Form factor approach to the correlation functions of critical models

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Collaborators : *N. Kitanine, K.K. Kozlowski, N.A. Slavnov, V. Terras*.

- "Form factors of the XXZ Heisenberg spin-1/2 finite chain" *Nucl. Phys. B 554, 647-678 (1999)*
- "Algebraic Bethe ansatz approach to the asymptotic behavior of correlation functions" *J. Stat. Mech. P04003 (2009)*
- "On the thermodynamic limit of form factors in the massless XXZ Heisenberg chain" *J. Math. Phys. 50, 095209 (2009)*
- "On the thermodynamic limit of particle-holes form factors in the massless XXZ Heisenberg chain" *J. Stat. Mech. P05028 (2011)*
- "Form factor approach to the correlation functions of critical models" *J. Stat. Mech. P12010 (2011)*

Our favorite example : the XXZ Heisenberg chain

The *XXZ* spin-1/2 Heisenberg chain **in a magnetic field** is a quantum interacting model defined on a one-dimensional lattice with M sites, with Hamiltonian,

$$H_{XXZ} = \sum_{m=1}^{M} \{\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta(\sigma_m^z \sigma_{m+1}^z - 1)\} - \mathbf{h} \sum_{m=1}^{M} \sigma_m^z$$

Quantum space of states : $\mathcal{H} = \otimes_{m=1}^{M} \mathcal{H}_m$, $\mathcal{H}_m \sim \mathbb{C}^2$, $\dim \mathcal{H} = 2^M$.

$\sigma_m^{x,y,z}$: local spin operators (in the spin-$\frac{1}{2}$ representation) at site m
They act as the corresponding Pauli matrices in the space $\mathcal{H}_m$ and as the identity operator elsewhere.

- periodic boundary conditions
- disordered regime, $|\Delta| < 1$ and $h < h_c$

Correlation functions of critical (integrable) models

- **Asymptotic results predictions**
  - Luttinger liquid approximation / C.F.T. and finite size effects
    Luther and Peschel, Haldane, Cardy, Affleck, ... Lukyanov, ...
- **Exact results (XXZ, NLS, ...)**
  - Free fermion point $\Delta = 0$: Lieb, Shultz, Mattis, Wu, McCoy, Sato, Jimbo, Miwa . . .
  - From 1984: Izergin, Korepin . . . (first attempts using ABA)
  - General Δ: (form factors and building blocks)
    $\star$ 1992-96 Jimbo, Miwa . . . $\longrightarrow$ for infinite chain from QG
    $\star$ 1999 Kitanine, M, Terras $\longrightarrow$ for finite and infinite chain from ABA
  - Several developments for the last twelve years: Temperature case, numerics and actual experiments, master equation representation, some asymptotics, fermionic structures, etc.

$\hookrightarrow$ **Compute explicitly relevant physical correlation functions?**

$\hookrightarrow$ **Connect to the CFT limit from the exact results on the lattice?**

Correlation function : general strategies

At zero temperature only the ground state $|\omega\rangle$ contributes :

$$g_{12} = \langle\omega|\theta_1\theta_2|\omega\rangle$$

Two main strategies to evaluate such a function:

(i) compute the action of local operators on the ground state $\theta_1\theta_2|\omega\rangle = |\tilde{\omega}\rangle$ and then calculate the resulting scalar product:

$$g_{12} = \langle\omega|\tilde{\omega}\rangle$$

(ii) insert a sum over a complete set of eigenstates $|\omega_i\rangle$ to obtain a sum over one-point matrix elements (form factor type expansion) :

$$g_{12} = \sum_i \langle\omega|\theta_1|\omega_i\rangle \cdot \langle\omega_i|\theta_2|\omega\rangle$$

Correlation functions : ABA approach

1. **Diagonalise the Hamiltonian using ABA**
   $\longrightarrow$ key point : Yang-Baxter algebra $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$
   $\longrightarrow$ $|\,\psi_g\,\rangle = B(\lambda_1)\dots B(\lambda_N)|\,0\,\rangle$ with $\mathcal{Y}(\lambda_j;\{\lambda\}) = 0$ (Bethe eq.)

2. **Act with local operators on eigenstates**
   $\longrightarrow$ solve the quantum inverse problem (1999):
   $$\sigma_j^{(\alpha)} = (A+D)^{j-1}X^{(\alpha)}(A+D)^{-j} \textit{ with } X^{(\alpha)} = A, B, C, D$$
   $\longrightarrow$ use Yang-Baxter commutation relations

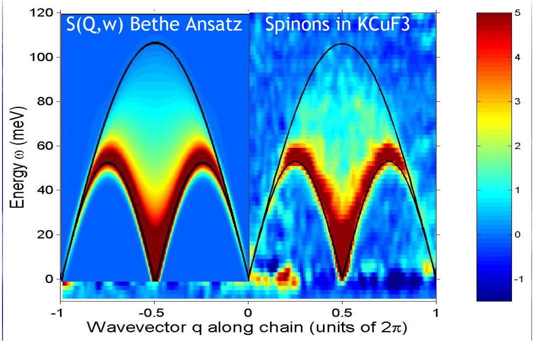
3. **Compute the resulting scalar products** (determinant representation)
   $\longrightarrow$ determinant representation for form factors of the finite chain
   $\longrightarrow$ elementary building blocks of correlation functions as multiple integrals in the thermodynamic limit (2000)

4. **Two-point function:** sum up elementary blocks or form factors?
   $\longrightarrow$ master equation representation in finite volume
   $\longrightarrow$ numerical sum of form factors : dynamical structure factors

5. **Analysis of the two-point functions (2008-2011):**
   $\longrightarrow$ series expansion (multiple integrals) and large distance asymptotics
   $\longrightarrow$ analysis of correlation functions from form factor series

Numerical summation of form factor series (XXX)

- **Structure factors define the dynamics of the models**
- **They can be measured experimentally**



$S(Q,\omega)$ is the dynamical spin-spin structure factor. The Bethe ansatz curve is computed for a chain of 500 sites (with J.- S. Caux) compared to the experimental curve obtained by A. Tennant in Berlin by neutron scattering. Colors indicate the value of the function $S(Q,\omega)$.

Results from master equation (multiple integrals)

Generating function

$$Q^{\kappa}_{1,m} = \prod_{n=1}^{m}\left(\frac{1+\kappa}{2} + \frac{1-\kappa}{2}\cdot\sigma^z_n\right) \quad \textit{with} \;\; \kappa = e^{\beta}$$

Asymptotic behavior (RH techniques applied to multiple integrals)

$$\langle e^{\beta \mathcal{Q}_{1m}}\rangle = \underbrace{G^{(0)}(\beta,m)[1+o(1)]}_{\text{non-oscillating terms}} + \underbrace{\sum_{\sigma=\pm} G^{(0)}(\beta+2i\pi\sigma,m)[1+o(1)]}_{\text{oscillating terms}}$$

$$G^{(0)}(\beta,m) = C(\beta)\, e^{m\beta D}\, m^{\frac{\beta^2}{2\pi^2}Z(q)^2}$$

- $Z(\lambda)$ is the dressed charge $\quad Z(\lambda) + \int_{-q}^{q}\frac{\mathrm{d}\mu}{2\pi}K(\lambda-\mu)\,Z(\mu) = 1$
- D is the average density $\quad D = \int_{-q}^{q}\rho(\mu)\mathrm{d}\mu = \frac{1-\langle\sigma^z\rangle}{2} = \frac{p_F}{\pi}$
- The coefficient $C(\beta)$ is given as the ratio of four Fredholm determinants.
- sub-leading oscillating terms restore the $2\pi i$-periodicity in β related to periodicity in Fredholm determinant of generalized sine kernel

2-point function asymptotic behavior

$$\langle\sigma^z_1\sigma^z_{m+1}\rangle = (2D-1)^2 - \frac{2Z(q)^2}{\pi^2 m^2} + 2|F_{\sigma^z}|^2\cdot\frac{\cos(2mp_F)}{m^{2Z(q)^2}} + o\left(\frac{1}{m^2},\frac{1}{m^{2Z(q)^2}}\right)$$

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Form factors strike back!

The umklapp form factor

$$\lim_{N,M\to\infty}\left(\frac{M}{2\pi}\right)^{2\mathcal{Z}^2}\frac{|\langle\psi(\{\mu\})|\sigma^z|\psi(\{\lambda\})\rangle|^2}{\|\psi(\{\mu\})\|^2\cdot\|\psi(\{\lambda\})\|^2}=|F_{\sigma^z}|^2.$$

with

$$2\mathcal{Z}^2=Z(q)^2+Z(-q)^2$$

- $\{\lambda\}$ are the Bethe parameters of the ground state
- $\{\mu\}$ are the Bethe parameters for the excited state with one particle and one hole on opposite sides of the Fermi boundary (umklapp type excitation).
- the critical exponents for the form factor behavior (in terms of size M) and for the correlation function (in terms of distance) are equal!

$\hookrightarrow$ Higher terms in the asymptotic expansion will involve n - particle/holes form factors corresponding to $2np_F$ oscillations and properly normalized form factors will be related to the corresponding amplitudes

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The form factor series for critical models

$$\langle\, \psi_g \,|\mathcal{O}_1(x')\mathcal{O}_2(x'+x)|\, \psi_g \,\rangle = \sum_{|\,\psi'\rangle} \mathcal{F}^{(1)}_{\psi_g\,\psi'}(x')\, \mathcal{F}^{(2)}_{\psi'\,\psi_g}(x+x')$$

$$\mathcal{F}^{(1)}_{\psi_g\,\psi'}(z) = \langle\, \psi_g \,|\mathcal{O}_1(z)|\, \psi' \,\rangle \quad \mathcal{F}^{(2)}_{\psi'\,\psi_g}(z) = \langle\, \psi' \,|\mathcal{O}_2(z)|\, \psi_g \,\rangle$$

$\hookrightarrow$ Main difficulty : form factors scale to zero in the infinite size limit ($L \to \infty$) for critical models reflecting the conformal dimensions of the local operators involved and of the creation operators for the excited state ψ'.

$$\mathcal{F}^{(1)}_{\psi_g\,\psi'}(x') \cdot \mathcal{F}^{(2)}_{\psi'\,\psi_g}(x+x') = L^{-\theta}\, e^{ix\mathcal{P}_{ex}}\, \mathcal{A}(\psi', \psi_g)$$

$\hookrightarrow$ Analyze the form factor series for very large (but finite) system size L, in the asymptotic regime where the distance x becomes large with $x << L$; the thermodynamic limit $L \to \infty$ being taken only at the end of the computation.

$\hookrightarrow$ We need to identify the states giving the leading behavior of such series, to obtain the corresponding form factors, their behavior in these limits and finally to compute the final (huge) multiple sums!

The form factor series in the large distance limit (1)

$$\langle \mathcal{O}_1(x')\,\mathcal{O}_2(x+x')\rangle_{ph} = \lim_{L\to\infty} \sum_{\{\mu_p\},\{\mu_h\}} L^{-\theta}\, e^{ix\mathcal{P}_{ex}}\, \mathcal{S}(\{\mu_p\},\{\mu_h\})\,\mathcal{D}(\{\mu_p\},\{\mu_h\}|\{p\},\{h\}).$$

In the large distance limit $x \to \infty$, the oscillatory character of these sums localizes them, in the absence of any other saddle point of the oscillating exponent (such saddle points will appear in the time dependent case), around small excitations on the Fermi boundaries $\pm q$.

$\hookrightarrow$Hence we consider so-called critical n particle-hole excited state $\{\mu_{\ell_a}\}$ for which the rapidities $\{\mu_{p_a}\}$ and $\{\mu_{h_a}\}$ of such a state accumulate on the two end-points of the Fermi-zone in the thermodynamic limit. Form factors corresponding to any such a state are called critical form factors

In the thermodynamic limit, there will be $n_p^{\pm}$ particles whose rapidities are equal to $\pm q$ and $n_h^{\pm}$ holes whose rapidities are equal to $\pm q$ with $n_p^+ + n_p^- = n_h^+ + n_h^- = n$. A given critical excited state belong to the $\mathbf{P}_\ell$ class if the distribution of particles and holes on the Fermi boundaries is such that

$$n_p^+ - n_h^+ = n_h^- - n_p^- = \ell, \qquad -n \le \ell \le n.$$

Then such an excited state has momentum $2\ell p_F$ in the thermodynamic limit and its associated critical form factor will also be said to belong to the $\mathbf{P}_\ell$ class.

The form factor series in the large distance limit (2)

Localization of the form factor series by analogy with the multiple integral situation :

$$I_n(x) = \int\limits_{\mathbb{R}\setminus[-q,q]} d^n\mu_p \int\limits_{[-q,q]} d^n\mu_h \, f(\{\mu_p\},\{\mu_h\}) \prod_{j=1}^{n} e^{ix(p(\mu_{p_j})-p(\mu_{h_j}))}, \qquad x \to \infty,$$

$\hookrightarrow$ If $f(\{\mu_p\},\{\mu_h\})$ is a holomorphic function and $p(\mu)$ has no saddle points on the integration contours then the large x asymptotic analysis reduces to the calculation of the integral in small vicinities of the endpoints, where $f(\{\mu_p\},\{\mu_h\})$ can be replaced by $f(\{\pm q\},\{\pm q\})$.

$\hookrightarrow$ If $f(\{\mu_p\},\{\mu_h\})$ has integrable singularities at $\pm q$, for example $f(\{\mu_p\},\{\mu_h\}) = (q-\mu_{h_1})^{\nu_+}(\mu_{h_1}+q)^{\nu_-} f_{reg}(\{\mu_p\},\{\mu_h\})$, then one has to keep the singular factors $(q \mp \mu_{h_1})^{\nu_\pm}$ as they are but we can replace the regular part $f_{reg}(\{\mu_p\},\{\mu_h\})$ by an appropriate constant $f_{reg}(\{\pm q\},\{\pm q\})$.

The form factor series in the large distance limit (3)

$\hookrightarrow$ By analogy with these oscillating multiple integrals we obtain:

- the smooth part of the form factor $\mathcal{S}(\{\mu_p\},\{\mu_h\})$ doesn't introduce any singularities and thus, the corresponding rapidities can be set equal to their values in the $\mathbf{P}_\ell$ class. Likewise can be done for the part of the discrete form factor depending smoothly on the rapidities $\mathcal{D}(\{\mu_p\},\{\mu_h\}|\{p\},\{h\})$.
- we need to keep and to treat explicitly the summation of the integer dependent part of $\mathcal{D}(\{\mu_p\},\{\mu_h\}|\{p\},\{h\})$.

$\hookrightarrow$ Thus, the main contribution to the asymptotic behavior of the correlation function is produced by the critical form factors. Hereby the smooth part becomes a constant depending only on the $\mathbf{P}_\ell$ class of the form factor. The discrete part $\mathcal{D}$ plays the role of the singular factors $(q \mp \mu_{h_1})^{\nu\pm}$ in the integral. Hence fixing the $\mathbf{P}_\ell$ class of critical from factors we should still take the sum over all the excited states within this class.

Warning: The form factor sums cannot have the sense of integral sums even for L large. Indeed, it produces eventually the factor L^{θ_ℓ} that makes the final result finite; however, θ_ℓ being in general not an integer, such a sum hardly reduces to a Riemann sum. This feature explains the difficulties already noted in the literature while trying to use the form factor approach directly in the continuum limit for massless models.

Spin-spin correlation functions as sum over form factors

$$\langle \sigma_1^s \, \sigma_{m+1}^{s'} \rangle = \sum_{|\,\psi'\,\rangle} \mathcal{F}^{(s)}_{\psi_g \psi'}(1) \cdot \mathcal{F}^{(s')}_{\psi' \psi_g}(m+1) \quad \text{with} \quad \mathcal{F}^{(s)}_{\psi\psi'}(m) = \frac{\langle\, \psi \,|\sigma^s_m|\, \psi' \,\rangle}{||\psi|| \cdot ||\psi'||}$$

Eigenstates parametrized by solutions of logarithmic Bethe equations:

$$\widehat{\xi}(|\{\mu_\ell\}) \equiv \frac{p_0()}{2\pi} - \frac{1}{2\pi M}\sum_{k=1}^{N} \theta(-\mu_{\ell_k}) + \frac{N+1}{2M} = \frac{\ell_j}{M}, \quad j = 1, \ldots, N$$

$p_0(\lambda) = i \log \frac{\sinh(\frac{i\zeta}{2}+\lambda)}{\sinh(\frac{i\zeta}{2}-\lambda)}$ (bare momentum); $\quad \theta(\lambda) = i \log \frac{\sinh(i\zeta+\lambda)}{\sinh(i\zeta-\lambda)}$ (bare phase)

$\ell_j \in \mathbb{Z}$; $\quad \widehat{\xi}(\lambda|\{\mu_\ell\})$: counting function associated to the set of roots $\{\mu_\ell\}$

- **ground state** $|\,\psi_g\,\rangle$**:** $N = N_0$ fixed by h, $\ell_j = j$, $j = 1, \ldots, N_0$
  all roots λ_j are real and densely fill a symmetric interval $[-q, q]$ (the Fermi zone) in the thermodynamic limit $M \to \infty$ with density $\rho(\lambda)$
- **particle/hole excited states** $|\,\psi'\,\rangle$ ($N = N_0$ for σ^z or $N_0 + 1$ for σ^-) :
  roots μ_{ℓ_j} are real $\ell_j = j$ for $j \in \{1, \ldots, N\} \setminus \{h_1, \ldots, h_n\}$ and $\ell_{h_j} = p_j \notin \{1, \ldots, N\}$
  - $\rightsquigarrow$ particle/hole rapidities: $\widehat{\xi}(\mu_{p_a}|\{\mu_\ell\}) = \frac{p_a}{M}$, $\quad \widehat{\xi}(\mu_{h_a}|\{\mu_\ell\}) = \frac{h_a}{M}$
  - $\rightsquigarrow$ 'background' rapidities: $\widehat{\xi}(\mu_j|\{\mu_\ell\}) = \frac{j}{M}$, $\quad j \in \{1, \ldots, N\}$
  - $\rightsquigarrow$ shift function: $\mu_j - \lambda_j \underset{M\to\infty}{\sim} \frac{F(\lambda_j)}{M\rho(\lambda_j)}$

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  roots μ_{ℓ_j} are real $\ell_j = j$ for $j \in \{1, \dots, N\} \setminus \{h_1, \dots, h_n\}$ and $\ell_{h_j} = p_j \notin \{1, \dots, N\}$
  - ⇝ particle/hole rapidities: $\widehat{\xi}(\mu_{p_a}|\{\mu_\ell\}) = \frac{p_a}{M}$, $\widehat{\xi}(\mu_{h_a}|\{\mu_\ell\}) = \frac{h_a}{M}$
  - ⇝ 'background' rapidities: $\widehat{\xi}(\mu_j|\{\mu_\ell\}) = \frac{j}{M}$, $j \in \{1, \dots, N\}$
  - ⇝ shift function: $\mu_j - \lambda_j \underset{M\to\infty}{\sim} \frac{F(\lambda_j)}{M\rho(\lambda_j)}$

Spin-spin correlation functions as sum over form factors

$$\langle \sigma_1^s \, \sigma_{m+1}^{s'} \rangle = \sum_{|\,\psi'\,\rangle} \mathcal{F}^{(s)}_{\psi_g \psi'}(1) \cdot \mathcal{F}^{(s')}_{\psi' \psi_g}(m+1) \quad \text{with} \quad \mathcal{F}^{(s)}_{\psi\psi'}(m) = \frac{\langle\, \psi \,|\sigma_m^s|\, \psi' \,\rangle}{||\psi|| \cdot ||\psi'||}$$

Eigenstates parametrized by solutions of logarithmic Bethe equations:

$$\widehat{\xi}(\mu_{\ell_j}|\{\mu_\ell\}) \equiv \frac{p_0(\mu_{\ell_j})}{2\pi} - \frac{1}{2\pi M} \sum_{k=1}^{N} \theta(\mu_{\ell_j} - \mu_{\ell_k}) + \frac{N+1}{2M} = \frac{\ell_j}{M}, \quad j = 1, \dots, N$$

$p_0(\lambda) = i \log \frac{\sinh(\frac{i\zeta}{2}+\lambda)}{\sinh(\frac{i\zeta}{2}-\lambda)}$ (bare momentum); $\theta(\lambda) = i \log \frac{\sinh(i\zeta+\lambda)}{\sinh(i\zeta-\lambda)}$ (bare phase)

$\ell_j \in \mathbb{Z}$; $\widehat{\xi}(\lambda|\{\mu_\ell\})$: counting function associated to the set of roots $\{\mu_\ell\}$

- **ground state** $|\,\psi_g\,\rangle$: $N = N_0$ fixed by h, $\ell_j = j$, $j = 1, \dots, N_0$
  all roots λ_j are real and densely fill a symmetric interval $[-q, q]$ (the Fermi zone) in the thermodynamic limit $M \to \infty$ with density $\rho(\lambda)$

- **particle/hole excited states** $|\,\psi'\,\rangle$ ($N = N_0$ for σ^z or $N_0 + 1$ for σ^-) : roots μ_{ℓ_j} are real $\ell_j = j$ for $j \in \{1, \dots, N\} \setminus \{h_1, \dots, h_n\}$ and $\ell_{h_j} = p_j \notin \{1, \dots, N\}$
  - ⇝ particle/hole rapidities: $\widehat{\xi}(\mu_{p_a}|\{\mu_\ell\}) = \frac{p_a}{M}$, $\widehat{\xi}(\mu_{h_a}|\{\mu_\ell\}) = \frac{h_a}{M}$
  - ⇝ 'background' rapidities: $\widehat{\xi}(\mu_j|\{\mu_\ell\}) = \frac{j}{M}$, $j \in \{1, \dots, N\}$
  - ⇝ shift function: $\mu_j - \lambda_j \underset{M\to\infty}{\sim} \frac{F(\lambda_j)}{M\rho(\lambda_j)}$

Particle-hole form factors

- Finite-size form factors can be exactly computed as determinants of elementary functions (Kitanine, M, Terras, NPB 1999):

$$\langle \psi' | \sigma^s_m | \psi_g \rangle = e^{im\mathcal{P}_{\rm ex}} \cdot \det_N \Omega^{(s)}(\{\lambda\},\{\mu\})$$

with $|\psi_g\rangle \equiv |\psi(\{\lambda\})\rangle$, $|\psi'\rangle \equiv |\psi(\{\mu\})\rangle$

- dependence in distance m in phase factor given by relative momentum between ground state and excited state:

$$\mathcal{P}_{\rm ex} = \sum_{j=1}^{N} p_0(\mu_{\ell_j}) - \sum_{j=1}^{N_0} p_0(\lambda_j) = \frac{2\pi}{M} \sum_{k=1}^{n} (p_k - h_k)$$

labelled in terms of particle/hole integers

- "singularities" at $\lambda_j = \mu_{\ell_k}$ $\rightsquigarrow$ factor out Cauchy determinant:

$$\det_N \Omega^{(s)}(\{\lambda\},\{\mu\}) = \det_N \left[\frac{1}{\sinh(\mu_{\ell_k} - \lambda_j)}\right] \cdot \underbrace{\det_N \widetilde{\Omega}^{(s)}(\{\lambda\},\{\mu\})}_{\text{non singular}}$$

Thermodynamic limit of particle-hole form factors

1. Non-singular part $\longrightarrow$ smooth thermodynamic limit $\mathcal{S}(\{\mu_p\};\{\mu_h\})[F]$

$$\sum_{j=1}^{N}[f(\mu_{\ell_j})-f(\lambda_j)]=\sum_{j=1}^{n}[f(\mu_{p_j})-f(\mu_{h_j})]+\sum_{j=1}^{N}[f(\mu_j)-f(\lambda_j)]$$

$$\xrightarrow[M\to\infty]{}\sum_{j=1}^{n}[f(\mu_{p_j})-f(\mu_{h_j})]+\int_{-q}^{q}f'(\lambda)F(\lambda)\,d\lambda$$

2. Singular part (Cauchy)

⋆ multiply and divide by counting function $\widehat{\xi}$ to go to integers

$$\frac{1}{\sinh(\lambda-\mu)}=\underbrace{\frac{\widehat{\xi}(\lambda)-\widehat{\xi}(\mu)}{\sinh(\lambda-\mu)}}_{\substack{\text{smooth function}\\ cf.1}}\times\underbrace{\frac{1}{\widehat{\xi}(\lambda)-\widehat{\xi}(\mu)}}_{\substack{\propto\text{ difference of integers}\\ \text{(possibly shifted by function F)}}}$$

⋆ (some of) the products are ratios of Gamma functions

ex: $$\prod_{k=1}^{n}\frac{\Gamma(h_k+F(\mu_{h_k}))\,\Gamma(N+1-h_k-F(\mu_{h_k}))}{\Gamma(h_k)\,\Gamma(N+1-h_k)},\quad h_k\in\{1,\dots,N\}$$

$\rightsquigarrow$ use Stirling if p's and h's are integers far away from Fermi surface

$\rightsquigarrow$ large-size behavior depends on the number of particles/holes collapsing on the Fermi surface

Thermodynamic limit of particle-hole form factors

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ex: $\widehat{\xi}(\mu_j)-\widehat{\xi}(\mu_k)=\frac{1}{M}(j-k)$
$\widehat{\xi}(\mu_j)-\widehat{\xi}(\lambda_k)\sim\frac{1}{M}(j-k+F(\lambda_k))$

⋆ (some of) the products are ratios of Gamma functions

ex: $\prod_{k=1}^{n}\frac{\Gamma(h_k+F(\mu_{h_k}))\,\Gamma(N+1-h_k-F(\mu_{h_k}))}{\Gamma(h_k)\,\Gamma(N+1-h_k)}, \quad h_k\in\{1,\dots,N\}$

⇝ use Stirling if p's and h's are integers far away from Fermi surface

⇝ large-size behavior depends on the number of particles/holes

Thermodynamic limit of particle-hole form factors

1. Non-singular part $\rightarrow$ smooth thermodynamic limit $\mathcal{S}(\{\mu_p\};\{\mu_h\})[F]$

$$\sum_{j=1}^{N}[f(\mu_{\ell_j}) - f(\lambda_j)] = \sum_{j=1}^{n}[f(\mu_{p_j}) - f(\mu_{h_j})] + \sum_{j=1}^{N}[f(\mu_j) - f(\lambda_j)]$$

$$\underset{M\rightarrow\infty}{\longrightarrow} \sum_{j=1}^{n}[f(\mu_{p_j}) - f(\mu_{h_j})] + \int_{-q}^{q} f'(\lambda)F(\lambda)\,d\lambda$$

2. Singular part (Cauchy)

   ⋆ multiply and divide by counting function $\widehat{\xi}$ to go to integers

$$\frac{1}{\sinh(\lambda-\mu)} = \underbrace{\frac{\widehat{\xi}(\lambda)-\widehat{\xi}(\mu)}{\sinh(\lambda-\mu)}}_{\substack{\text{smooth function}\\ cf.1}} \times \underbrace{\frac{1}{\widehat{\xi}(\lambda)-\widehat{\xi}(\mu)}}_{\substack{\propto \text{ difference of integers}\\ \text{(possibly shifted by function F)}}}$$

   ⋆ (some of) the products are ratios of Gamma functions

$$\text{ex: } \prod_{k=1}^{n} \frac{\Gamma(h_k + F(\mu_{h_k}))\,\Gamma(N+1-h_k-F(\mu_{h_k}))}{\Gamma(h_k)\,\Gamma(N+1-h_k)}, \quad h_k \in \{1,\ldots,N\}$$

   $\rightsquigarrow$ use Stirling if p's and h's are integers far away from Fermi surface

   $\rightsquigarrow$ large-size behavior depends on the number of particles/holes collapsing on the Fermi surface

Thermodynamic limit of particle-hole form factors

- 1srt example: particles and holes far away from the Fermi boundaries

$$\lim_{N,M\to\infty} \frac{\Gamma(p_k - N + F(\mu_{p_k}))\,\Gamma(p_k)\,\Gamma(N+1-h_k - F(\mu_{h_k})\,\Gamma(h_k + F(\mu_{h_k}))}{\Gamma(p_k - N)\,\Gamma(p_k + F(\mu_{p_k}))\,\Gamma(N+1-h_k)\,\Gamma(h_k)}$$

$$= \lim_{N,M\to\infty} \left(\frac{p_k - N}{p_k}\right)^{F(\mu_{p_k})} \left(\frac{N-h_k}{h_k}\right)^{-F(\mu_{h_k})}$$

$$= \lim_{N,M\to\infty} \left(\frac{\widehat{\xi}(\mu_{p_k}) - \widehat{\xi}(q)}{\widehat{\xi}(\mu_{p_k}) - \widehat{\xi}(-q)}\right)^{F(\mu_{p_k})} \left(\frac{\widehat{\xi}(q) - \widehat{\xi}(\mu_{h_k})}{\widehat{\xi}(\mu_{h_k}) - \widehat{\xi}(-q)}\right)^{-F(\mu_{h_k})}$$

$$= \left(\frac{p(\mu_{p_k}) - p(q)}{p(\mu_{p_k}) - p(-q)}\right)^{F(\mu_{p_k})} \left(\frac{p(q) - p(\mu_{h_k})}{p(\mu_{h_k}) - p(-q)}\right)^{-F(\mu_{h_k})}$$

(since $\widehat{\xi}(\lambda) \to \frac{p(\lambda)+p(q)}{2\pi}$ with $p(\lambda) = 2\pi \int_0^\lambda \rho(\mu)\, d\mu$: dressed momemtum)

$\rightsquigarrow$ smooth function of μ_{p_k}, μ_{h_k} giving sub-dominant contributions

- 2nd example: critical state of $\mathbf{P}_\ell$ class (particles and holes all on the Fermi boundaries) $n_p^\pm$ particles, resp. $n_h^\pm$ holes, with rapidities equal to $\pm q$ such that $n_p^+ + n_p^- = n_h^+ + n_h^- = n$ and $n_p^+ - n_h^+ = n_h^- - n_p^- = \ell$, $\ell \in \mathbb{Z}$

$$p_j = p_j^+ + N \quad \text{if } \mu_{p_j} = q, \qquad p_j = 1 - p_j^- \text{ if } \mu_{p_j} = -q$$

Thermodynamic limit of particle-hole form factors

- 1srt example: particles and holes far away from the Fermi boundaries

$$\lim_{N,M\to\infty} \frac{\Gamma(p_k - N + F(\mu_{p_k}))\,\Gamma(p_k)\,\Gamma(N+1-h_k - F(\mu_{h_k})\,\Gamma(h_k + F(\mu_{h_k}))}{\Gamma(p_k - N)\,\Gamma(p_k + F(\mu_{p_k}))\,\Gamma(N+1-h_k)\,\Gamma(h_k)}$$
$$= \left(\frac{p(\mu_{p_k}) - p(q)}{p(\mu_{p_k}) - p(-q)}\right)^{F(\mu_{p_k})} \left(\frac{p(q) - p(\mu_{h_k})}{p(\mu_{h_k}) - p(-q)}\right)^{-F(\mu_{h_k})}$$

$\rightsquigarrow$ smooth function of μ_{p_k}, μ_{h_k} giving sub-dominant contributions

- 2nd example: critical state of $\mathbf{P}_\ell$ class (particles and holes all on the Fermi boundaries) $n_p^\pm$ particles, resp. $n_h^\pm$ holes, with rapidities equal to $\pm q$ such that $n_p^+ + n_p^- = n_h^+ + n_h^- = n$ and $n_p^+ - n_h^+ = n_h^- - n_p^- = \ell, \ \ \ell \in \mathbb{Z}$

$$p_j = p_j^+ + N \quad \text{if } \mu_{p_j} = q, \qquad p_j = 1 - p_j^- \ \text{ if } \mu_{p_j} = -q$$
$$h_j = N + 1 - h_j^+ \ \text{ if } \mu_{h_j} = q, \qquad h_j = h_j^- \quad \text{if } \mu_{h_j} = -q$$

$$\prod_{k=1}^{n} \frac{\Gamma(p_k - N + F(\mu_{p_k}))\,\Gamma(p_k)\,\Gamma(N+1-h_k - F(\mu_{h_k})\,\Gamma(h_k + F(\mu_{h_k}))}{\Gamma(p_k - N)\,\Gamma(p_k + F(\mu_{p_k}))\,\Gamma(N+1-h_k)\,\Gamma(h_k)}$$
$$\sim N^{-\ell[F(q)+F(-q)]} \prod_{k=1}^{n_p^+} \frac{\Gamma(p_k^+ + F(q))}{\Gamma(p_k^+)} \prod_{k=1}^{n_p^-} \frac{\Gamma(p_k^- + F(-q))}{\Gamma(p_k^-)} \prod_{k=1}^{n_h^+} \frac{\Gamma(h_k^+ - F(q))}{\Gamma(h_k^+)} \prod_{k=1}^{n_h^-} \frac{\Gamma(h_k^- + F(-q))}{\Gamma(h_k^-)}$$

$\rightsquigarrow$ decreasing exponent is modified + discrete structure in finite part

Thermodynamic limit of particle-hole critical form factors

Inside a given $\mathbf{P}_\ell$ class :

$\rightsquigarrow$ phase factors $\mathcal{P}_{\text{ex}}$ depend on the particular state we consider:

$$\mathcal{P}_{\text{ex}} = \frac{2\pi}{M} \sum_{k=1}^{n} (p_k - h_k) = 2\ell k_F + \frac{2\pi}{M} \mathcal{P}_{\text{ex}}^{(d)} \qquad (k_F\text{: Fermi momentum } p(q))$$

$$\mathcal{P}_{\text{ex}}^{(d)} = \sum_{j=1}^{n_p^+} (p_j^+ - 1) + \sum_{j=1}^{n_h^+} h_j^+ - \sum_{j=1}^{n_p^-} p_j^- - \sum_{j=1}^{n_h^-} (h_j^- - 1)$$

$\rightsquigarrow$ smooth parts $\mathcal{S}(\{\mu_p\};\{\mu_h\})[F]$ are all the same

$\rightsquigarrow$ critical exponents θ_ℓ are all the same

$\rightsquigarrow$ finite discrete parts depend on the particular state we consider (they are expressed in terms of particle/hole integers around the Fermi zone)

$\hookrightarrow$ all critical form factors inside a same $\mathbf{P}_\ell$ class can be expressed in terms of the simplest form factor of the class (the ℓ-shifted state $|\,\psi_\ell\,\rangle$ with integers $\ell_j = j + \ell$) by just taking in consideration the modification of the discrete part

$$\mathcal{F}^{(s)}_{\psi_g\,\psi'}(1)\cdot\mathcal{F}^{(s')}_{\psi'\,\psi_g}(m+1) \sim e^{2im\ell k_F}\, e^{\frac{2\pi i m}{M}\mathcal{P}_{\text{ex}}^{(d)}}\, M^{-\theta_\ell^{(ss')}} \left[\mathcal{F}^{(s)}_{\psi_g\,\psi'}\cdot\mathcal{F}^{(s')}_{\psi'\,\psi_g}\right]_{\text{finite}}$$

with $\theta_\ell = (F_+ + \ell)^2 + (F_- + \ell)^2$
and

$$\left[\mathcal{F}^{(s)}_{\psi_g\,\psi'}\cdot\mathcal{F}^{(s')}_{\psi'\,\psi_g}\right]_{\text{finite}} = \left[\mathcal{F}^{(s)}_{\psi_g\,\psi_\ell}\cdot\mathcal{F}^{(s')}_{\psi_\ell\,\psi_g}\right]_{\text{finite}} \cdot \frac{G^2(1+F_+)G^2(1-F_-)}{G^2(1+\ell+F_+)G^2(1-\ell-F_-)}$$

Thermodynamic limit of particle-hole critical form factors

$\hookrightarrow$ all critical form factors inside a same $\mathbf{P}_\ell$ class can be expressed in terms of the simplest form factor of the class (the ℓ-shifted state $|\,\psi_\ell\,\rangle$ with integers $\ell_j = j + \ell$) by just taking in consideration the modification of the discrete part

$$\mathcal{F}^{(s)}_{\psi_g\,\psi'}(1)\cdot\mathcal{F}^{(s')}_{\psi'\,\psi_g}(m+1) \sim e^{2im\ell k_F}\, e^{\frac{2\pi i m}{M}\mathcal{P}^{(d)}_{\text{ex}}}\, M^{-\theta^{(ss')}_\ell}\left[\mathcal{F}^{(s)}_{\psi_g\,\psi'}\cdot\mathcal{F}^{(s')}_{\psi'\,\psi_g}\right]_{\text{finite}}$$

with $\theta_\ell = (F_+ + \ell)^2 + (F_- + \ell)^2$

and

$$\left[\mathcal{F}^{(s)}_{\psi_g\,\psi'}\cdot\mathcal{F}^{(s')}_{\psi'\,\psi_g}\right]_{\text{finite}} = \left[\mathcal{F}^{(s)}_{\psi_g\,\psi_\ell}\cdot\mathcal{F}^{(s')}_{\psi_\ell\,\psi_g}\right]_{\text{finite}}\cdot\frac{G^2(1+F_+)G^2(1-F_-)}{G^2(1+\ell+F_+)G^2(1-\ell-F_-)}$$
$$\times R_{n_p^+,n_h^+}(\{p^+\},\{h^+\}|F_+)\; R_{n_p^-,n_h^-}(\{p^-\},\{h^-\}|-F_-)$$

$F_+ = F(q) + N - N_0,\quad F_- = F(-q),\ G(z)$ the Barnes function

$$R_{n,n'}(\{p\},\{h\}|F) = \frac{\prod_{j>k}^{n}(p_j-p_k)^2\prod_{j>k}^{n'}(h_j-h_k)^2}{\prod_{j=1}^{n}\prod_{k=1}^{n'}(p_j+h_k-1)^2}\left[\frac{\sin(\pi F)}{\pi}\right]^{2n'}$$
$$\times\prod_{k=1}^{n}\frac{\Gamma^2(p_k+F)}{\Gamma^2(p_k)}\prod_{k=1}^{n'}\frac{\Gamma^2(h_k-F)}{\Gamma^2(h_k)}$$

Summation over critical form factors

$$\langle \sigma_1^s \, \sigma_{m+1}^{s'} \rangle_{cr} = \lim_{M\to\infty} \sum_{\ell=-\infty}^{\infty} \sum_{|\,\psi'\,\rangle \text{ in } \mathbf{P}_\ell \text{ class}} \mathcal{F}^{(s)}_{\psi_g \psi'}(1) \cdot \mathcal{F}^{(s')}_{\psi' \psi_g}(m+1)$$

$$= \lim_{M\to\infty} \sum_{\ell=-\infty}^{\infty} e^{2im\ell k_F} \, M^{-\theta_\ell^{(ss')}} \left[\mathcal{F}^{(s)}_{\psi_g\,\psi_\ell}\,\mathcal{F}^{(s')}_{\psi_\ell\,\psi_g}\right]_{\text{finite}} \prod_{\epsilon=\pm} \frac{G^2(1+\epsilon F_\epsilon)}{G^2(1+\epsilon\ell+\epsilon F_\epsilon)}$$

$$\times \underbrace{\sum_{\substack{\{p\},\{h\} \\ n_p^+ - n_h^+ = \ell}} e^{\frac{2\pi i m}{M}\mathcal{P}_{ex}^{(d)}} \prod_{\epsilon=\pm} R_{n_p^\epsilon, n_h^\epsilon}(\{p^\epsilon\},\{h^\epsilon\}|\epsilon F_\epsilon)}_{\text{sum over all possible configurations of integers in the } \mathbf{P}_\ell \text{ class}}$$

$$\sum_{\substack{n_p,n_h=0 \\ n_p-n_h=\ell}}^{\infty} \sum_{\substack{p_1<\cdots<p_{n_p} \\ p_a\in\mathbb{N}^*}} \sum_{\substack{h_1<\cdots<h_{n_h} \\ h_a\in\mathbb{N}^*}} e^{\frac{2\pi i m}{M}\left[\sum_{j=1}^{n_p}(p_j-1)+\sum_{k=1}^{n_h} h_k\right]} R_{n_p,n_h}(\{p\},\{h\}|F)$$

$$= \frac{G^2(1+\ell+F)}{G^2(1+F)} \, \frac{e^{\frac{i\pi m}{M}\ell(\ell-1)}}{\left(1-e^{\frac{2i\pi m}{M}}\right)^{(F+\ell)^2}}$$

$\rightsquigarrow$ Proof? Yes....but we realized afterwards that this is just the fundamental identity for Z-measures on partitions (Borodin, Olshanski, Kerov,...). Combinatorics for generic anisotropy Δ in XXZ chain!

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$$\times \underbrace{\sum_{\substack{\{p\},\{h\}\\ n_p^+ - n_h^+ = \ell}} e^{\frac{2\pi i m}{M}\mathcal{P}^{(d)}_{ex}} \prod_{\epsilon=\pm} R_{n_p^\epsilon, n_h^\epsilon}(\{p^\epsilon\},\{h^\epsilon\}|\epsilon F_\epsilon)}_{\text{sum over all possible configurations of integers in the } \mathbf{P}_\ell \text{ class}}$$

$$\sum_{\substack{n_p,n_h=0\\ n_p-n_h=\ell}}^{\infty} \sum_{\substack{p_1<\cdots<p_{n_p}\\ p_a\in\mathbb{N}^*}} \sum_{\substack{h_1<\cdots<h_{n_h}\\ h_a\in\mathbb{N}^*}} e^{\frac{2\pi i m}{M}\left[\sum_{j=1}^{n_p}(p_j-1)+\sum_{k=1}^{n_h} h_k\right]} R_{n_p,n_h}(\{p\},\{h\}|F)$$

$$= \frac{G^2(1+\ell+F)}{G^2(1+F)} \, \frac{e^{\frac{i\pi m}{M}\ell(\ell-1)}}{\left(1-e^{\frac{2i\pi m}{M}}\right)^{(F+\ell)^2}}$$

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$$\langle \sigma_1^{s}\, \sigma_{m+1}^{s'} \rangle_{\rm cr} = \lim_{M\to\infty} \sum_{\ell=-\infty}^{\infty} \sum_{|\,\psi'\,\rangle \text{ in } \mathbf{P}_\ell \text{ class}} \mathcal{F}^{(s)}_{\psi_g \psi'}(1) \cdot \mathcal{F}^{(s')}_{\psi' \psi_g}(m+1)$$

$$= \lim_{M\to\infty} \sum_{\ell=-\infty}^{\infty} e^{2im\ell k_F}\, M^{-\theta_\ell^{(ss')}} \left[\mathcal{F}^{(s)}_{\psi_g\,\psi_\ell}\, \mathcal{F}^{(s')}_{\psi_\ell\,\psi_g}\right]_{\rm finite} \prod_{\epsilon=\pm} \frac{G^2(1+\epsilon F_\epsilon)}{G^2(1+\epsilon\ell+\epsilon F_\epsilon)}$$

$$\times \underbrace{\sum_{\substack{\{p\},\{h\}\\ n_p^+ - n_h^+ = \ell}} e^{\frac{2\pi i m}{M}\mathcal{P}^{(d)}_{ex}} \prod_{\epsilon=\pm} R_{n_p^\epsilon, n_h^\epsilon}(\{p^\epsilon\},\{h^\epsilon\}|\epsilon F_\epsilon)}_{\substack{\text{sum over all possible configurations of integers}\\ \text{in the } \mathbf{P}_\ell \text{ class}}}$$

$$\sum_{\substack{n_p,n_h=0\\ n_p-n_h=\ell}}^{\infty} \sum_{\substack{p_1<\cdots<p_{n_p}\\ p_a\in\mathbb{N}^*}} \sum_{\substack{h_1<\cdots<h_{n_h}\\ h_a\in\mathbb{N}^*}} e^{\frac{2\pi i m}{M}\left[\sum_{j=1}^{n_p}(p_j-1)+\sum_{k=1}^{n_h} h_k\right]} R_{n_p,n_h}(\{p\},\{h\}|F)$$

$$= \frac{G^2(1+\ell+F)}{G^2(1+F)} \frac{e^{\frac{i\pi m}{M}\ell(\ell-1)}}{\left(1-e^{\frac{2i\pi m}{M}}\right)^{(F+\ell)^2}}$$

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Summation formula: sketch of our proof

Summation formula

$$f_\ell(\nu, w) = w^{\ell(\ell-1)/2} \frac{G^2(1+\ell+\nu)}{G^2(1+\nu)} (1-w)^{-(\nu+\ell)^2}$$

where G is the Barnes G-function: $G(z+1) = \Gamma(z)G(z)$

and
$$f_\ell(\nu, w) \equiv \sum_{\substack{n_p, n_h = 0 \\ n_p - n_h = \ell}}^{\infty} \sum_{\substack{p_1 < \cdots < p_{n_p} \\ p_a \in \mathbb{N}^*}} \sum_{\substack{h_1 < \cdots < h_{n_h} \\ h_a \in \mathbb{N}^*}} w^{\sum_{j=1}^{n_p}(p_j - 1) + \sum_{k=1}^{n_h} h_k} \left(\frac{\sin \pi\nu}{\pi}\right)^{2n_h}$$
$$\times \frac{\prod_{j>k}^{n_p}(p_j - p_k)^2 \prod_{j>k}^{n_h}(h_j - h_k)^2}{\prod_{j=1}^{n_p}\prod_{k=1}^{n_h}(p_j + h_k - 1)^2} \prod_{j=1}^{n_p} \frac{\Gamma^2(p_j + \nu)}{\Gamma^2(p_j)} \prod_{k=1}^{n_h} \frac{\Gamma^2(h_k - \nu)}{\Gamma^2(h_k)}$$

1. the **case** $\ell \neq 0$ can be obtained from $\ell = 0$ by a "background shift" :

   recast the sum over all possible excitations over the Dirac sea $\mathbb{Z}^-$
   (particles $p_j \in \mathbb{Z}^{+*}$, holes $1 - h_j \in \mathbb{Z}^-$, with $n_p - n_h = \ell$)
   as a sum over excitations over a shifted Dirac sea $\mathbb{Z} \cap]-\infty, \ell]$:
   particles $\ell + \tilde{p}_j$, holes $1 + \ell - \tilde{h}_j$, with $\tilde{n}_p - \tilde{n}_h = 0$

$$\Longrightarrow \quad f_\ell(\nu, w) = w^{\ell(\ell-1)/2} \frac{G^2(1+\ell+\nu)}{G^2(1+\nu)} f_0(\nu + \ell, w)$$

Summation formula: sketch of the proof (case $\ell = 0$)

Summation formula in the case $\ell = 0$

$$f_0(\nu, w) = (1 - w)^{-\nu^2}$$

where G is the Barnes G-function: $G(z+1) = \Gamma(z)G(z)$

and $$f_0(\nu, w) \equiv \sum_{n=0}^{\infty} \sum_{\substack{p_1 < \cdots < p_n \\ p_a \in \mathbb{N}^*}} \sum_{\substack{h_1 < \cdots < h_n \\ h_a \in \mathbb{N}^*}} w^{\sum_{j=1}^{n}(p_j + h_j - 1)} \left(\frac{\sin \pi\nu}{\pi} \right)^{2n}$$
$$\times \left[\det_n \frac{1}{p_j + h_k - 1} \right]^2 \prod_{j=1}^{n} \frac{\Gamma^2(p_j + \nu)}{\Gamma^2(p_j)} \frac{\Gamma^2(h_j - \nu)}{\Gamma^2(h_j)}$$

- ⋆ $f_0(-\nu, w) = f_0(\nu, w) \quad (p_j \leftrightarrow h_j) \quad \rightarrow \quad$ restrict to the case $\nu \geq 0$
- ⋆ Get an infinite determinant formula for $f_0(\nu, w)$:

$$f_0(\nu, w) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{h_1, \dots, h_n = 1}^{\infty} \det_{\substack{j=1,\dots,n \\ k=1,\dots,n}} V(h_j, h_k) = \det_{\substack{j=1,\dots,\infty \\ k=1,\dots,\infty}} [\delta_{jk} + V(j,k)]$$

with
$$V(j,k) = \left(\frac{\sin \pi\nu}{\pi} \right)^2 \frac{\Gamma(j-\nu)\Gamma(k-\nu)}{\Gamma(j)\Gamma(k)} \sum_{p=0}^{\infty} \frac{w^{p+(j+k)/2}}{(p+j)(p+k)} \frac{\Gamma^2(p+1+\nu)}{\Gamma^2(p+1)}$$

Summation formula: sketch of the proof (case $\ell = 0$)

Summation formula in the case $\ell = 0$

$$f_0(\nu, w) = (1-w)^{-\nu^2}$$

where G is the Barnes G-function: $G(z+1) = \Gamma(z)G(z)$

and $$f_0(\nu, w) \equiv \sum_{n=0}^{\infty} \sum_{\substack{p_1<\cdots<p_n \\ p_a\in\mathbb{N}^*}} \sum_{\substack{h_1<\cdots<h_n \\ h_a\in\mathbb{N}^*}} w^{\sum_{j=1}^{n}(p_j+h_j-1)} \left(\frac{\sin \pi\nu}{\pi}\right)^{2n} \times \left[\det_n \frac{1}{p_j+h_k-1}\right]^2 \prod_{j=1}^{n} \frac{\Gamma^2(p_j+\nu)}{\Gamma^2(p_j)} \frac{\Gamma^2(h_j-\nu)}{\Gamma^2(h_j)}$$

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Summation formula: sketch of the proof (case $\ell = 0$, $\nu \in \mathbb{N}$)

- If $\nu = N$ is a positive integer, the determinant becomes finite:

$$f_0(N, w) = \det_{\substack{j=1,\dots,N \\ k=1,\dots,N}} [\delta_{jk} + V(j,k)]$$

$$V(j,k) = \frac{w^{(j+k)/2}}{\prod_{\substack{m=1 \\ m\neq j}}^{N}(j-m)\,\prod_{\substack{m=1 \\ m\neq k}}^{N}(k-m)} \sum_{p=0}^{\infty} \frac{w^p}{(p+j)(p+k)} \prod_{m=1}^{N}(p+m)^2$$

- It can be simplified using the identity:

$$\det(I+V) = \frac{\det(AA^T + AVA^T)}{[\det A]^2} \quad \text{with} \quad A_{jk} = w^{-k/2}\, k^{j-1}$$

and setting $w = e^{-t}$:

$$f_0(N, e^{-t}) = \frac{e^{-tN(N+1)/2}}{\prod_{k=1}^{N-1}(k!)^2} \det_N \left[\partial_t^{j+k-2} \frac{e^{Nt}}{1-e^{-t}} \right]$$

- It can be rewritten as the homogeneous limit of a Cauchy determinant:

$$f_0(N, e^{-t}) = \lim_{\substack{u_1,\dots u_n \to 0 \\ v_1,\dots v_N \to 0}} \frac{e^{-tN(N+1)/2}}{\prod_{j<k}(u_j-u_k)(v_j-v_k)} \det_N \left[\frac{e^{N(t+u_j+v_k)}}{1-e^{-t-u_j-v_k}} \right]$$

Compute the determinant, take the homogeneous limit and get the result!

Summation formula: sketch of the proof (case $\ell = 0$, $\nu \notin \mathbb{N}$)

- The infinite determinant in terms of Toeplitz and Hankel matrices:

$$f_0(\nu, w) = \det\left[I + T^{-1}[a]\, H[a]\, H[\tilde{b}]\, T^{-1}[b]\right]$$

with $$a(z) = \sum_{-\infty}^{\infty} \frac{w^{n/2}}{n+\nu}\, z^n, \qquad b(z) = \sum_{-\infty}^{\infty} \frac{w^{-n/2}}{n-\nu}\, z^n$$

and $$H_{jk}[a] \equiv [a]_{j+k-1} = -H_{jk}[\tilde{b}] \equiv -[b]_{-j-k+1} = \frac{w^{(j+k-1)/2}}{j+k-1+\nu},$$

$$T_{jk}[a] \equiv [a]_{j-k} = \frac{w^{(j-k)/2}}{j-k+\nu}, \qquad T_{jk}[b] \equiv [b]_{j-k} = \frac{w^{-(j-k)/2}}{j-k-\nu}$$

- Use properties of Toeplitz matrices:
  - ⋆ $T[ab] = T[a]\, T[b] + H[a]\, H[\tilde{b}]$

    $\Rightarrow\ f_0(\nu, w) = \det\left[T^{-1}[a]\, T[ab]\, T^{-1}[b]\right]$
  - ⋆ Wiener-Hopf factorization:

$$a(z) = a_+(z)\, a_-(z) \quad \text{with} \quad \begin{cases} a_+(z) = \exp\left\{\sum_{k=1}^{\infty} z^k [\log a]_k\right\} \\ a_-(z) = \exp\left\{\sum_{k=1}^{\infty} z^{-k} [\log a]_{-k}\right\} \end{cases}$$

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$$\text{with} \quad \begin{cases} [\log a]_n = \delta_{n,0} \log \frac{\pi}{\sin \pi\nu} + (1-\delta_{n,0}) \frac{\nu}{n} w^{n/2} \\ [\log b]_n = \delta_{n,0} \log \left(-\frac{\pi}{\sin \pi\nu} \right) - (1-\delta_{n,0}) \frac{\nu}{n} w^{n/2} \end{cases}$$

  - ⋆ $\det [T^{-1}[a_+]\, T[a_+ b_-]\, T^{-1}[b_-]] = \exp \left[\sum_{k=1}^{\infty} k\, [\log a]_k [\log b]_{-k} \right]$ (Widom)

$$\Rightarrow \; f_0(\nu, w) = \exp\left\{ -\sum_{n=1}^{\infty} \frac{\nu^2}{n} w^n \right\} = (1-w)^{-\nu^2}$$

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Correlation function $\langle \sigma_1^z \, \sigma_{m+1}^z \rangle$

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = -\frac{1}{2\pi^2} \, \partial_\alpha^2 \mathbf{D}_m^2 \langle e^{2\pi i \alpha \mathcal{Q}_m} \rangle \Big|_{\alpha=0} - 2D + 1$$

where $\mathbf{D}_m^2$ is the second lattice derivative, D is the average density, and

$$\mathcal{Q}_m = \frac{1}{2} \sum_{k=1}^{m} (1 - \sigma_k^z)$$

$\rightsquigarrow$ study form factors $\langle \, \psi_\alpha(\{\mu\}) \, | e^{2\pi i \alpha \mathcal{Q}_m} | \, \psi_g \, \rangle$ where $| \, \psi_\alpha(\{\mu\}) \, \rangle$ is an α-deformed Bethe state, with $\{\mu\}$ solution of

$$M p_0(\mu_{\ell_j}) - \sum_{k=1}^{N} \theta(\mu_{\ell_j} - \mu_{\ell_k}) = 2\pi \Big(\ell_j + \alpha - \frac{N+1}{2} \Big)$$

For the $\mathbf{P}_\ell$ class:

- excitation momentum $2\alpha k_F + \mathcal{P}_{\text{ex}}$
- shift functions $F_\pm$: $F_- = F_+ = \alpha \mathcal{Z} + \ell(\mathcal{Z} - 1)$ with $\mathcal{Z} = Z(\pm q)$
  where $Z(\lambda)$ is the dressed charge given by
  $$Z(\lambda) + \frac{1}{2\pi} \int_{-q}^{q} d\mu \, \frac{\sin 2\zeta}{\sinh(\lambda - \mu + i\zeta) \sinh(\lambda - \mu - i\zeta)} \, Z(\mu) = 1$$
- exponent $\theta_{\alpha+\ell}$: $\theta_{\alpha+\ell} = 2[(\alpha + \ell)\mathcal{Z}]^2$,

Correlation function $\langle \sigma_1^z \, \sigma_{m+1}^z \rangle$

$\rightsquigarrow$ leading asymptotic terms for all oscillating harmonics:

$$\langle e^{2\pi i \alpha \mathcal{Q}_m} \rangle_{cr} = \sum_{\ell=-\infty}^{\infty} |\mathcal{F}_{\alpha+\ell}|^2_{\text{finite}} \, \frac{e^{2im(\alpha+\ell)k_F}}{(2\pi m)^{\theta_{\alpha+\ell}}}$$

with $\quad \theta_{\alpha+\ell} = 2[(\alpha+\ell)\mathcal{Z}]^2,$

and $\quad |\mathcal{F}_{\alpha+\ell}|^2_{\text{finite}} = \lim_{M\to\infty} M^{\theta_{\alpha+\ell}} \frac{|\langle \psi_g | \psi_{\alpha+\ell} \rangle|^2}{||\psi_g||^2 \, ||\psi_{\alpha+\ell}||^2},$

where $|\, \psi_{\alpha+\ell} \,\rangle$ is the $(\alpha+\ell)$-shifted ground state

Rm: terms $\ell = 0, \pm 1$ coincide with results from master equation analysis

$\rightsquigarrow$ leading asymptotic terms for the two-point function:

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle_{\text{cr}} = (2D-1)^2 - \frac{2\mathcal{Z}^2}{\pi^2 m^2} + 2 \sum_{\ell=1}^{\infty} |\mathcal{F}_\ell^z|^2_{\text{finite}} \, \frac{\cos(2m\ell k_F)}{(2\pi m)^{2\ell^2 \mathcal{Z}^2}}$$

with $\quad |\mathcal{F}_\ell^z|^2_{\text{finite}} = \lim_{M\to\infty} M^{2\ell^2\mathcal{Z}^2} \frac{|\langle\, \psi_g \,|\sigma_1^z|\, \psi_\ell \,\rangle|^2}{||\psi_g||^2 \, ||\psi_\ell||^2},$

where $|\, \psi_\ell \,\rangle$ is the ℓ-shifted ground state

Correlation function $\langle \sigma_1^+ \sigma_{m+1}^- \rangle$

$\leadsto$ critical excited states of the $\mathbf{P}_\ell$ class in the (N_0+1)-sector

- critical values of the shift function in the $\mathbf{P}_\ell$ class:

$$F_- = \ell(\mathcal{Z}-1) - \frac{1}{2\mathcal{Z}}, \qquad F_+ = \ell(\mathcal{Z}-1) + \frac{1}{2\mathcal{Z}}$$

- critical exponents: $\theta_\ell = 2\ell^2\mathcal{Z}^2 + \frac{1}{2\mathcal{Z}^2}$
- simplest form factor in the $\mathbf{P}_\ell$ class:

$$\left|\mathcal{F}_\ell^+\right|^2_{\text{finite}} = \lim_{M\to\infty} M^{(2\ell^2\mathcal{Z}^2+\frac{1}{2\mathcal{Z}^2})} \frac{|\langle \psi_g | \sigma_1^+ | \psi_\ell \rangle|^2}{||\psi_g||^2 \, ||\psi_\ell||^2}$$

where $|\,\psi_\ell\,\rangle$ is the ℓ-shifted ground state in the (N_0+1)-sector

$\leadsto$ leading asymptotic terms for the two-point function:

$$\langle \sigma_1^+ \sigma_{m+1}^- \rangle_{cr} = \frac{(-1)^m}{(2\pi m)^{\frac{1}{2\mathcal{Z}^2}}} \sum_{\ell=-\infty}^{\infty} (-1)^\ell \, |\mathcal{F}_\ell^+|^2_{\text{finite}} \, \frac{e^{2im\ell k_F}}{(2\pi m)^{2\ell^2\mathcal{Z}^2}}$$

Results for the XXZ chain

2-point functions

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle_{\rm cr} = (2D-1)^2 - \frac{2\mathcal{Z}^2}{\pi^2 m^2} + 2\sum_{\ell=1}^{\infty} |\mathcal{F}_\ell^z|^2_{\rm finite} \frac{\cos(2m\ell k_F)}{(2\pi m)^{2\ell^2 \mathcal{Z}^2}}$$

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- $\mathcal{Z} = Z(q)$ where $Z(\lambda)$ is the dressed charge
$$Z(\lambda) + \int_{-q}^{q} \frac{\mathrm{d}\mu}{2\pi} K(\lambda-\mu)\, Z(\mu) = 1$$
- D is the average density $\quad D = \int_{-q}^{q} \rho(\mu)\mathrm{d}\mu = \frac{1-\langle \sigma^z \rangle}{2} = \frac{k_F}{\pi}$
- $$|\mathcal{F}_\ell^z|^2_{\rm finite} = \lim_{M\to\infty} M^{2\ell^2 \mathcal{Z}^2} \frac{|\langle \psi_g | \sigma_1^z | \psi_\ell \rangle|^2}{\langle \psi_g | \psi_g \rangle \langle \psi_\ell | \psi_\ell \rangle}$$
- $$|\mathcal{F}_\ell^+|^2_{\rm finite} = \lim_{M\to\infty} M^{\left(2\ell^2 \mathcal{Z}^2 + \frac{1}{2\mathcal{Z}^2}\right)} \frac{|\langle \psi_g | \sigma_1^+ | \psi_\ell \rangle|^2}{\langle \psi_g | \psi_g \rangle \langle \psi_\ell | \psi_\ell \rangle}$$

Further results and open questions

- **Further results**
  - Time dependent case for the Bose gas (simpler model: no bound-states) (to appear)
    $\rightsquigarrow$ contribution of a saddle point away from the Fermi surface
  - Asymptotics for large distances in the temperature case (contact with QTM method)
    $\rightsquigarrow$ see Kozlowski, M, Slavnov *J. Stat. Mech. P12010 (2011)*
  - Arbitrary n-point correlation functions in the CFT limit (to appear)
  - In fact all the derivation applies to a large class of non integrable models as well

- **Some open problems...**
  - Sub-leading terms for each harmonics?
  - Time dependent case for XXZ : needs careful treatment of bound-states (complex roots)
  - Deeper links with TASEP, Z-measures, ...?