $W_n \sim c(p)\, n^{a(p)} K(p)^n$ then $Q_n \sim \tilde{c}(p)\, n^{a(p)} \left(\frac{1}{1 - K(p)} \right)^n$



Proof: Express N_n as a residue and compute by method of Hankel contour. □

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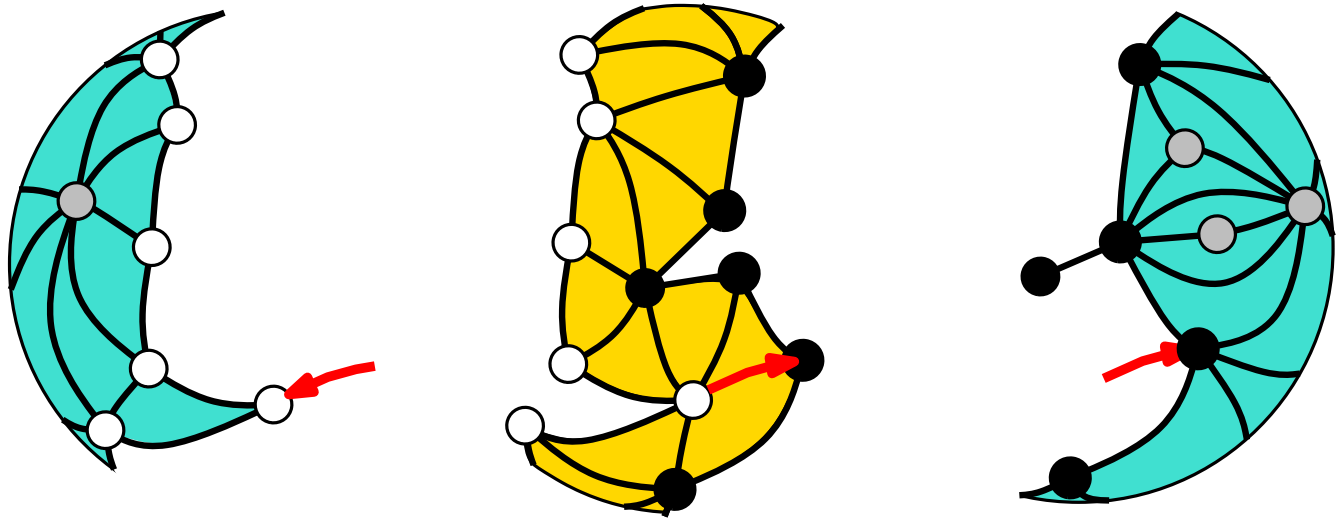


Corollary: The weight for interface of white length n is $\sim \hat{c}(p)\ n^{a(p)+a(1-p)} \left(\frac{K(p)}{1-K(p)}\right)^n$ where $1/K(p)$ is the largest root of $(18px^3-9x^3+5x^3\sqrt{3}+6x^2\sqrt{3}-48\sqrt{3}x+32\sqrt{3})(2px-x+\sqrt{3}x-2\sqrt{3})$.

$\Rightarrow$**polynomially** in $n^{-10/3}$ if $p=1/2$, and **exponentially** if $p\neq 1/2$.

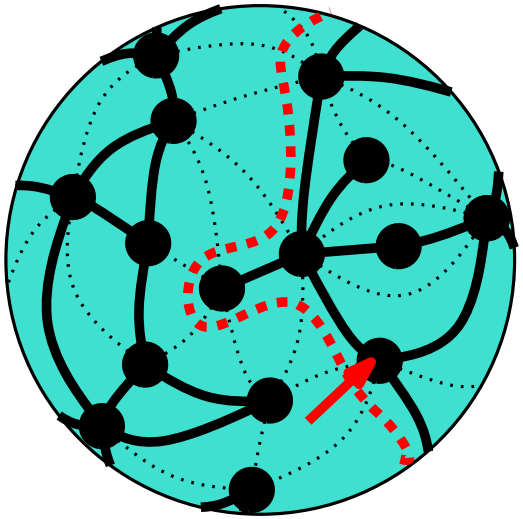
Average size on each side of loops

Theorem: The mean number of edges on the black side of an interface of length n is of order

$$\begin{array}{ll} n & \text{if } p < 1/2, \\ n^{4/3} & \text{if } p = 1/2, \\ n^2 & \text{if } p < 1/2. \end{array}$$

Bond percolation

The counting problem for bond percolation.



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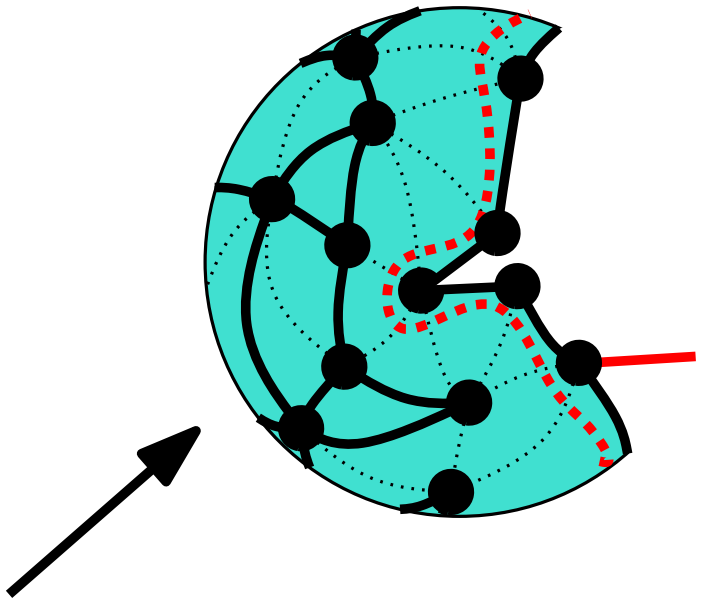
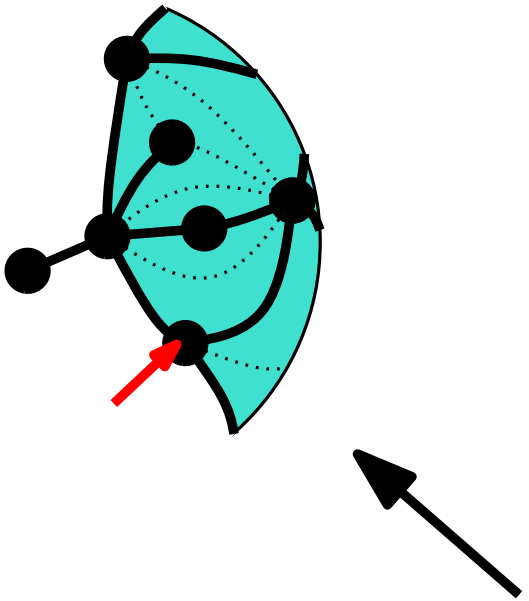


Edges incident to
outer vertices are closed.



Outer edges are open.

Bond percolation

The counting problem for bond percolation.

Theorem. Let $p_c = \frac{2\sqrt{3}-1}{11} \approx 2.22$.

• The total weight $W_n, \hat{W}_n$ of triangulations with outer degree n with outer edges open (resp. with edges incident to outer vertices closed) satisfies:

$$W_n \sim c(p)\, n^{-3/2}\, K(p)^n \quad \text{and} \quad \hat{W}_n \sim \hat{c}(p)\, n^{-5/2}\, \hat{K}(p)^n \quad \text{if } p < p_c,$$
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Bond percolation

The counting problem for bond percolation.

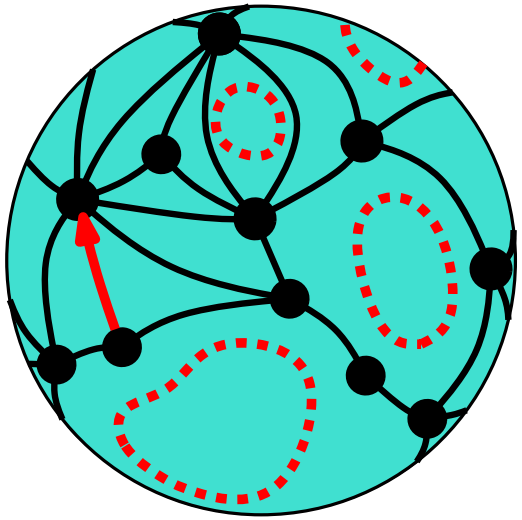
Theorem. Let $p_c = \frac{2\sqrt{3}-1}{11} \approx 2.22$.

• The probability that the bond-percolation interface started at the root-edge of a random percolated triangulation has length n decreases **polynomially** in $n^{-10/3}$ if $p = 1/2$, and **exponentially** if $p \neq 1/2$.

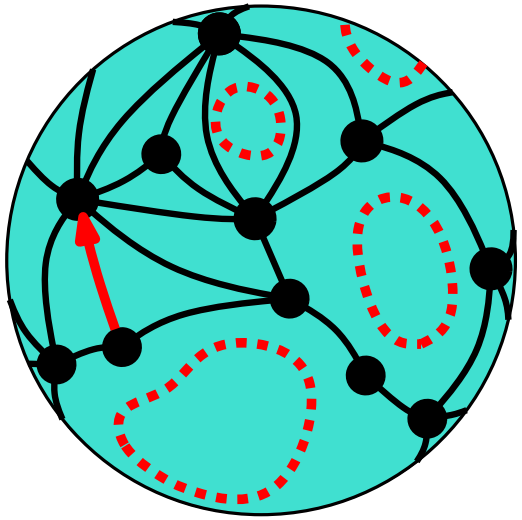
• The mean size of the two sides of an interface conditioned to have size n are of order

Closed side	Open side	
n	n^2	if $p < p_c$
$n^{4/3}$	$n^{4/3}$	if $p = p_c$
n^2	n	if $p < p_c$

The random gasket



What do we know about the gasket?



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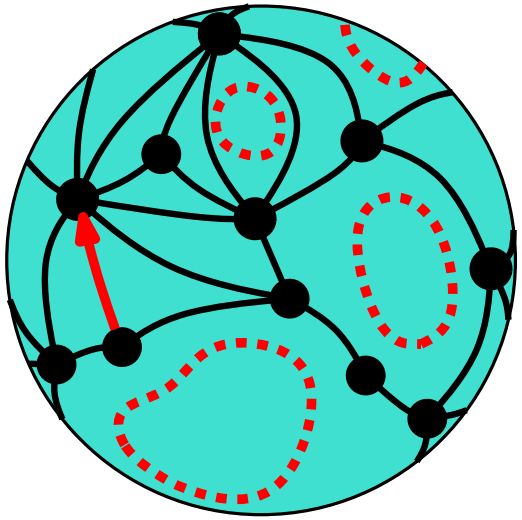

Lemma. The probability that the degree of a random face of the gasket has degree n is of order $n^{-10/3}$ at criticality, and has an exponential tail otherwise.

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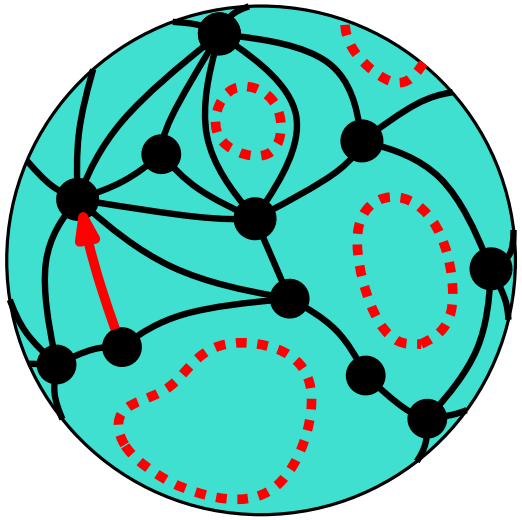

Lemma. The probability that the degree of a random face of the gasket has degree n is of order $n^{-10/3}$ at criticality, and has an exponential tail otherwise.

Proof. Let $F=$ face at the right of a uniformly random edge in gasket.

$$\mathbb{P}(\deg(F)=n)=\mathbb{P}(\deg(F_0)=n)=\frac{I_n}{Z},$$

where I_n is the total weight of percolated triangulations rooted on an interface of length n.

The gasket as a Boltzmann map

Def. Given weights $\mathbf{q} = (q_n)_{n>0}$, we call **q-Boltzmann map** a map chosen with probability proportional to $\prod_{f \text{ face}} q_{\deg(f)}$.

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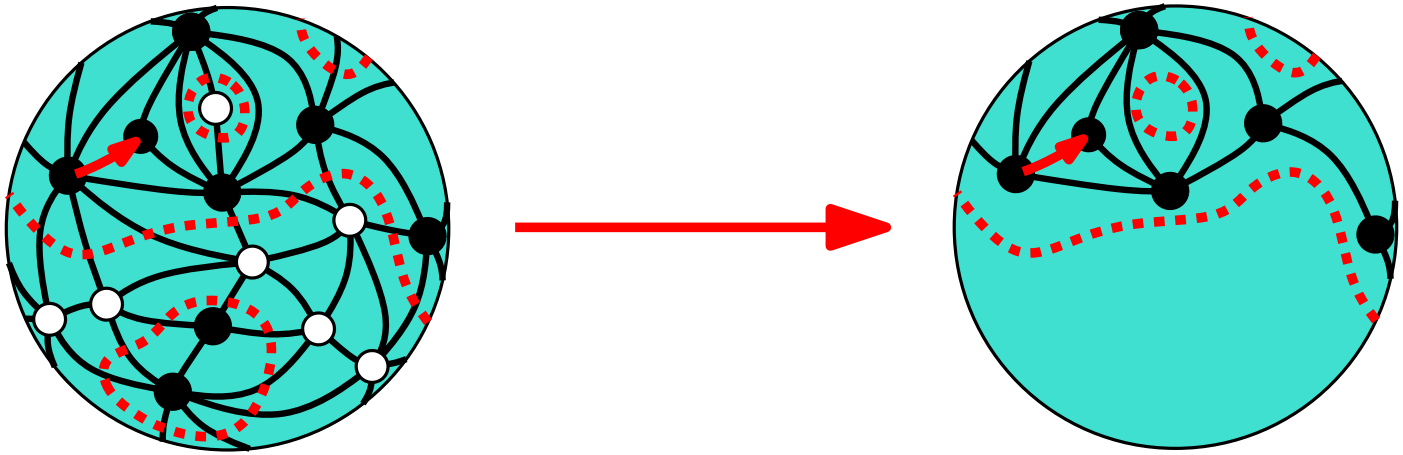


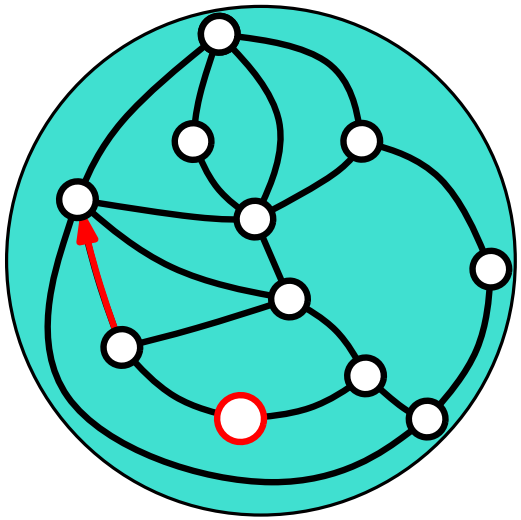
Remark. Boltzmann distribution on triangulations+percolations gives a gasket which is a **q**-Boltzmann map, where

$$q_n = p^{n/2-1}(z_0^{3/2}\delta_{n,3} + Q_n(p)).$$

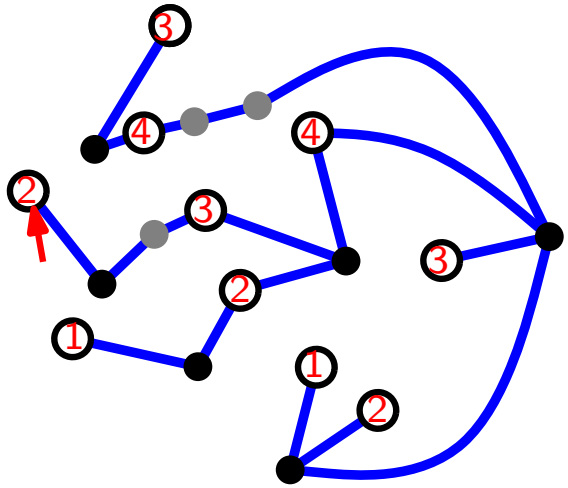
Total weight of necklace of outside length n + inside with white outer vertices.

Bijection with trees [Bouttier, Di Francesco, Guitter 04]

rooted pointed map



plane tree + labels



vertices ⟷ white vertices

faces ⟷ black vertices

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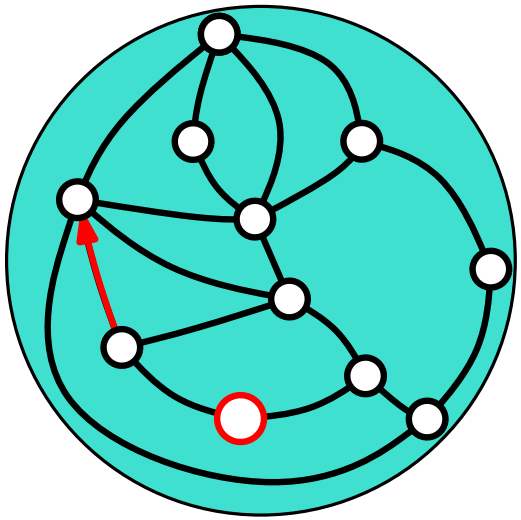

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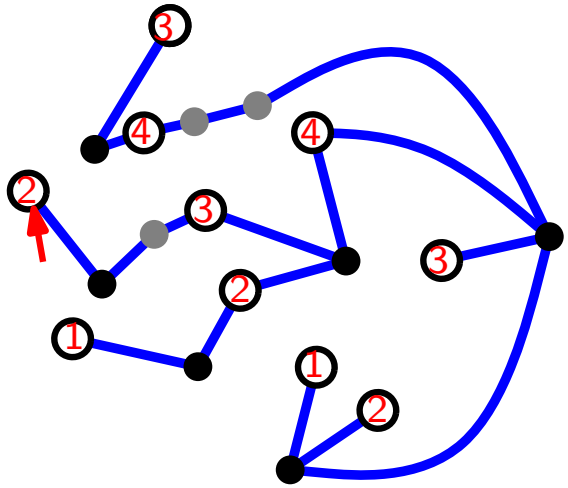

vertices ⟷ white vertices

faces ⟷ black vertices + gray vertices

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q-Boltzmann map ⟷ **q**-Galton-Watson tree

$$\mathbb{P}_{\circ}(i\bullet) = (1-1/R)^i/R$$

$$\mathbb{P}_{\bullet}(i\circ, j\bullet) = \binom{2i+j+1}{i+1,i,j} q_{2i+j+2} R^i S^j$$

$$\mathbb{P}_{\bullet}(i\circ, j\bullet) = \binom{2i+j}{i,i,j} q_{2i+j+1} R^i S^j$$

where R, S defined in terms of $\mathbf{q}$.

Criticality of the gasket

Def: We say that a weight sequence (q_n) is **critical** if the corresponding Galton Watson tree is critical
(that is, the matrix of reproduction-means has spectral radius 1).

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Theorem.

- The weights (q_n) defining the gasket are critical for $p \geq 1/2$.
- The size of the gasket has a tail in $n^{-20/7}$ if $p = 1/2$,
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  $n^{-5/2}$ if $p > 1/2$.

Proof of criticality.

- Criticality of (q_n) is equivalent to $R'(1) = \infty$ where $R(u), S(u)$ are the smallest non-negative solutions of
  $R(u) = 1 + \sum_{i,j} \binom{2i+j+1}{i+1,i,j} q_{2i+j+2} R(u)^{i+1} S(u)^j,$
  $S(u) = \sum_{i,j} \binom{2i+j}{i,i,j} q_{2i+j+1} R(u)^i S(u)^j.$
- The total weight of maps with outer degree n is $\int_0^1 \sum_{2i+j=n} \binom{n}{i,i,j} R(u)^i S(u)^j$ which behaves like $c\, n^{-3/n} \left(S(1) + 2\sqrt{R(1)}\right)^n$ unless $R'(1) = \infty$.

Scaling limit of the gasket

Theorem. Suppose (q_n) is a sequence of weights which is critical and such that $q_{2n+1} = 0$ and $q_{2n} \sim c\, n^{-\gamma-1/2}\, K^n$ with $\gamma \in (1,2)$.
Let $M_n =$ random metric space (V, d_{gr}) corresponding to a **q**-Boltzmann map conditioned to have n vertices.

Then $n^{-2\gamma} M_n$ converges in law (in the Gromov-Haussdorff topology) toward a compact metric space of Hausdorff dimension 2γ.

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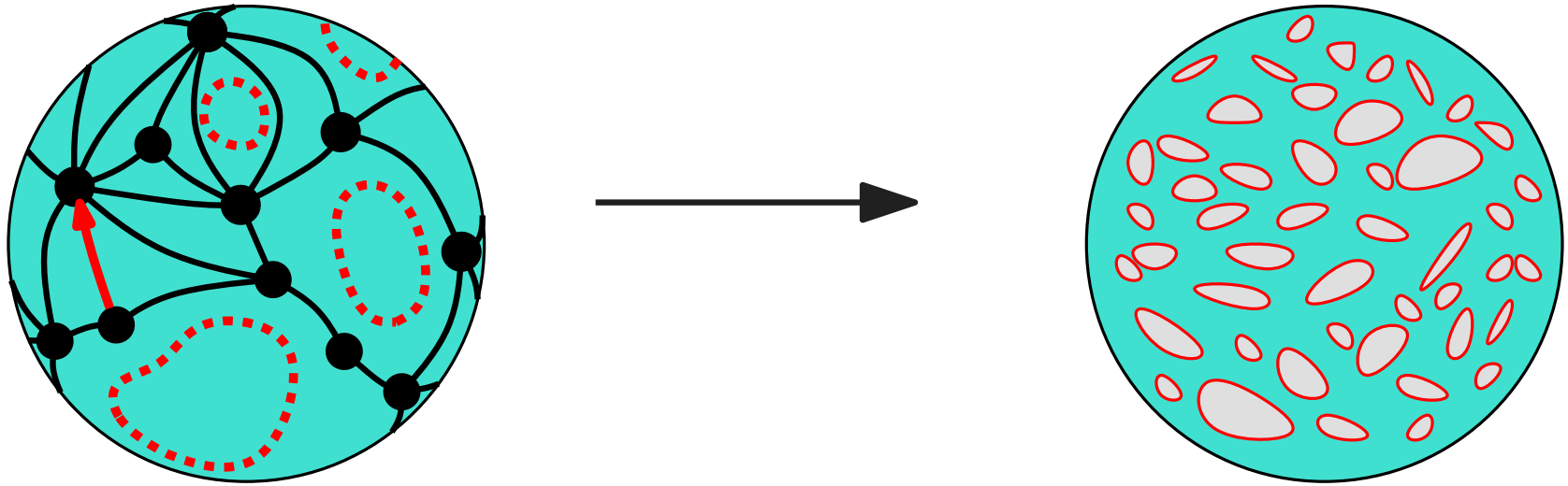
If the result extends to the non-bipartite setting, then the gasket of percolation converges toward the stable map of parameter $\gamma = 7/6$.



Thanks.