The scaling limit of uniform random plane quadrangulations

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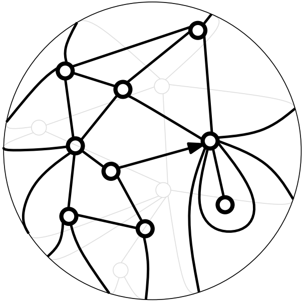
Plane maps

Definition

A plane map is an embedding of a connected, finite (multi)graph into the 2-dimensional sphere, considered up to orientation-preserving homeomorphisms of the sphere.



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$E(\mathbf{m})$ Edges
$F(\mathbf{m})$ Faces

$$\#V(\mathbf{m}) - \#E(\mathbf{m}) + \#F(\mathbf{m}) = 2 \quad \text{(Euler)}$$

A rooted map: one distinguished oriented edge.

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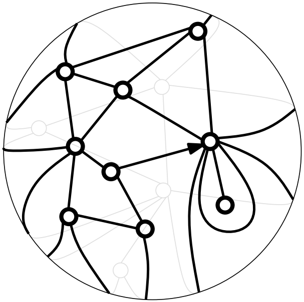

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General motivation

Maps appear naturally in many contexts

- Graph theory (4-color theorem, ...)
- Counting problems
  - by direct resolution of the equation solved by generating functions, using the quadratic method [Tutte, Bender-Canfield, Bousquet-Mélou-Jehanne, Bousquet-Mélou-Bernardi]
  - by using matrix integrals [t'Hooft, Brézin-Parisi-Itzykson-Zuber, ...]
  - by algebraic methods: representation theory of the symmetric group, algebraic geometry [Goulden-Jackson,...]
  - by bijective methods [Cori-Vauquelin, Schaeffer, Poulalhon, Bouttier-Di Francesco-Guitter, Bernardi, Chapuy, Fusy,...]
- Theoretical physics: random maps are natural models of random surfaces (discretization of 2D quantum gravity) [Polyakov, Kazakov, Kawai, Ambjørn & Watabiki, ...]
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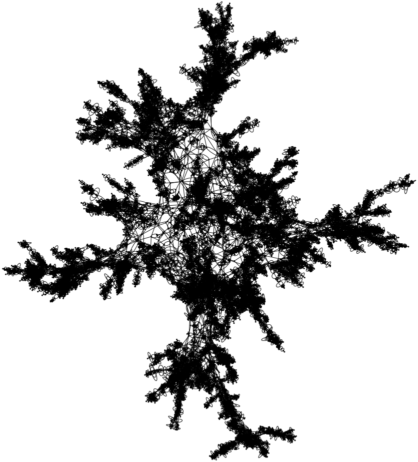
Simulation of a uniform random plane quadrangulation with 30000 vertices, by J.-F. Marckert



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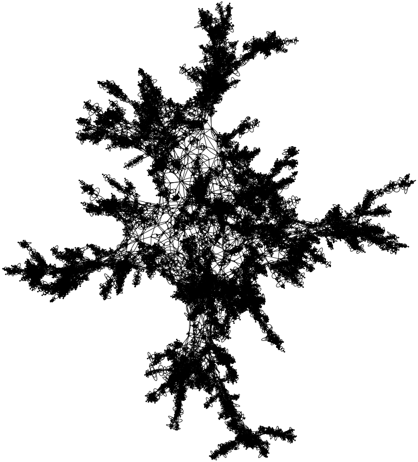

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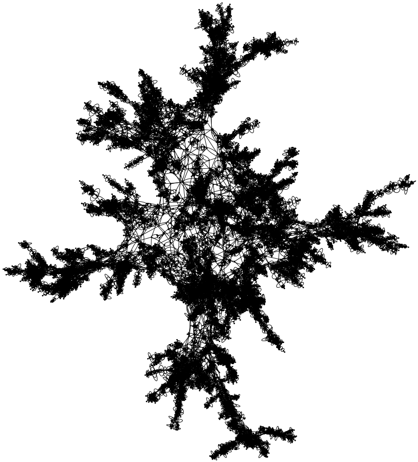

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Topologies on metric spaces

A natural framework for random metric spaces is to compare them using the Gromov-Hausdorff distance. If $(X, d), (X', d')$ are compact metric, let

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the infimum being taken over isometric embeddings of X, X' into a common metric space (Z, δ) and δ_H is the usual Hausdorff distance between compact subsets of Z.

Proposition

This endows the space $\mathbb{M}$ of isometry classes of compact spaces with a complete, separable distance.

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Convergence to the Brownian map

Theorem (M. (2011))

There exists a random metric space (S, D^*), called the Brownian map, such that the following convergence in distribution holds

$$(V(Q_n), (8n/9)^{-1/4} d_{Q_n}) \xrightarrow[n\to\infty]{(d)} (S, D^*)$$

as $n \to \infty$, for the Gromov-Hausdorff topology.

- This result has been proved independently by Le Gall (2011), *via* a different approach.
- Before this work, convergence was only known up to extraction of subsequences, but the uniqueness of the limiting law was open.
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- Chassaing-Schaeffer (2004) identify $n^{1/4}$ as the proper scaling and compute certain limiting functionals for random quadrangulations. Generalized by Marckert-M. (2007), M. (2008) to the larger class of Boltzmann random maps.
- Marckert-Mokkadem (2006) establish limit theorems (in a sense weaker than Gromov-Hausdorff), and introduce the Brownian map.
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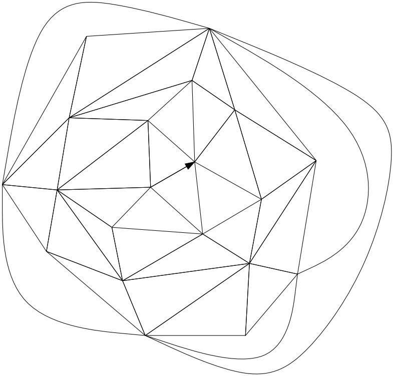
Other topologies

- Similar results are known when the sphere is replaced by the genus-g torus, building on bijective results by Chapuy-Marcus-Schaeffer (2007), Chapuy (2009).
- Bettinelli (2010) shows that subsequential limits of genus-g random bipartite quadrangulations exist, have Hausdorff dimension 4, and have the same topology as the g-torus.
- Similar results are known for plane quadrangulations with a boundary (Bettinelli).

Local limits

- In another direction, Angel-Schramm (2002) and Angel (2002) consider local limit results for random triangulations. They construct the so-called uniform infinite planar triangulation (UIPT). See also Krikun (2003,2005).
- Followed by work of Chassaing-Durhuus (2006) Ménard (2008), that generalize the bijective approaches in this infinite context.



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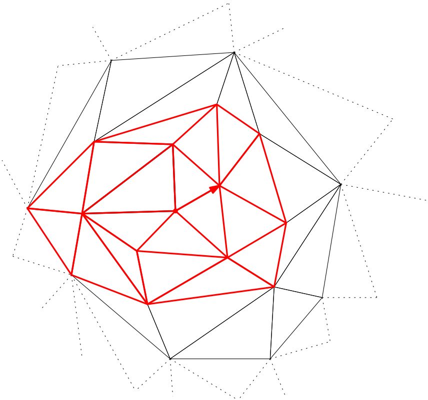

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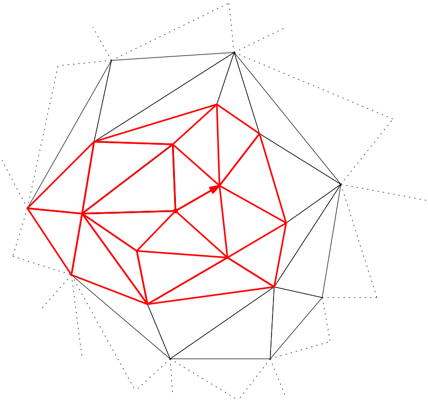
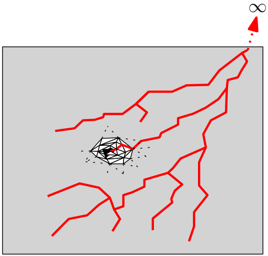


Geometry at infinity of the UIPQ

Theorem (Curien-Ménard-M. (2012))

- *There exists a sequence of vertices $p_1, p_2, \dots$ such that any infinite geodesic path goes through every but a finite number of the vertices $p_i, i \geq 1$.*
- *Moreover it holds that for every vertices x, y, $z \mapsto d_{\mathrm{gr}}(x,z) - d_{\mathrm{gr}}(y,z)$ takes the same value for every but a finite number of z's.*

This says that the Uniform Infinite Planar Quadrangulation has an essentially unique infinite geodesic path, that leads to a single point at infinity.



Schaeffer's bijection: coding maps with trees

- Let $\mathbf{T}_n$ be the set of rooted plane trees with n edges,pause
- $\mathbb{T}_n$ be the set of labeled trees $(\mathbf{t}, \mathbf{l})$ where $\mathbf{l} : V(\mathbf{t}) \to \mathbb{Z}$ satisfies $\mathbf{l}(\mathrm{root}) = 0$ and

$$|\mathbf{l}(u) - \mathbf{l}(v)| \leq 1\,, \qquad u, v \text{ neighbors}\,.$$

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The construction to follow yields a bijection between $\mathbb{T}_n \times \{0, 1\}$ and $\mathbf{Q}_n^*$, the set of rooted, pointed plane quadrangulations with n faces.

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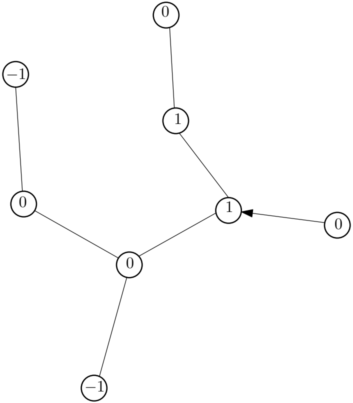

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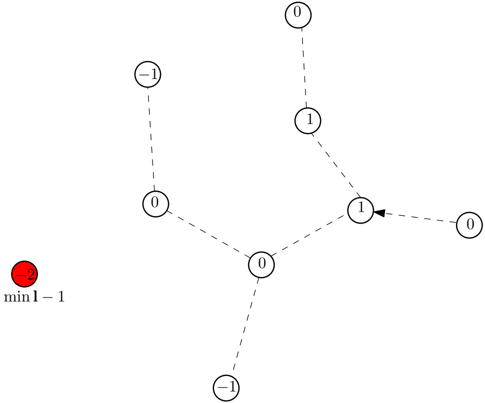

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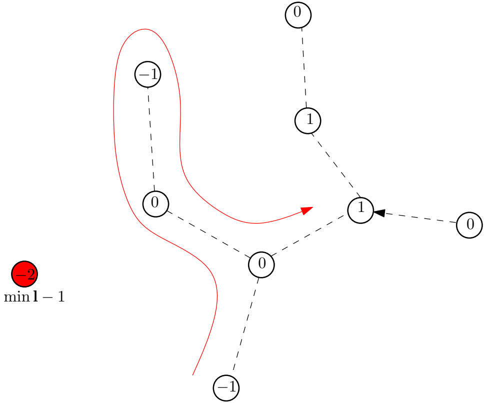

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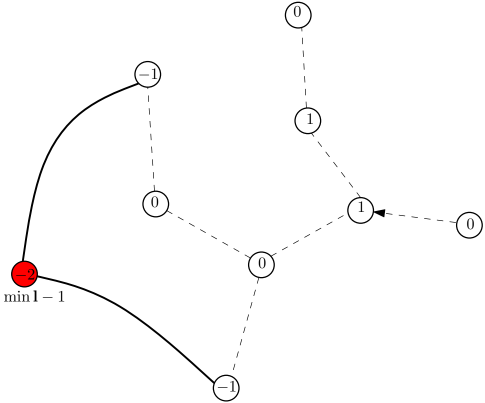

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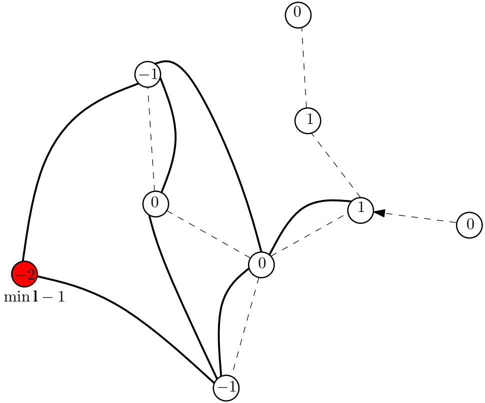

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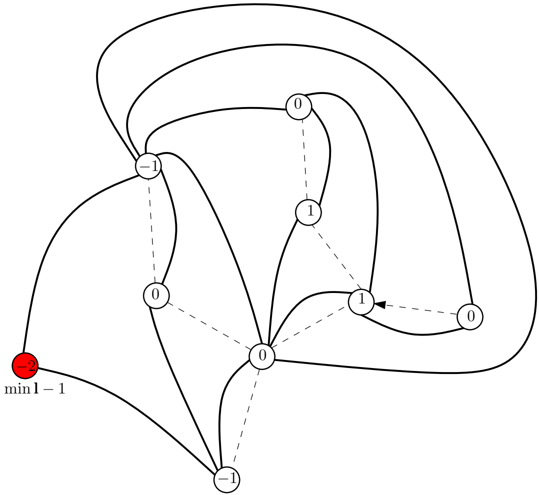

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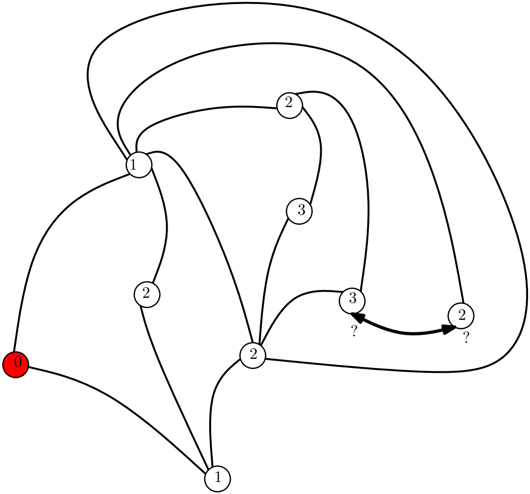


Note that the labels are geodesic distances in the map. Key formula:

$$d_{\mathbf{q}}(v_*, v) = \mathbf{l}(v) - \inf \mathbf{l} + 1$$

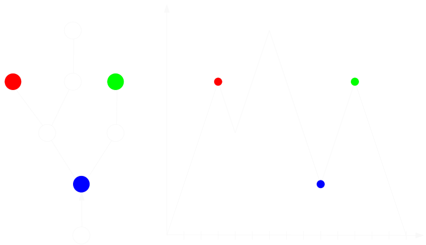
Scaling limits for plane trees: Aldous' CRT

- The Brownian tree arises as the scaling limit of many discrete random tree models, e.g. uniform random element T_n of $\mathbf{T}_n$:

$$(V(T_n), (2n)^{-1/2} d_{T_n}) \to \mathcal{T},$$

  for the Gromov-Hausdorff distance.

- Note that a tree with n edges can be encoded by a walk (Harris encoding): let $u_i, 0 \le i \le 2n$ be the $i+1$-th explored vertex in contour order (started at the root). Let C_i the height of u_i.



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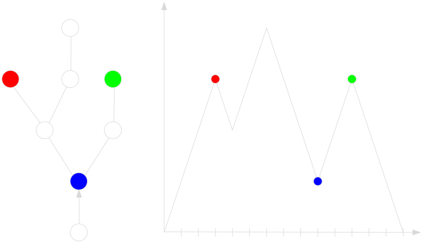

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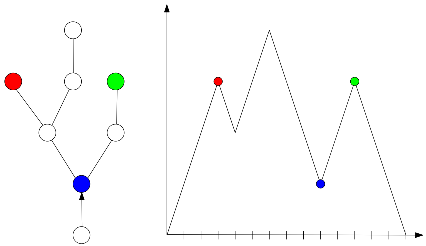

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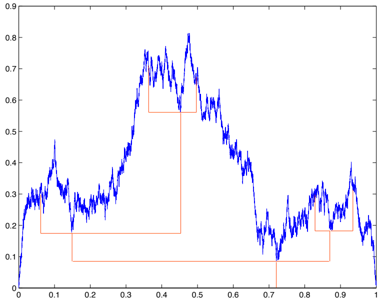
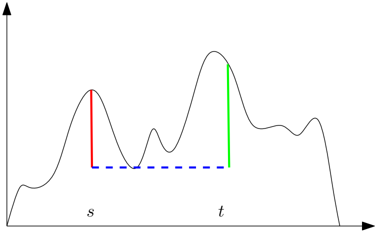
- Define

$$d_{\mathbb{e}}(s,t) = \mathbb{e}_s + \mathbb{e}_t - 2 \inf_{s \wedge t \le u \le s \vee t} \mathbb{e}_u .$$

This is a pseudo-distance on $[0,1]$. The continuum random tree is the quotient space $\mathcal{T}_{\mathbb{e}} = [0,1]/\sim_{\mathbb{e}}$, where $s \sim t \iff d_{\mathbb{e}}(s,t) = 0$. It defines an $\mathbb{R}$-tree.





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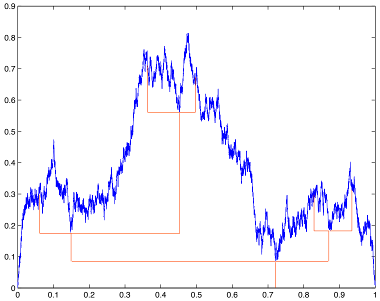
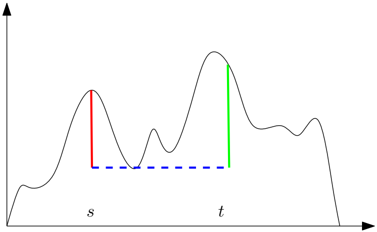
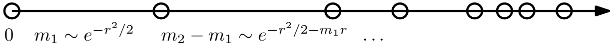
$$d_{\mathbb{e}}(s,t) = \mathbb{e}_s + \mathbb{e}_t - 2 \inf_{s \wedge t \leq u \leq s \vee t} \mathbb{e}_u \,.$$

This is a pseudo-distance on $[0,1]$. The continuum random tree is the quotient space $\mathcal{T}_{\mathbb{e}} = [0,1]/\sim_{\mathbb{e}}$, where $s \sim t \iff d_{\mathbb{e}}(s,t) = 0$. It defines an $\mathbb{R}$-tree.

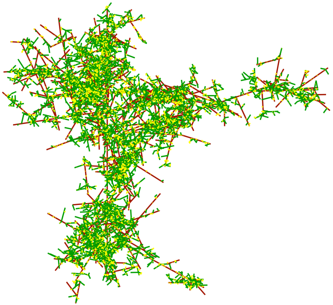




Stick-breaking construction



- Build $\mathcal{T}_\circ$ as an $\mathbb{R}$-tree, by grafting segments drawn from a Poisson measure on $\mathbb{R}_+$ with intensity $t\mathrm{d}t$ recursively at a uniform location in the tree constructed at each stage.
- Then let $\mathcal{T}$ be (the isometry class of) the metric completion of $\mathcal{T}_\circ$. It holds that $\mathcal{T} =_d \mathcal{T}_\mathbb{e}$.



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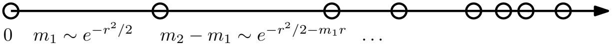

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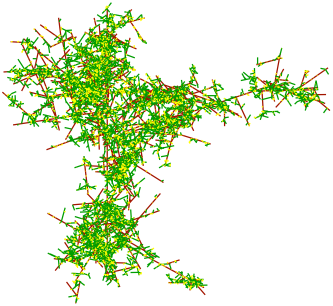

Brownian labels on the Brownian tree

- Once the tree is build, one can consider a white noise supported by the tree, or, equivalently, branching Brownian paths.
- Informally, we let Z be a centered Gaussian process run on $\mathcal{T}$, with covariance function

$$\mathrm{Cov}\,(Z_a, Z_b) = d_{\mathcal{T}}(\mathrm{root}, a \wedge b)\,,$$

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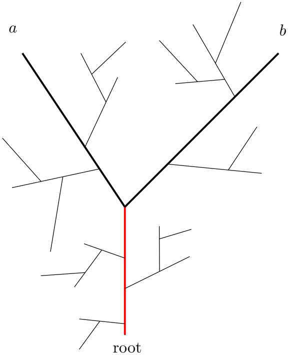

Convergence of labeled trees

- Let (T^n, ℓ^n) be uniform in $\mathbb{T}_n$. Then

$$\left(\frac{1}{\sqrt{2n}} T_n, \left(\frac{9}{8n}\right)^{1/4} \ell_n\right) \xrightarrow[n\to\infty]{(d)} (\mathcal{T}_{\mathbb{e}}, Z),$$

  e.g. in the sense of convergence of contour encoding functions.

- We want to apply to $(\mathcal{T}_{\mathbb{e}}, Z)$ a similar construction as Schaeffer's bijection. Assume

$$(T_n = V(Q_n) \setminus \{v_*\}, (8n/9)^{-1/4} d_{Q_n}) \xrightarrow[n\to\infty]{(d)} (\mathcal{T}_{\mathbb{e}}, D)$$

  where D is some random (pseudo-)distance on $\mathcal{T}_{\mathbb{e}}$ (a true distance on $\mathcal{T}_{\mathbb{e}}/\{D = 0\}$).

- The distance D should satisfy the continuous analog of the distance estimates to v_* in the discrete setting:

$$D(a, a_*) = Z_a - \inf Z$$

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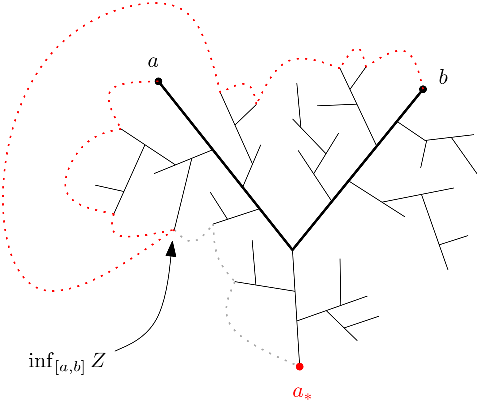
The Brownian map

- Set

$$D^\circ(a,b) = Z_a + Z_b - 2\max\left(\min_{[a,b]} Z, \min_{[b,a]} Z\right) .$$

This will be an upper-bound for $D(a,b)$, and equal to $D(a,b)$ whenever $a = a_*$ or more generally if a_*, a, b are aligned.



- A straightforward analog of D° gives an upper-bound for the distance in the discrete setting, by concatenating pieces of geodesics from a, b to a_*.

The Brownian map

- The function D° is not a pseudo-distance, so we set

$$D^*(a,b) = \inf\left\{\sum_{i=1}^{k-1} D^\circ(a_i, a_{i+1}) : a_1 = a, a_k = b\right\},$$

  the largest pseudo-distance on $\mathcal{T}_\mathbb{e}$ that is less than D°.
- The space (S, D^*) where

$$S = \mathcal{T}_\mathbb{e}/\{D^* = 0\}$$

  is called the Brownian map.
- The method to prove that $(V(Q_n), (8n/9)^{-1/4} d_{Q_n})$ converges in distribution to (S, D^*) is to show that the subsequential limit $(\mathcal{T}_\mathbb{e}, D)$ satisfies $D = D^*$.
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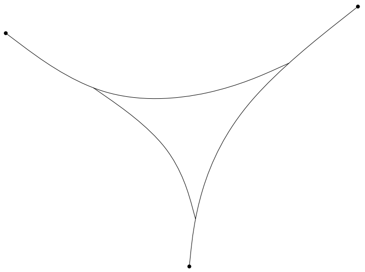
Idea of proof: Shape of the typical geodesics

- The main idea for the proof is to describe precisely the geodesic γ between two "generic" points x_1, x_2. More precisely, one must show that it is a patchwork of small segments of geodesic paths headed toward a_* (geodesics tend to stick).
- So we want to show that Γ, the set of points x on γ from which we can start a geodesic to a_* not meeting γ again, is a small set.



Proposition

There exists $\delta \in (0, 1)$ such that a.s. for every $\varepsilon > 0$, the set Γ can be covered with less than $\varepsilon^{-(1-\delta)}$ D-balls of radius ε. In particular $\dim_{\mathcal{H}}(\Gamma) < 1$.

Segments outside the "bad" purple set have coinciding D° and D^*-lengths.

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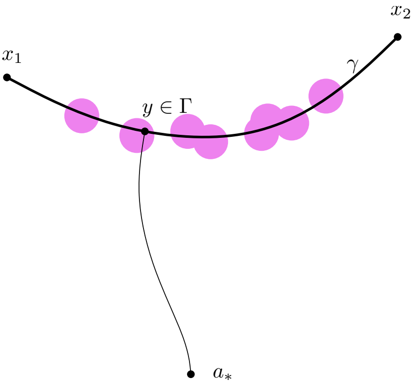

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Idea of proof: almost equivalence between D and D^*

The previous proposition is enough, once complemented by the following lemma.

Lemma

For every $\alpha \in (0,1)$, there exists a random $C \in (0,\infty)$ such that $D^*(a,b) \leq CD(a,b)^\alpha$ for every $a,b \in \mathcal{T}_{\mathbb{e}}$.

- This lemma comes from precise volume estimates of balls for both metrics D and D^* (Le Gall 2010).
- Covering Γ with at most $\varepsilon^{-(1-\delta)}$ balls for D breaks γ into segments $[a_i,b_i], 1 \leq i \leq K$ say, each of which has $D^*(a_i,b_i) = D(a_i,b_i)$. Then for $\alpha > 1-\delta$,

$$\begin{aligned} D^*(x_1,x_2) &\leq \sum_{i=1}^{K} D^*(a_i,b_i) + \varepsilon^{-(1-\delta)} \sup_{D(a,b)\leq 2\varepsilon} D^*(a,b) \\ &\leq \sum_{i=1}^{K} D(a_i,b_i) + C(2\varepsilon)^{\alpha-1+\delta} \leq D(x_1,x_2) + o(1) \end{aligned}$$

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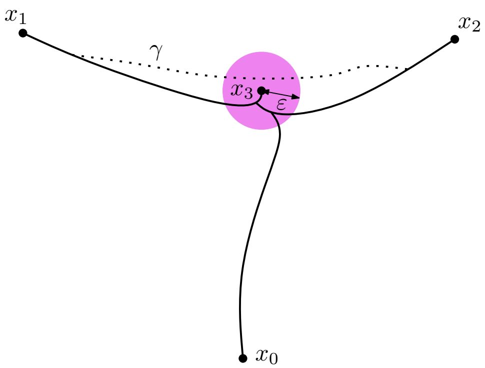
Idea of proof: Quickly separating geodesics



- A method to prove the main proposition is to approach points of Γ by points where geodesics perform a quick separation: Evaluate the probability that for 4 randomly chosen points x_0, x_1, x_2, x_3,
  - The three geodesics from x_3 to x_0, x_1, x_2 are disjoint outside of the ball of radius ε around x_3
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Proposition (codimension estimate)

The probability of the latter event is bounded above by $C\varepsilon^{3+\chi}$ for some $\chi > 0$.

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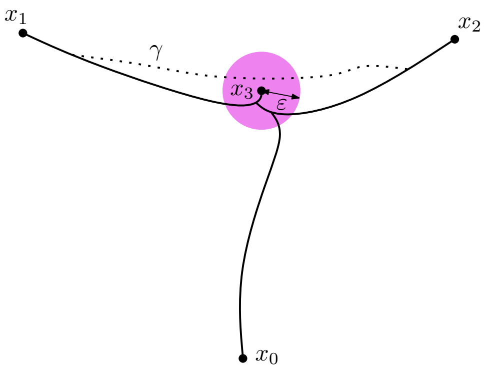

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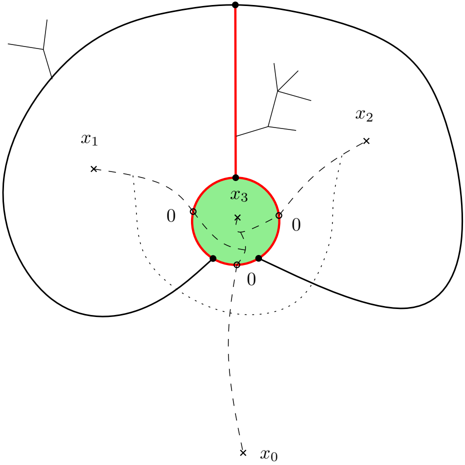
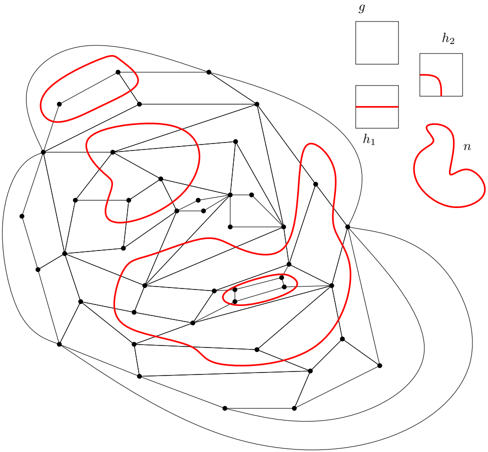
Idea of proof: Evaluating quickly avoiding geodesic configurations



- We then use a bijection (M. 2009) generalizing Schaeffer's by adding sources at x_0, x_1, x_2, x_3, measuring geodesic distances simultaneously from these points.
- Count labeled maps as in the picture: Labeled trees are branching out of a 3-regular graph with 4 faces.
- Red=non-negative labels. Green=labels are $\geq -2\varepsilon$ and $\ell(x_3) \leq -\varepsilon$. Dotted path is not a geodesic.

Loop models on random quadrangulations



weight $W^{(n)}_{g,h}(\mathbf{q}) = g^{12}h_1^{12}h_2^{16}n^4$

- Decorate a quadrangulation with "matter": a configuration of simple and mutually avoiding loops on the dual graph (Borot, Bouttier, Guitter).
- Emptying the interior of the loops, on obtains a Boltzmann random map, with distribution proportional to $\prod_{f\in F(\mathbf{m})} q_{\deg f/2}$, where

$$q_k = g\delta_{k2} + nh^{2k} \sum_{|\partial \mathbf{q}|=2k} W^{(n)}_{g,h}(\mathbf{q})$$

summing over decorated quadrangulations with a boundary

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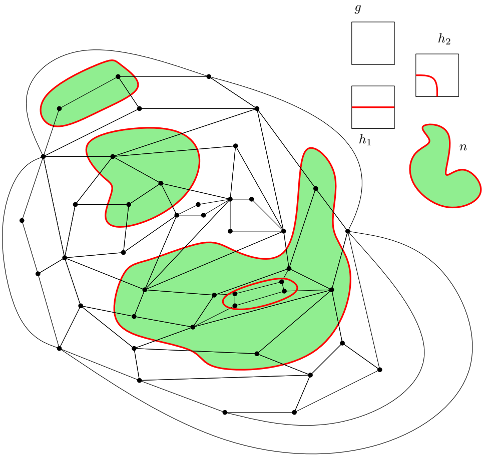

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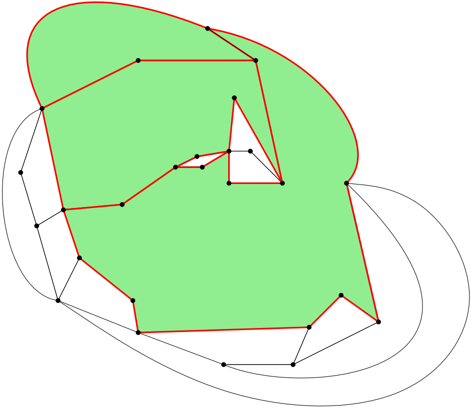

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Limits of maps with large faces: Stable maps

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- Instead of a topological sphere, these objects are random Sierpinsky carpets or gaskets.
- Conjectured links to Sheffield-Werner's CLE (candidate limits of loop models on Euclidean lattices!)

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