In memory of Philippe Flajolet
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Arctic curves of the six-vertex model

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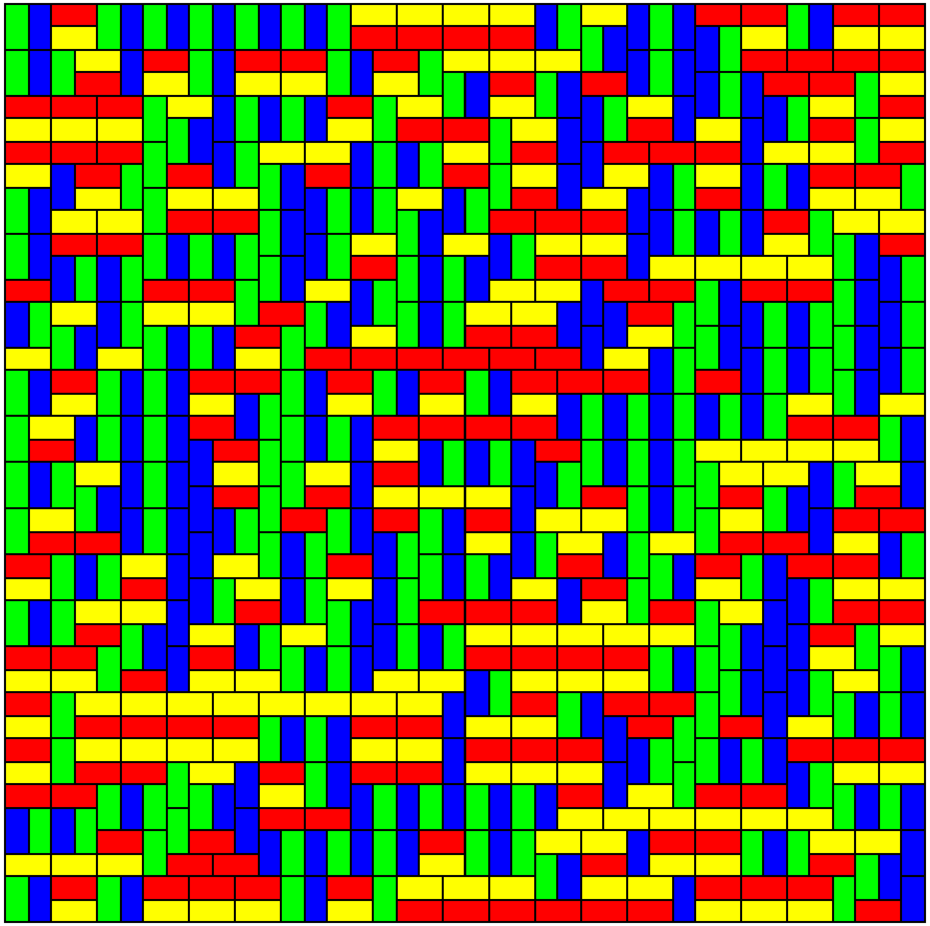
Domino tiling of a square

domino:= 2×1 tile



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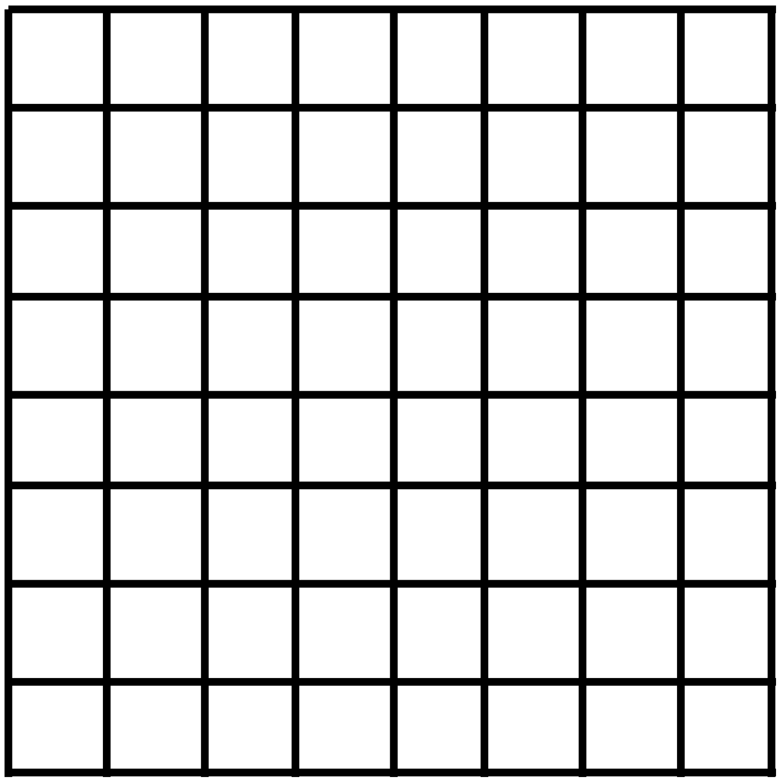
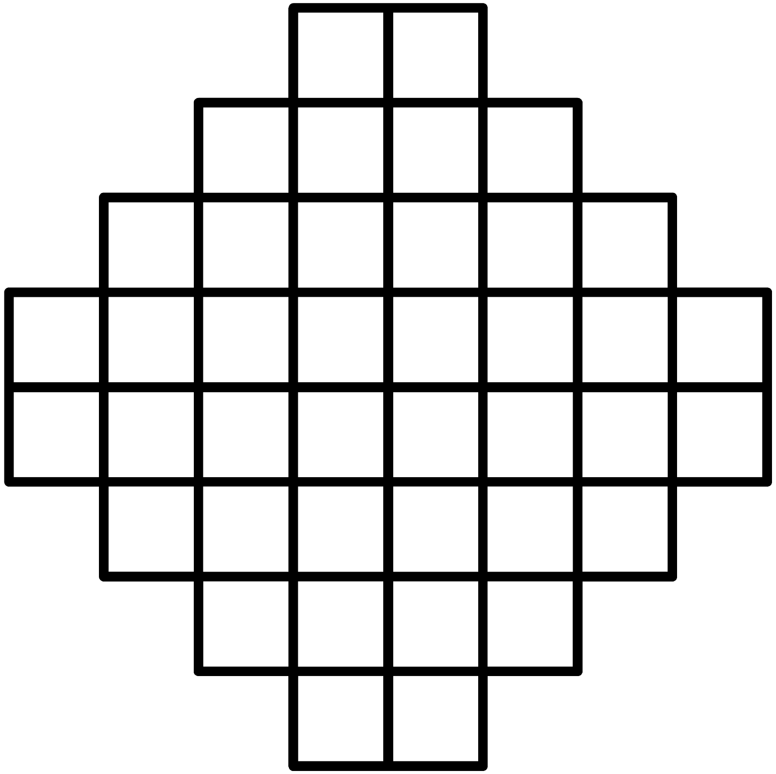


$N \times N$ square



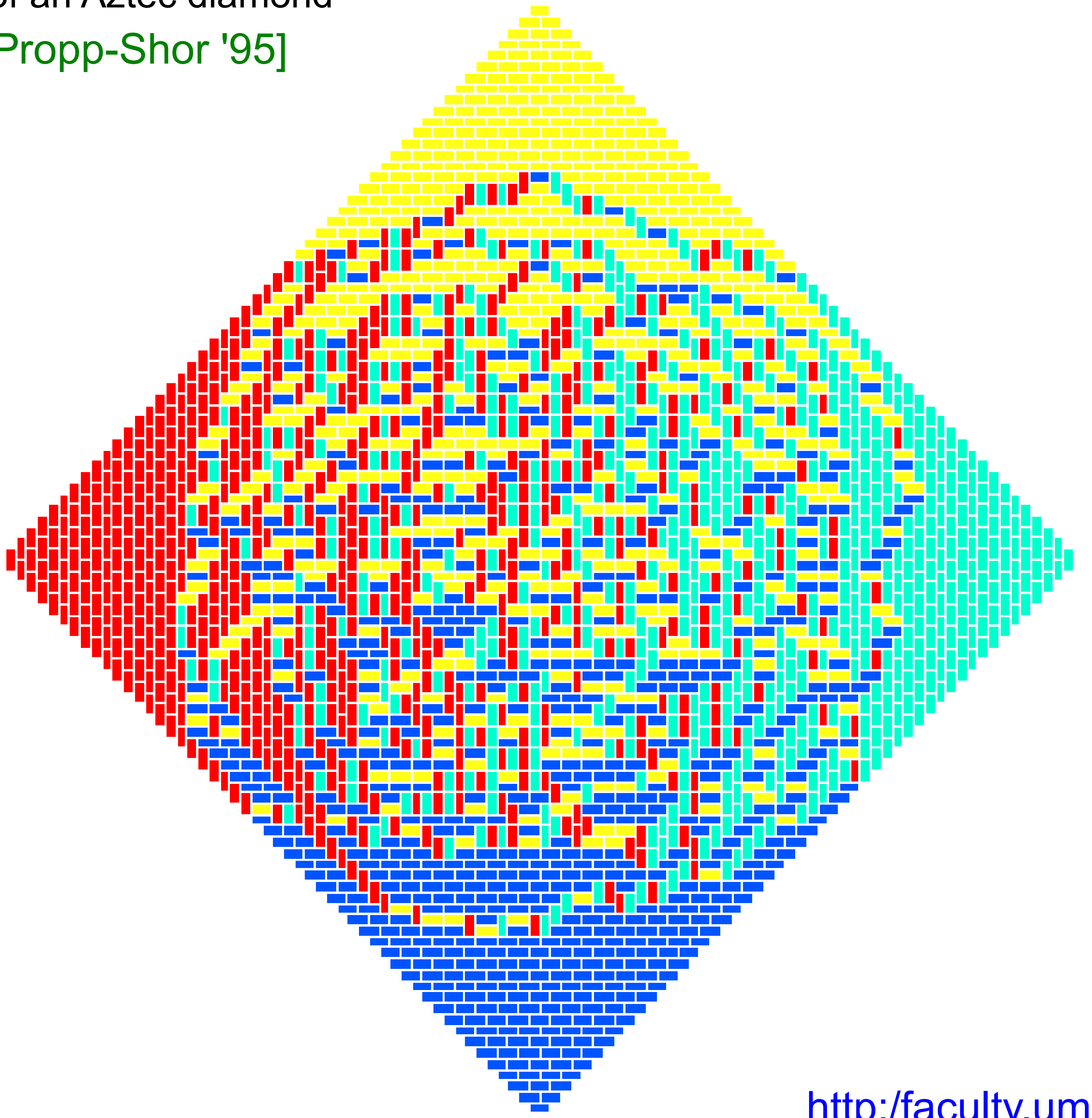
Aztec Diamond of order $N/2$

($N = 8$)

Domino tiling of an Aztec diamond

[Jockush-Propp-Shor '95]

$N = 64$



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The Arctic Circle Theorem

[Jockush-Propp-Shor '95]

$\forall \epsilon > 0, \quad \exists N$ such that "almost all" (i.e. with probability $P > 1 - \epsilon$) randomly picked domino tilings of $AD(N)$ have a temperate region whose boundary stays uniformly within distance ϵN from the circle of radius $N/\sqrt{2}$.

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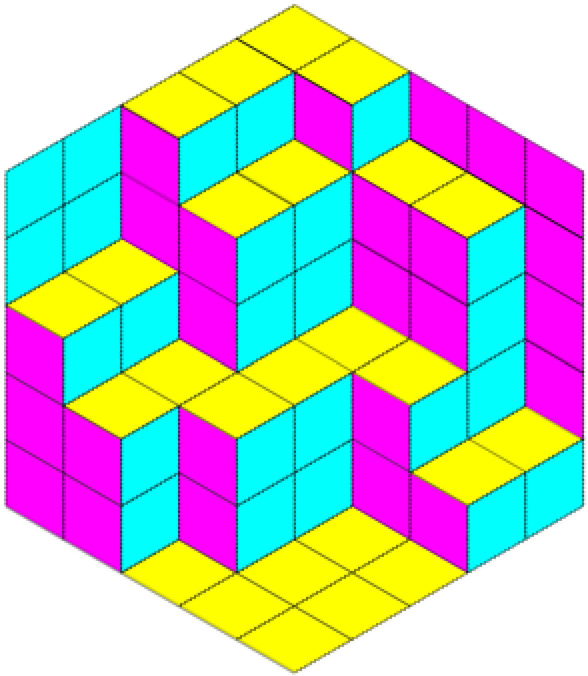
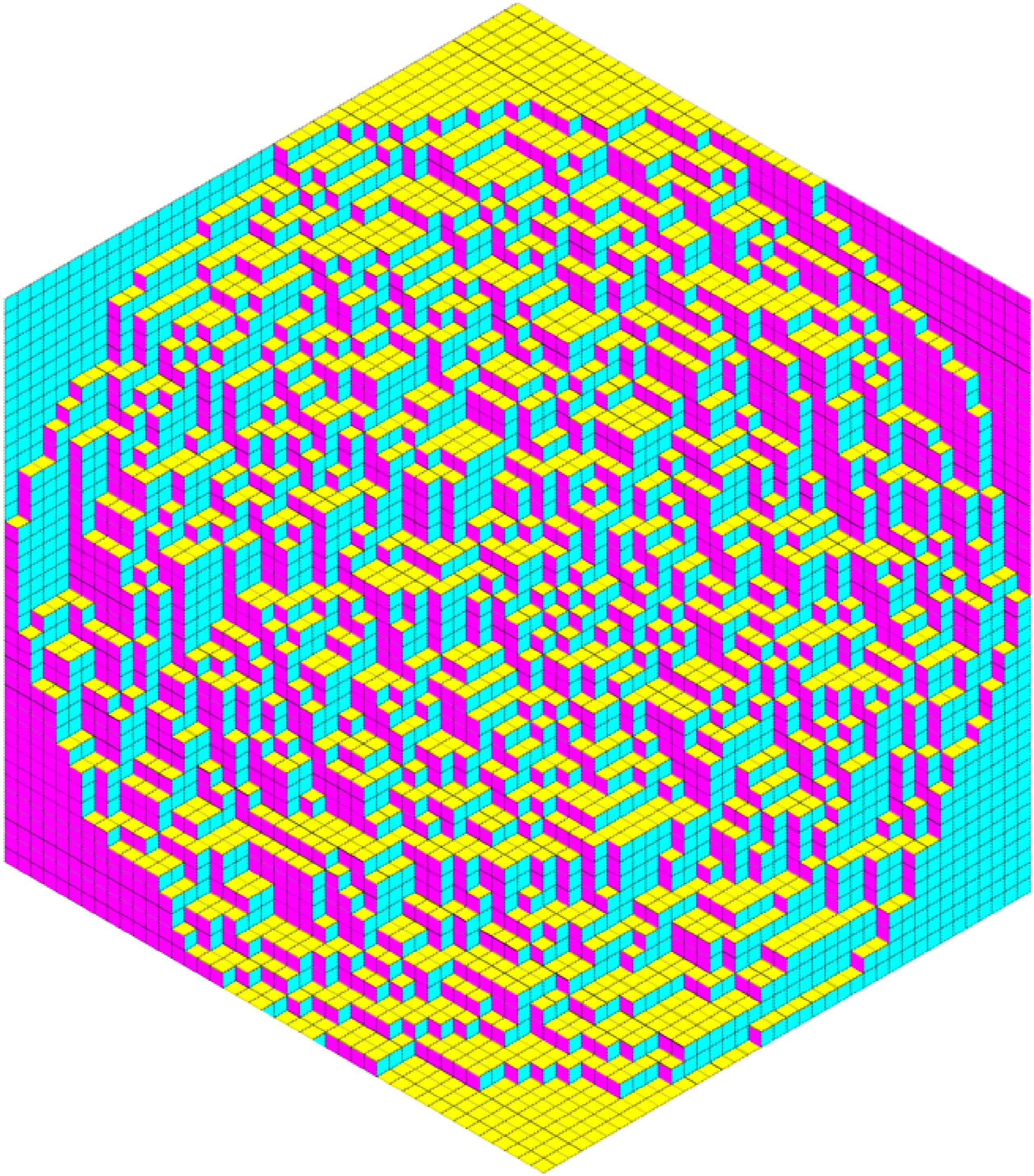
Fluctuations:

- boundary fluctuations $N^{1/3}$ [Johansson'00]
- fluctuations of boundary intersection with main diagonal obey Tracy-Widom distribution [Johansson'02]
- after suitable rescaling, boundary has limit as a random function, governed by an Airy stochastic process [Johansson'05]

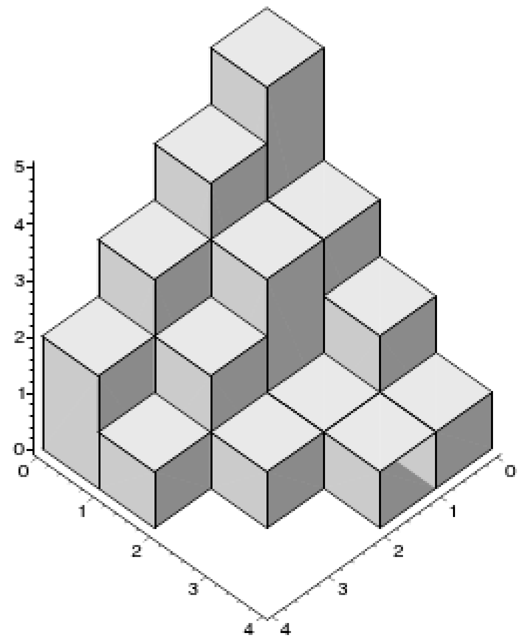
$\Rightarrow$ fluctuations are described by random matrix models

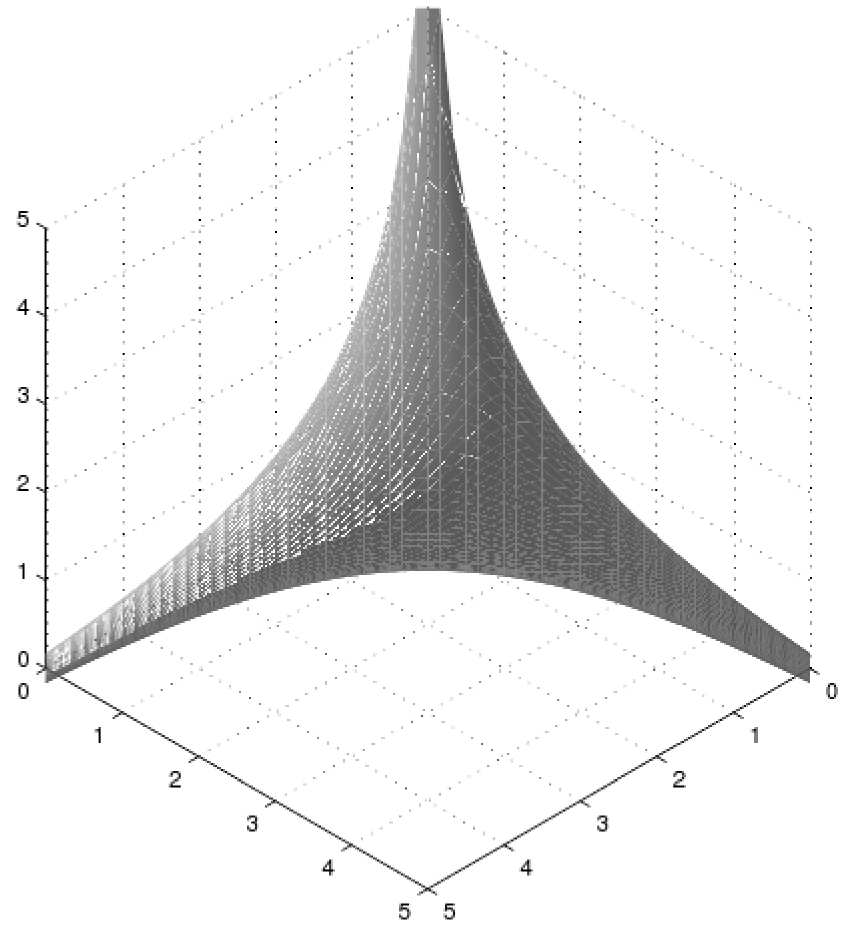
See also recent developments for the `Double Aztec Diamond and the Tacnode Process'
[Adler-Johansson-van Moerbeke '11]



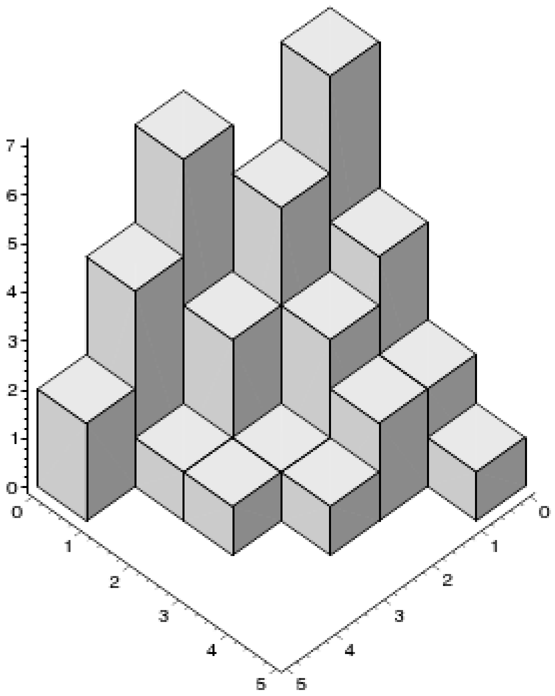


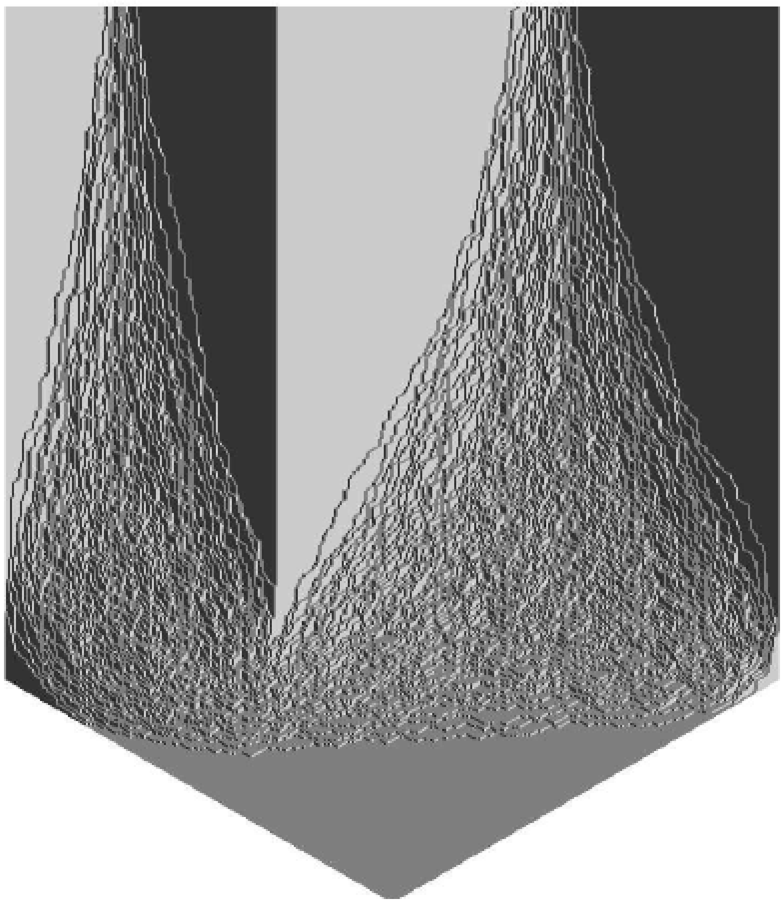
- Boxed plane partitions [Cohn-Larsen-Propp'98]





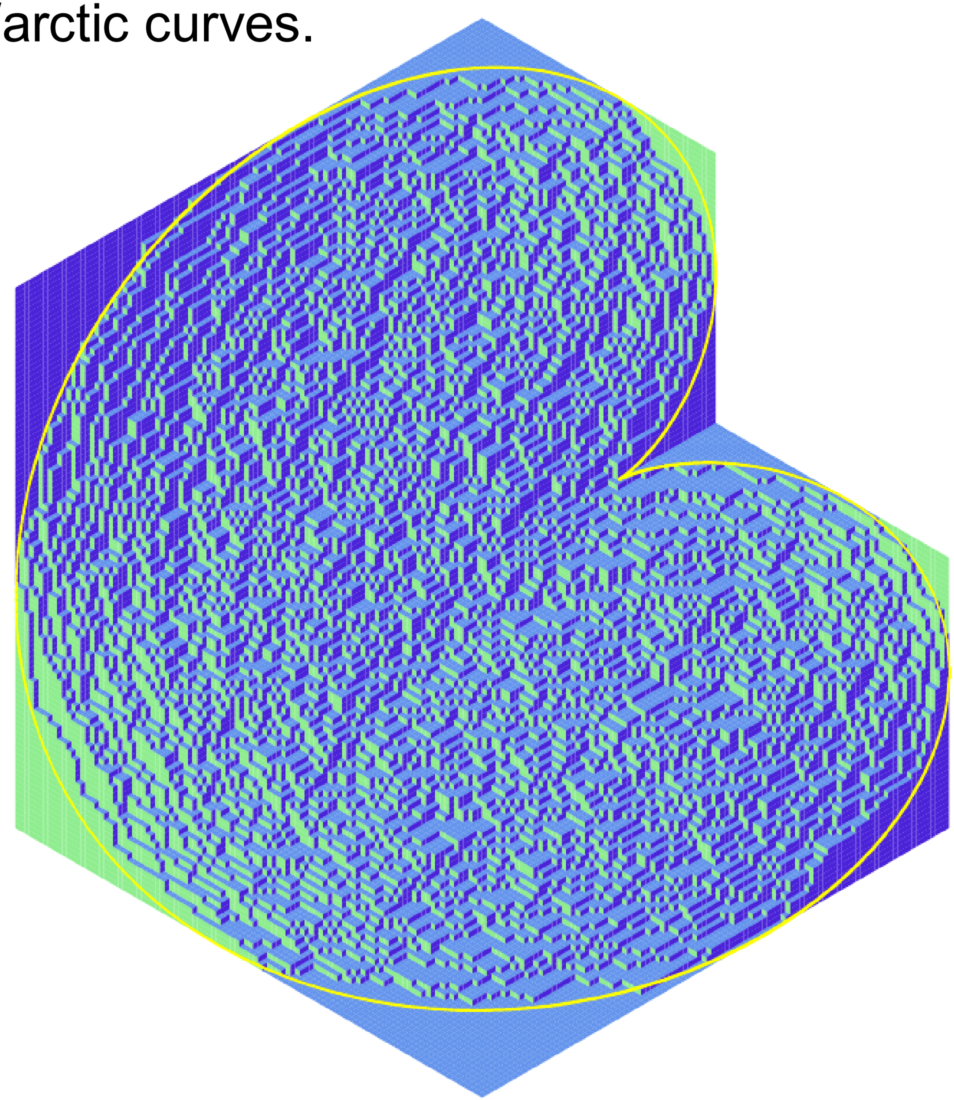
- Corner melting of a crystal [Ferrari-Spohn '02]
- Plane partitions [Cerf-Kenyon'01][Okounkov-Reshetikhin'01]





- Skewed plane partitions [Okounkov-Reshetikhin '05]
  [Boutillier-Mkrtchyan-Reshetikhin-Tingley '10]
  [Mkrtchyan '11]

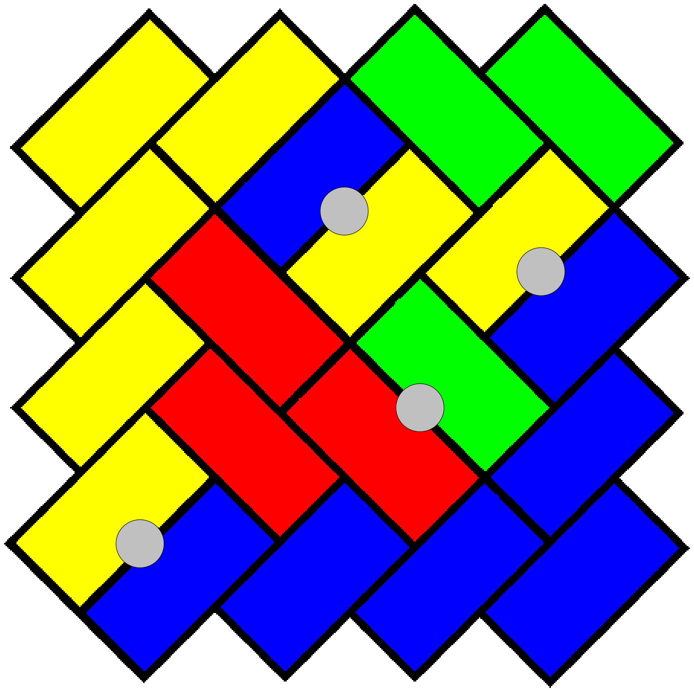
Actually all these models are avatars of the same model, `dimer covering of regular planar bipartite lattices', exhibiting emergence of phase separation, limit shapes, frozen boundaries/arctic curves.



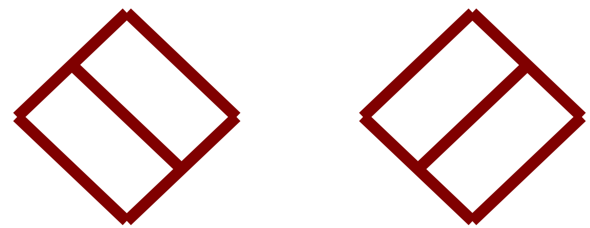
A beautiful unified theory has been provided for regular planar bipartite graphs [Kenyon, Sheffield, Okounkov, '03-'05], with deep implications in algebraic geometry and algebraic combinatorics.

Introducing an interaction between dimers

[Elkies-Kuperberg-Larsen-Propp'92]



Assign a nontrivial weight to:

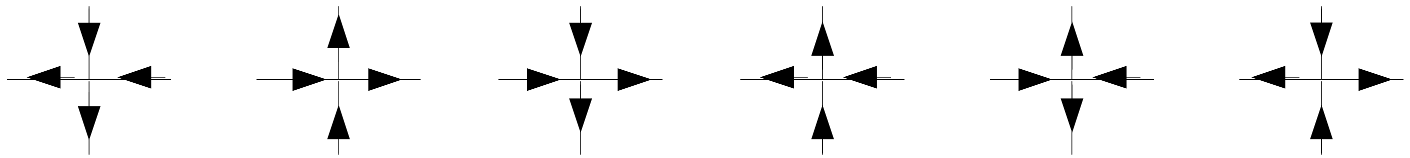


Six-vertex model with Domain Wall b.c.

An <u>exactly solvable</u> model of statistical mechanics

The six-vertex model

[Lieb '67] [Sutherland'67]



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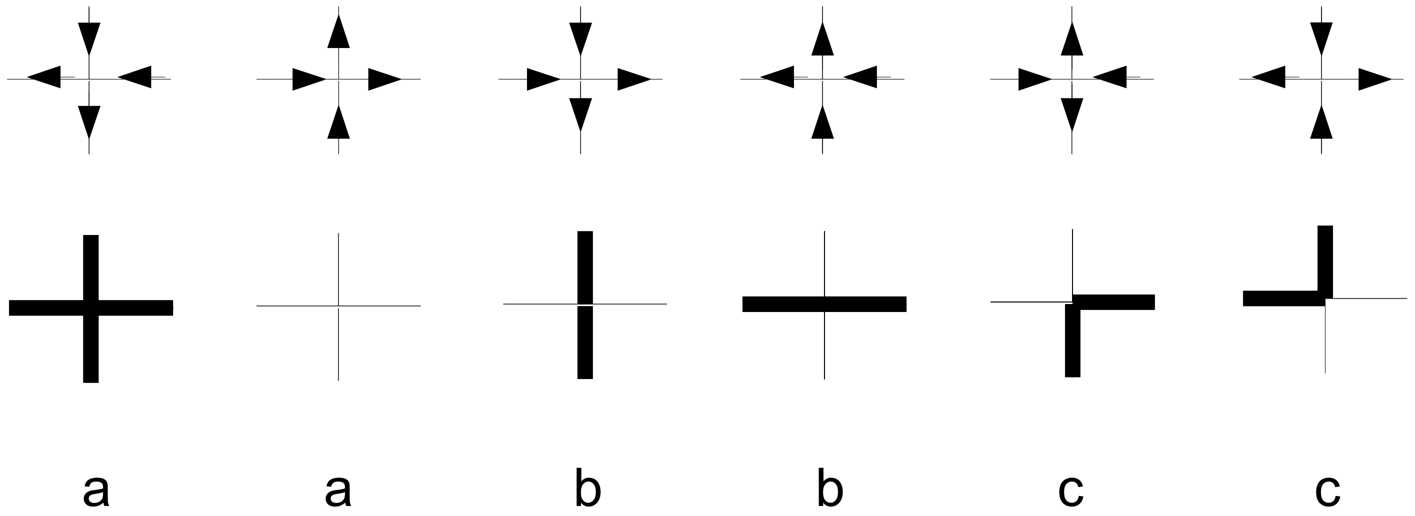

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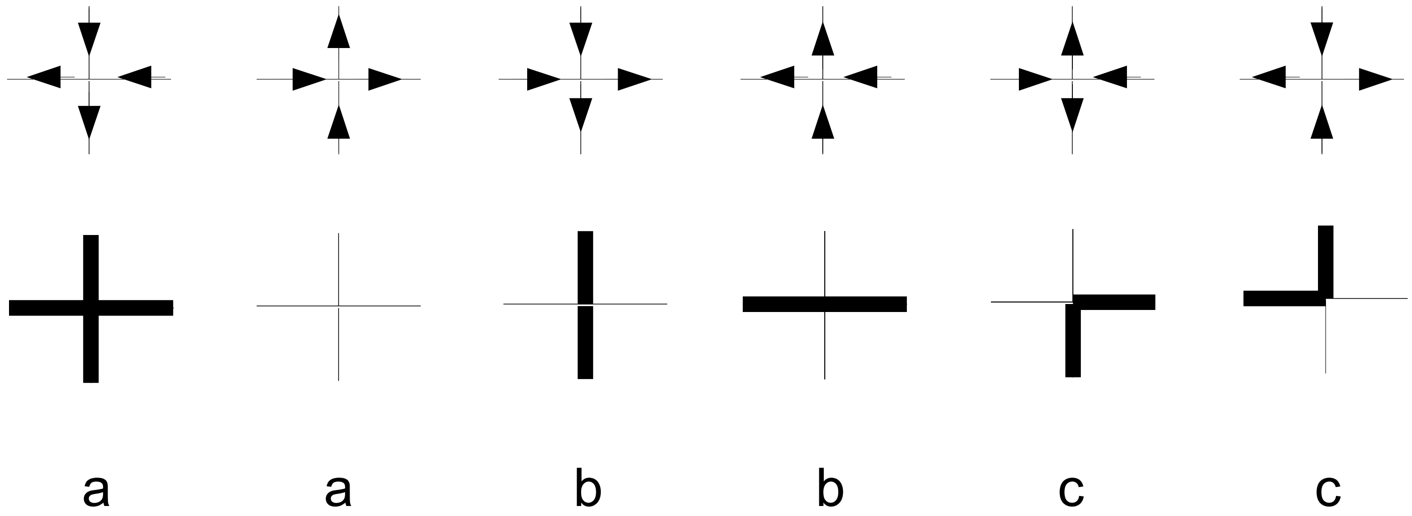
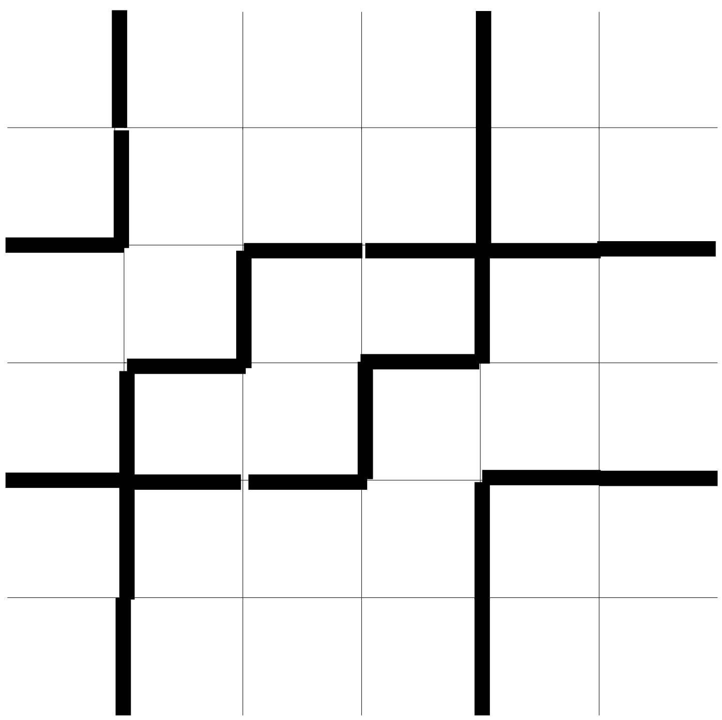




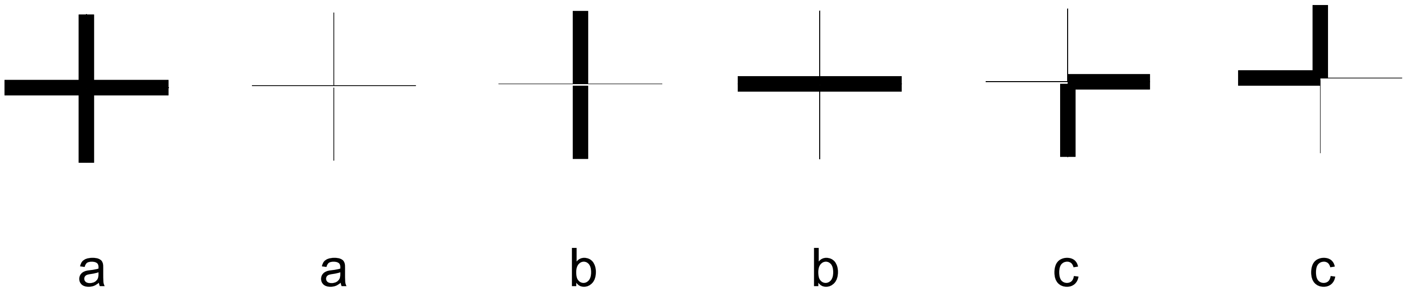
$$\begin{aligned} a &= \sin(\lambda + \eta) \\ b &= \sin(\lambda - \eta) \\ c &= \sin 2\eta \end{aligned}$$

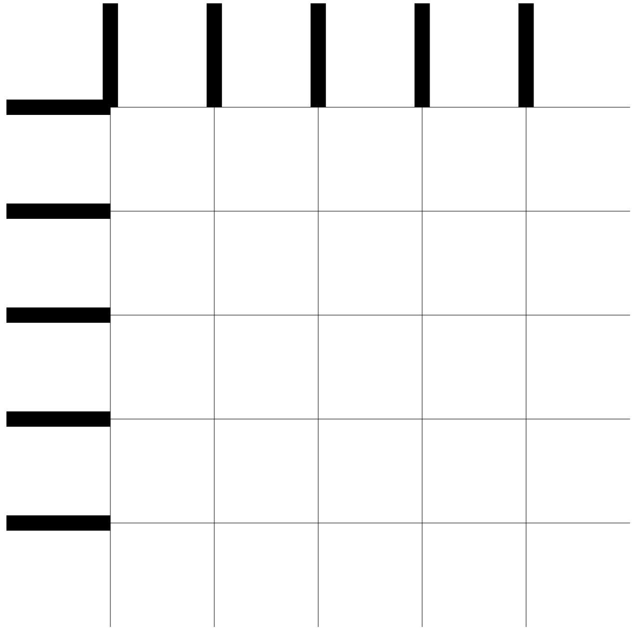
$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}$$

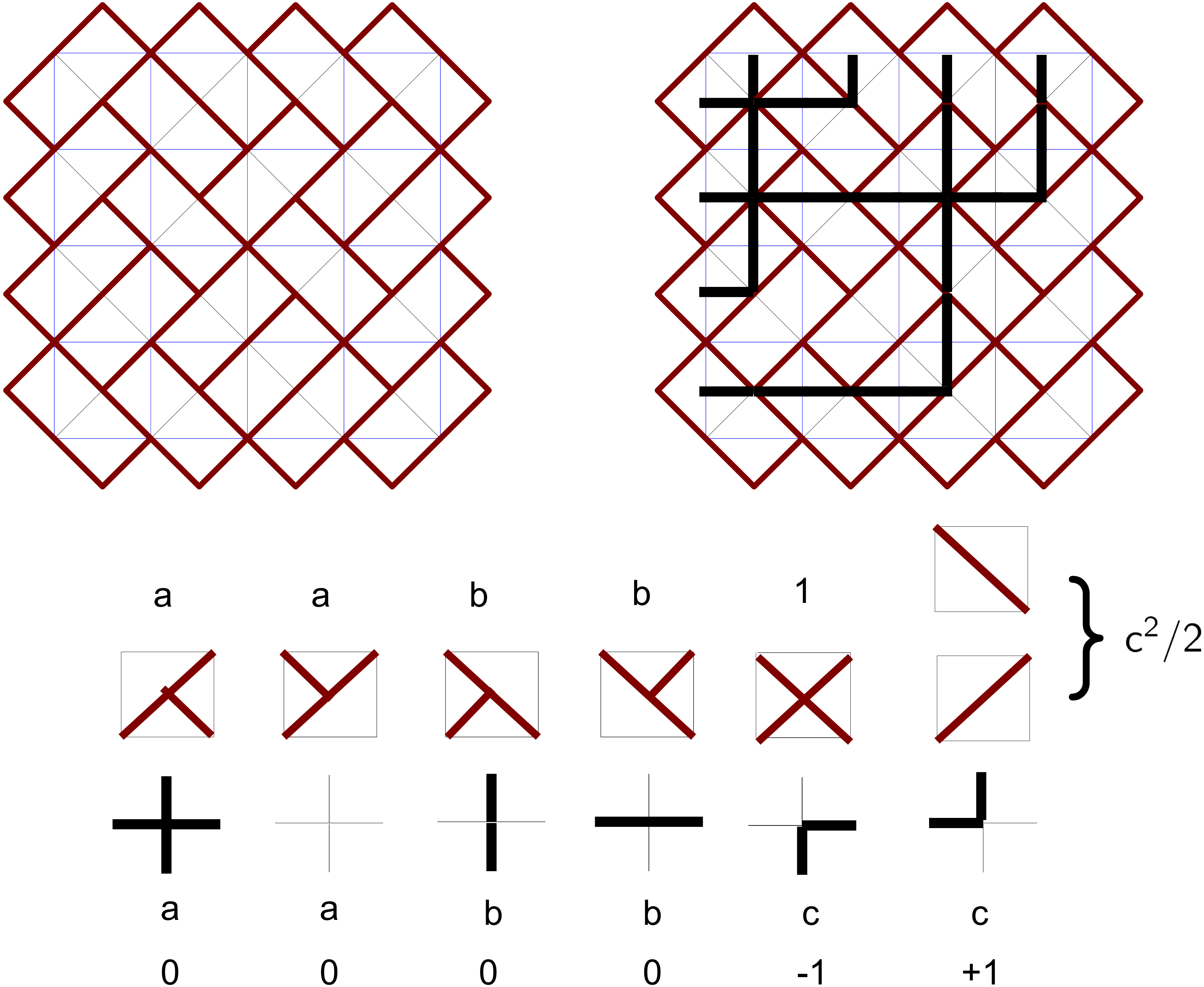
$$t = \frac{b}{a}$$

The Domain Wall six-vertex model

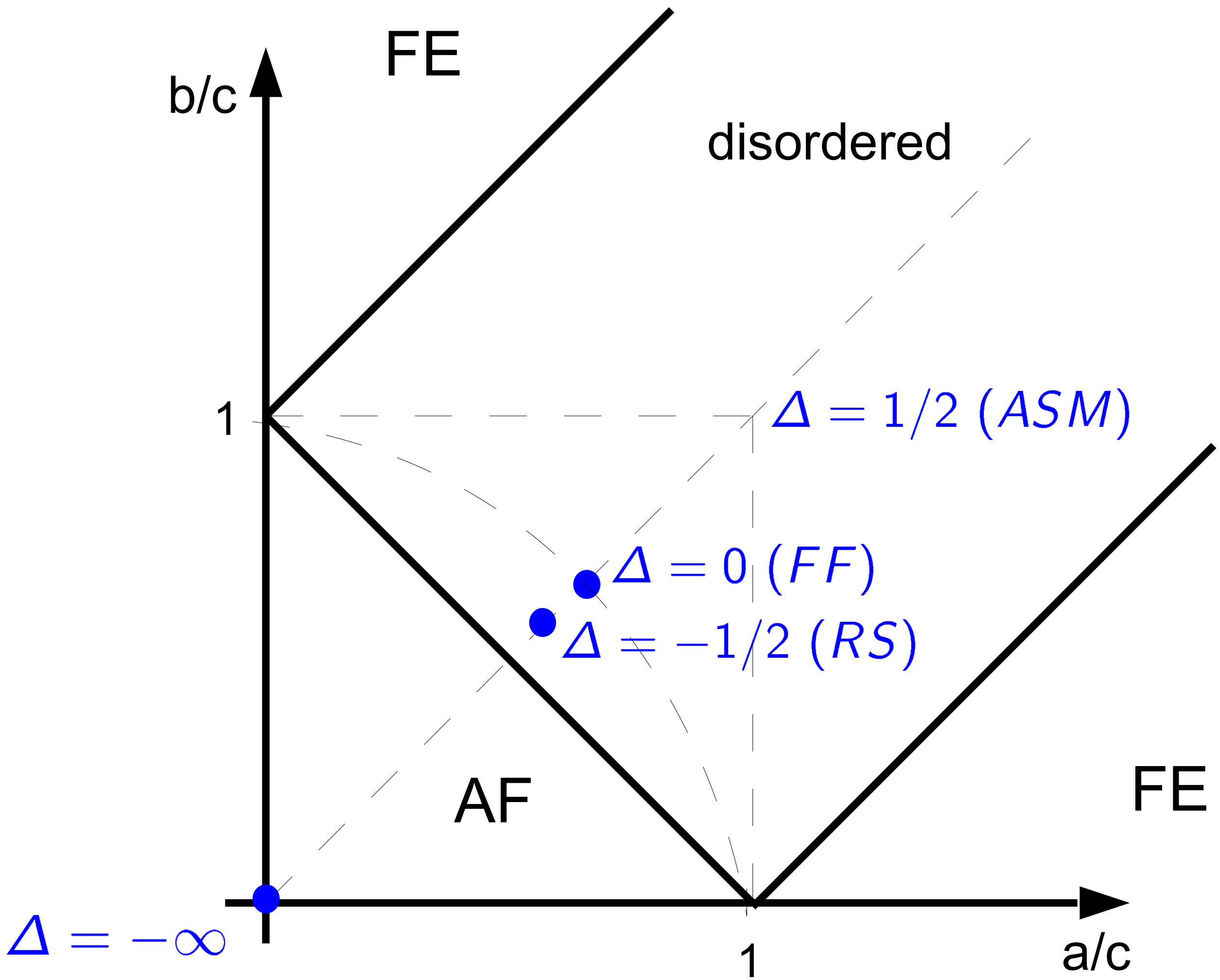
[Korepin '82]



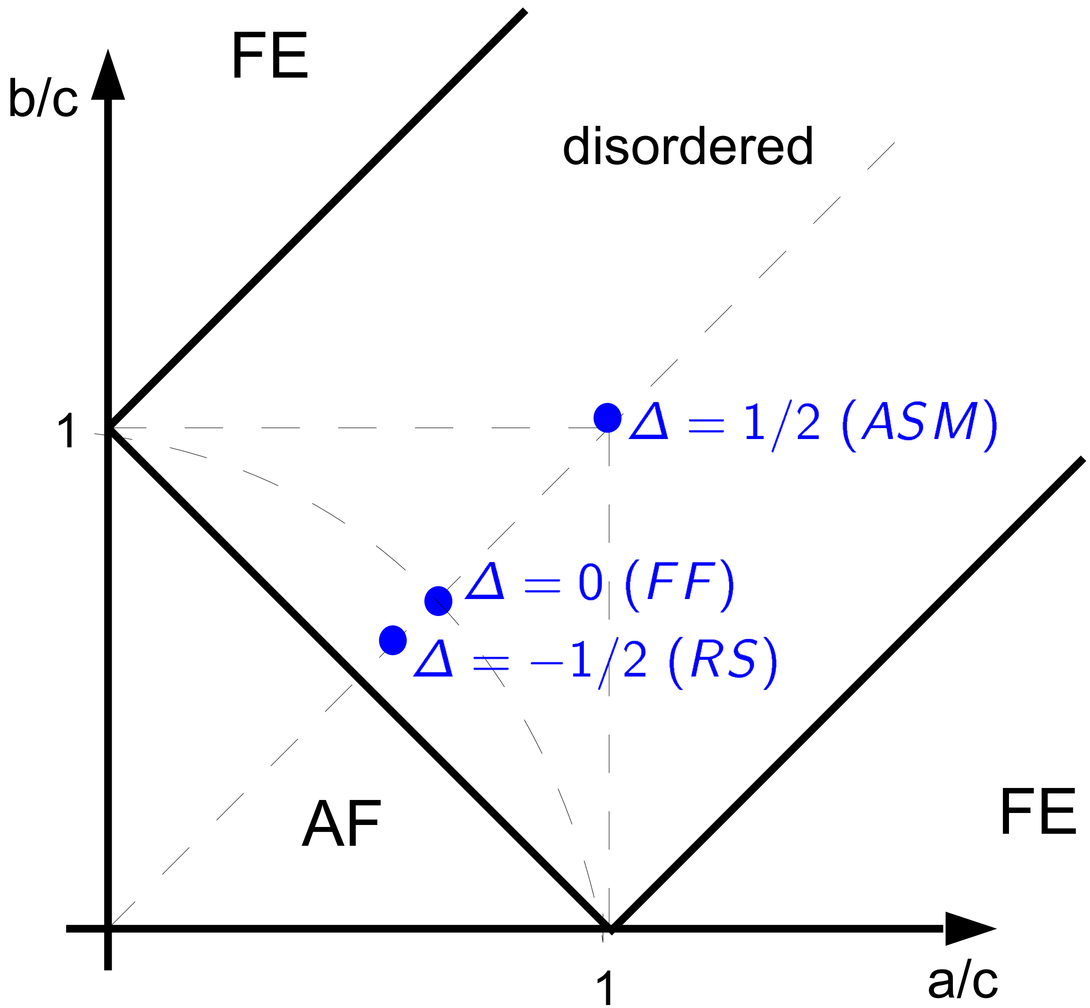




a
a
b
b
1
c2/2
a
a
b
b
c
c
0
0
0
0
-1
+1



FE
b/c
disordered
1
$\Delta = 1/2\ (ASM)$
$\Delta = 0\ (FF)$
$\Delta = -1/2\ (RS)$
AF
FE
$\Delta = -\infty$
1
a/c

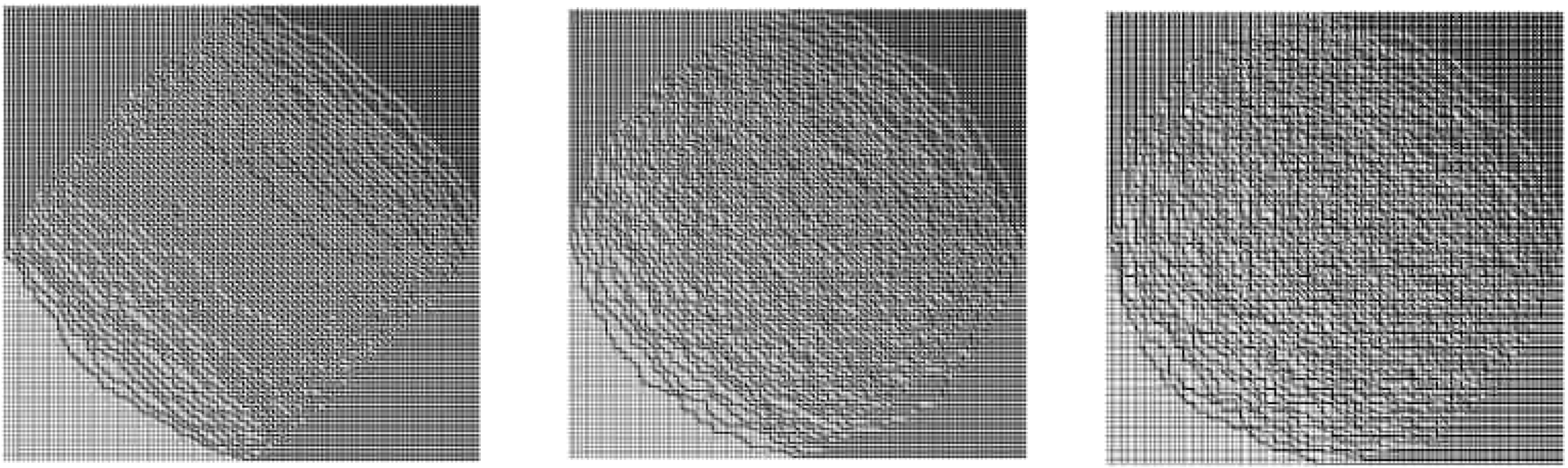


With Domain-Wall b.c., for $\Delta = 0$ we have the Arctic Circle Theorem.

And for generic Δ and t? And for generic regions? And what about fluctuations?

Domain Wall six-vertex model: numerical results

[Eloranta'99] [Zvonarev-Syluasen'04] [Allison-Reshetikhin'05]



$\Delta = -3$ $\Delta = -0.92$ $\Delta = 0$ (free fermions)

$N = 225$ [Allison-Reshetikhin'05]

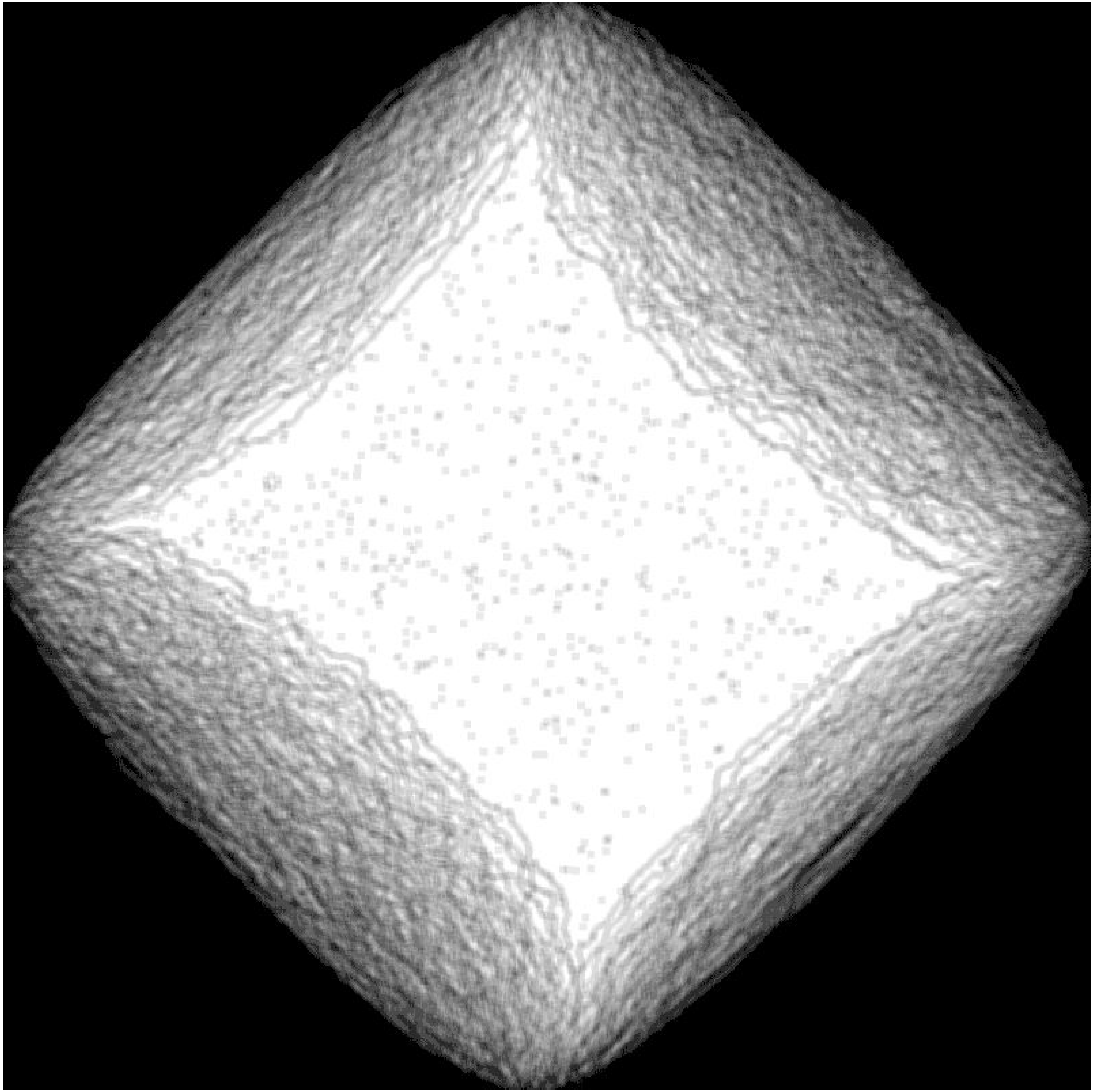
Area of disordered region increases with Δ.

(Anti-ferroelectric regime)

$N = 1000$

$\Delta = -3$

$t = 0.5$



White pixels represents c-vertices

[Allison-Reshetikhin'05]

Domain Wall six-vertex model: analytic results

(For generic Δ, not so many: translation invariance is broken!)

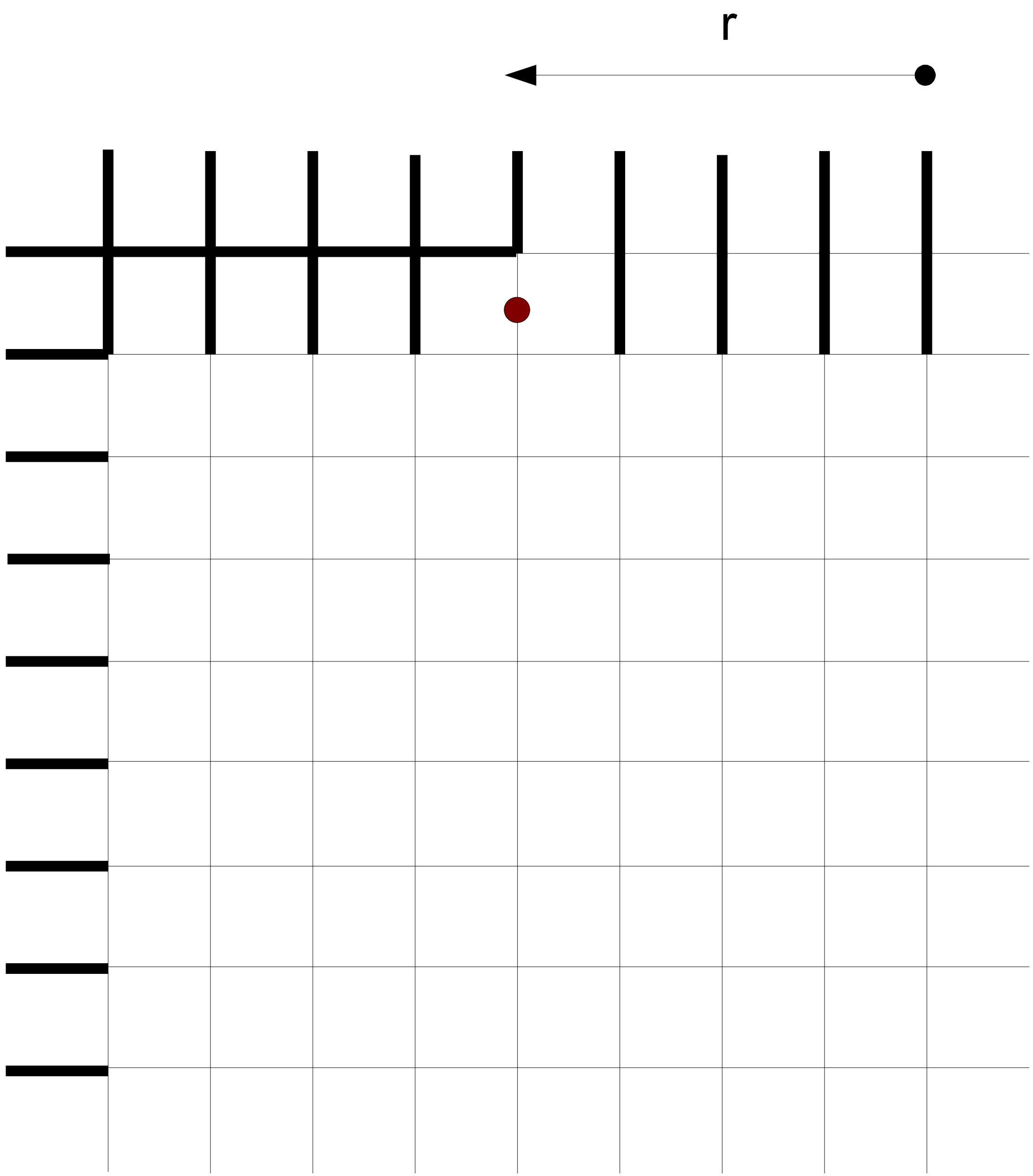
- Partition function:
  - I-K determinant representation and Hankel determinant representation for Z_N [Korepin'82] [Izergin'87]
  - Large N behaviour of Z_N :

    Bulk free energy: DWBC $\neq$ PBC
    [Korepin Zinn-Justin'00] [Zinn-Justin'01]
    [Bleher-Fokin-Liechty'05-'09]

- Boundary correlation functions:
  - one-point boundary correlation function
    [Bogoliubov-Pronko-Zvonarev'02]

One-point boundary correlation function

$H_N(r)$



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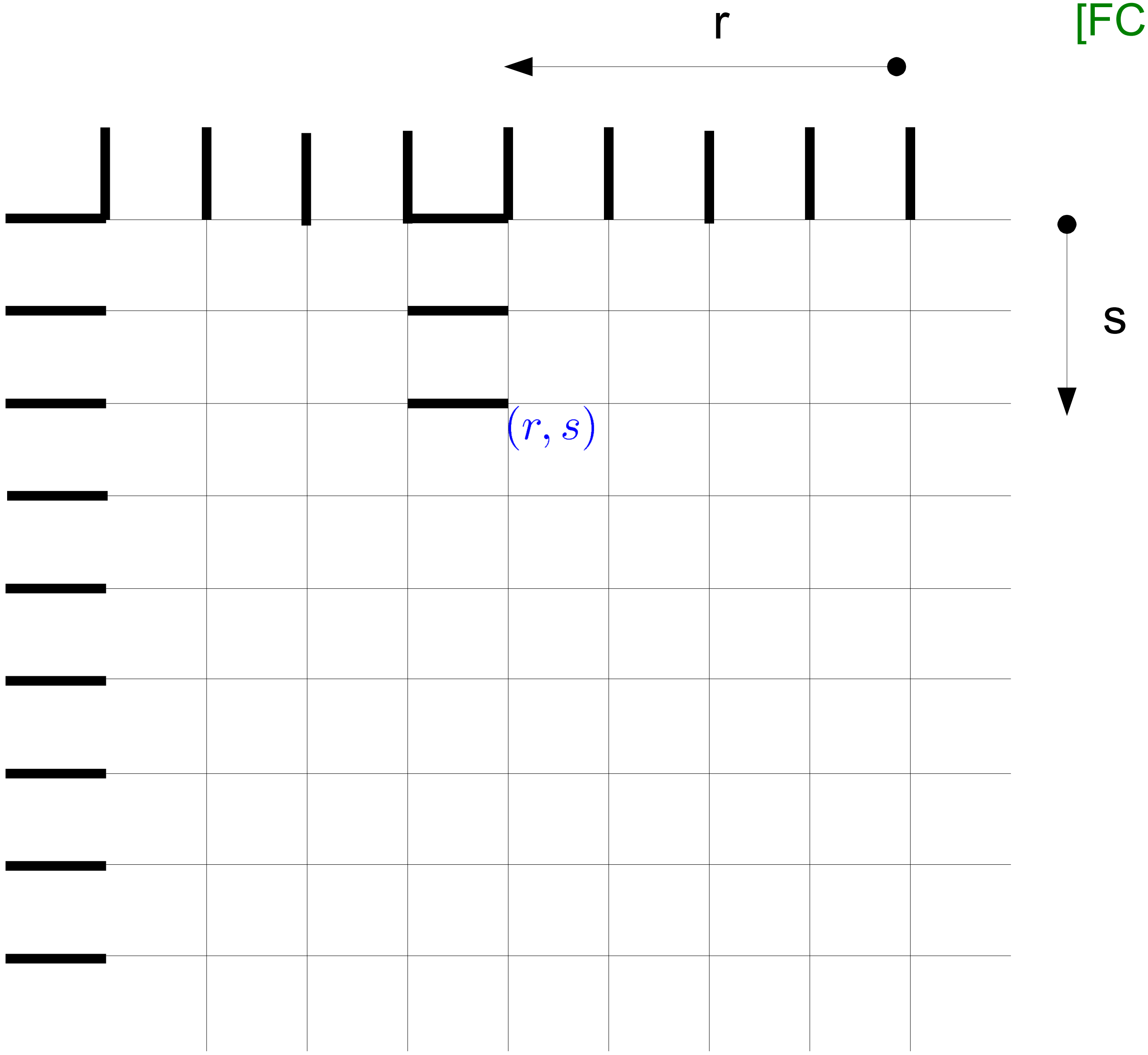
- Boundary correlation functions:
    - one-point boundary correlation function [Bogoliubov-Pronko-Zvonarev'02]
    - two-point boundary correlation function [FC-Pronko'05]

      (all these again in terms of $N \times N$ determinants)

- Bulk correlation functions: Nothing!

$F_N(r,s)$ Emptiness Formation Probability (EFP)

[FC-Pronko'08]



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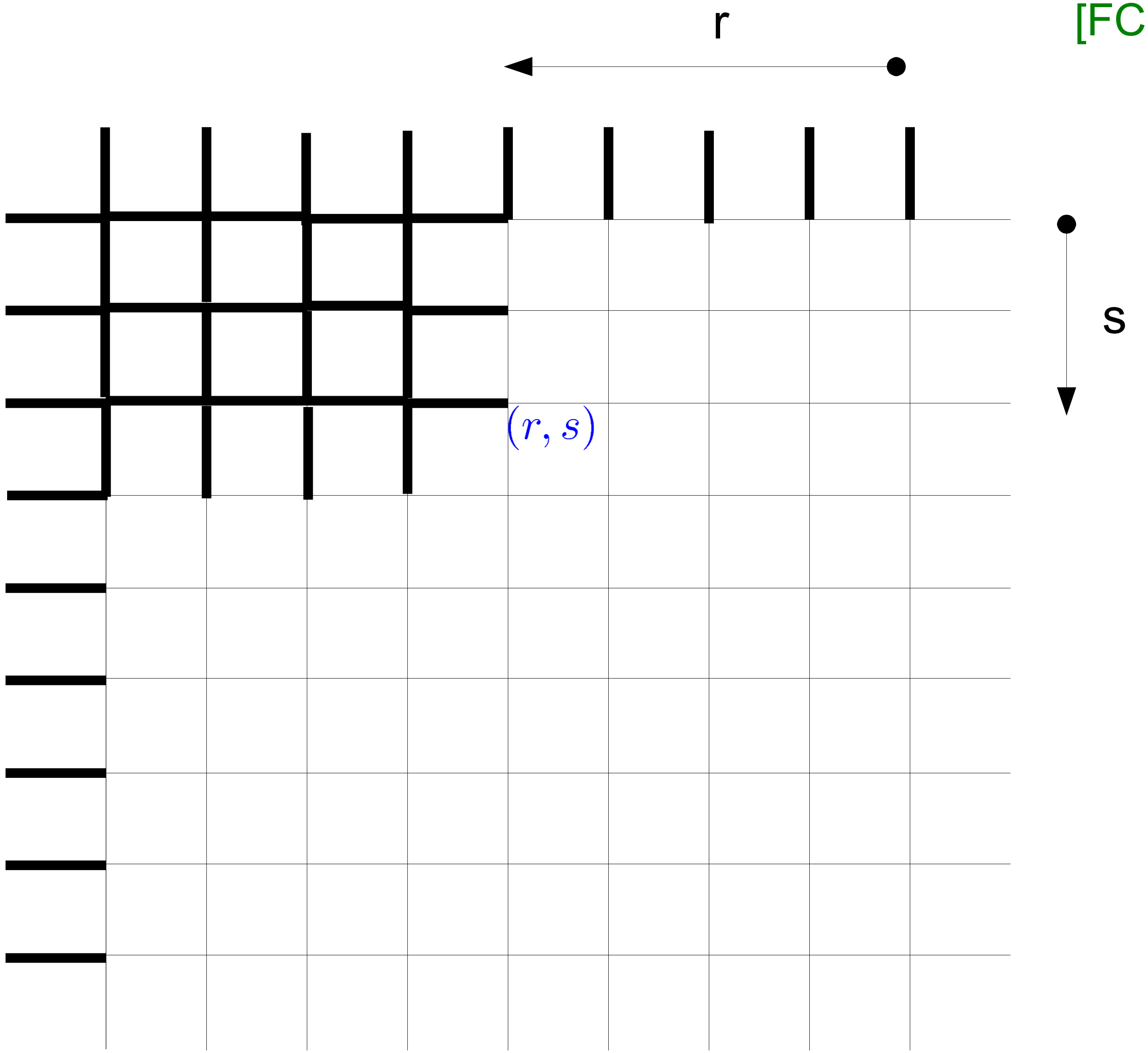

- Stepwise behaviour in correspondence of the Arctic curve
- Ability to discriminate only the top-left portion of the curve

Multiple Integral Representation for EFP

[FC-Pronko'08]

Define the generating function for the 1-point boundary correlator:

$$h_N(z) := \sum_{r=1}^{N} H_N(r) z^{r-1}\,, \qquad h_N(1) = 1\,.$$

Now define, for $s = 1, \dots, N$:

$$h_N^{(s)}(z_1, \dots, z_s) := \frac{1}{\Delta_s(z_1, \dots, z_s)} \det_{1 \le j,k \le s} \left[h_{N-s+k}(z_j)(z_j - 1)^{k-1} z_j^{s-k} \right]$$

- The functions $h_N^{(s)}(z_1, \dots, z_s)$ are totally symmetric polynomials of order $N-1$ in $z_1, \dots, z_s$.
- They define a new, alternative representation (with respect to Izergin-Korepin determinant) for the partition function Z_N.

Two important properties of $h_N^{(s)}(z_1, \dots, z_s)$:

$$h_N^{(s)}(z_1, \dots, z_{s-1}, 0) = h_N(0) h_{N-1}^{(s-1)}(z_1, \dots, z_{s-1})\,,$$

$$h_N^{(s)}(z_1, \dots, z_{s-1}, 1) = h_N^{(s-1)}(z_1, \dots, z_{s-1})\,.$$

Multiple Integral Representation for EFP

[FC-Pronko'08]

The following Multiple Integral Representation is valid ($r, s = 1, 2, \ldots, N$):

$$F_N^{(r,s)} = \frac{(-1)^s Z_s}{s!(2\pi i)^s a^{s(s-1)} c^s} \oint_{C_0} \cdots \oint_{C_0} d^s z \prod_{j=1}^{s} \frac{[(t^2 - 2t\Delta)z_j + 1]^{s-1}}{z_j^r (z_j - 1)^s}$$

$$\times \prod_{\substack{j,k=1 \\ j \neq k}}^{s} \frac{z_k - z_j}{t^2 z_j z_k - 2t\Delta z_j + 1} h_{N,s}(z_1, \ldots, z_s) h_{s,s}(u(z_1), \ldots, u(z_s))$$

where $u(z) := -\frac{z-1}{(t^2-2t\Delta)z+1}$.

Ingredients:

- Quantum Inverse Scattering Method to obtain a recurrence relation, which is solved in terms of a determinant representation on the lines of Izergin-Korepin formula;
- Orthogonal Polynomial and Random Matrices technologies to rewrite it as a multiple integral.

Remark:

Similar expressions occurs for correlation function in ASEP [Tracy-Widom'08-'11].

Scaling limit of EFP

[FC-Pronko'10]

Evaluate: $F(x,y) := \lim_{N\to\infty} F_N(xN, yN)$ $\quad x, y \in [0,1]$

in the limit: $N, r, s \to \infty$ $\quad \frac{r}{N} = x \quad \frac{s}{N} = y$

using Saddle-Point method.

Saddle-point equations:

$$
\begin{aligned}
&- \frac{s}{z_j - 1} - \frac{r}{z_j} + \frac{s(t^2 - 2\Delta t)}{(t^2 - 2\Delta t) z_j + 1} - \sum_{\substack{k=1\\k\neq j}}^{s} \Bigg(\frac{t^2 z_k - 2\Delta t}{t^2 z_j z_k - 2\Delta t z_j + 1} \\
&\qquad + \frac{t^2 z_k}{t^2 z_j z_k - 2\Delta t z_k + 1} + \frac{2}{z_k - z_j} \Bigg) + \frac{\partial \ln h_{N,s}(z_1, \dots, z_s)}{\partial z_j} \\
&\qquad\qquad - \frac{t^2 - 2\Delta t + 1}{[(t^2 - 2\Delta t) z_j + 1]^2} \, \frac{\partial \ln h_{s,s}(u_1, \dots, u_s)}{\partial u_j} = 0,
\end{aligned}
$$

$$(j = 1, \dots, s)$$

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Scaling limit of EFP

[FC-Pronko'10]

NB1:

- $s \times s$ Vandermonde determinant
- s-order pole at $z = 1$

Penner Random Matrix model

[Penner'88]

NB2:

- By construction, in the scaling limit, EFP is 1 in the frozen region, and 0 in the disordered one, with a stepwise behaviour in correspondence of the Arctic curve.
- From the structure of the Multiple Integral Representation, such stepwise behaviour can be ascribed to the position of the SPE roots with respect to the pole at $z = 1$.
- The considered generalized Penner model allows for condensation of `almost all' SPE roots at $z = 1$. [Tan'92] [Ambjorn-Kristjansen-Makeenko'94]

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$\Longrightarrow$ Penner Random Matrix model [Penner'88]

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Condensation of `almost all' SPE roots at $z = 1$ $\Longleftrightarrow$ Arctic Curves

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Condensation of `almost all' SPE roots at $z = 1$ $\Leftrightarrow$ Arctic Curves

Mathematically, the condition of total condensation (i.e. the Arctic curve) is given by:

$$\frac{y}{z-1} - \frac{x}{z} - \frac{y t^2}{t^2 z - 2\Delta t + 1} + \lim_{N\to\infty} \frac{1}{N} \partial_z \ln h_N(z) = 0$$

must have two coinciding real roots in interval: $z \in [1, \infty]$.

Evaluation of $h_N(z)$ (disordered regime $|\Delta| < 1$)

[FC-Pronko'10]

For $|\Delta| < 1$, we have

$$h_N(z(\zeta)) \underset{N\to\infty}{\sim} \left[\frac{\sin\gamma(\lambda-\eta)}{\gamma\sin(\lambda-\eta)}\right]^N \left[\frac{\sin(\zeta+\lambda-\eta)\sin(\gamma\zeta)}{\sin\gamma(\zeta+\lambda-\eta)\sin\zeta}\right]^N e^{o(N)}$$

where

$$z(\zeta) = \frac{\sin(\lambda+\eta)}{\sin(\lambda-\eta)}\frac{\sin(\zeta+\lambda-\eta)}{\sin(\zeta+\lambda+\eta)}, \qquad \text{and} \qquad \gamma := \frac{\pi}{\pi-2\eta}.$$

$$\Delta = \cos 2\eta \qquad t = \frac{\sin(\lambda-\eta)}{\sin(\lambda+\eta)}$$

NB: $z \in [1,+\infty)$ corresponds to $\zeta \in [0, \pi-\lambda-\eta)$

Evaluation of $h_N(z)$ (anti-ferroelectric regime $\Delta < -1$)

[FC-Pronko-Zinn-Justin'10]

For $\Delta < -1$, the large N behaviour of $h_N(z)$ is given by

$$h_N(z) \underset{N\to\infty}{\sim} \left[\frac{\vartheta_1(\gamma(\lambda+\eta))}{\gamma \sinh(\lambda+\eta)}\right]^N \left[\frac{\sinh(\zeta+\lambda+\eta)\vartheta_1(\gamma\zeta)}{\vartheta_1(\gamma(\zeta+\lambda+\eta))\sinh\zeta}\right]^N e^{o(N)}$$

where Jacobi Theta function ϑ_1 has nome $q = e^{\pi^2/(2\eta)}$.

We have

$$z(\zeta) = -\frac{\sinh(\eta-\lambda)}{\sinh(\eta+\lambda)}\frac{\sinh(\eta+\lambda+\zeta)}{\sinh(\eta-\lambda-\zeta)}, \qquad \text{and} \qquad \gamma := \frac{\pi}{2\eta}.$$

$$\Delta = -\cosh 2\eta \qquad\qquad t = \frac{\sinh(\eta+\lambda)}{\sinh(\eta-\lambda)}$$

NB: $z \in [1, +\infty)$ corresponds to $\zeta \in [0, \lambda+\eta)$

In both cases we get the Arctic curve in parametric form ($\zeta \in [0,\ \zeta_{max}]$):

$$
\begin{aligned}
x &= \frac{1}{\Phi(\zeta+\lambda-\eta,2\eta)\Psi(\zeta,2\eta)-\Psi(\zeta+\lambda-\eta,2\eta)\Phi(\zeta,2\eta)} \\
&\qquad \times \left\{ \left[\Psi(\zeta,\lambda-\eta)-\gamma^2\Psi(\gamma\zeta,\gamma(\lambda-\eta))\right]\Phi(\zeta,2\eta) \right. \\
&\qquad\qquad \left. -\left[\Phi(\zeta,\lambda-\eta)-\gamma\Phi(\gamma\zeta,\gamma(\lambda-\eta))\right]\Psi(\zeta,2\eta) \right\} , \\
y &= \frac{1}{\Phi(\zeta+\lambda-\eta,2\eta)\Psi(\zeta,2\eta)-\Psi(\zeta+\lambda-\eta,2\eta)\Phi(\zeta,2\eta)} \\
&\qquad \times \left\{ \left[\Psi(\zeta,\lambda-\eta)-\gamma^2\Psi(\gamma\zeta,\gamma(\lambda-\eta))\right]\Phi(\zeta+\lambda-\eta,2\eta) \right. \\
&\qquad\qquad \left. -\left[\Phi(\zeta,\lambda-\eta)-\gamma\Phi(\gamma\zeta,\gamma(\lambda-\eta))\right]\Psi(\zeta+\lambda-\eta,2\eta) \right\} .
\end{aligned}
$$

where

$$
\Phi(\mu) := \frac{\sin(2\eta)}{\sin(\mu+\eta)\sin(\mu-\eta)} , \qquad \gamma := \frac{\pi}{\pi-\arccos\Delta}
$$

$$
\Psi(\zeta) := \cot\zeta-\cot(\zeta+\lambda-\eta)-\gamma\cot\gamma\zeta+\gamma\cot\gamma(\zeta+\lambda-\eta) ,
$$

(Disordered regime, $-1<\Delta<1$)

or

$$
\Phi(\mu) := \frac{\sinh(2\eta)}{\sinh(\eta-\mu)\sinh(\eta+\mu)} , \qquad \gamma := \frac{\pi}{\operatorname{arccosh}(-\Delta)}
$$

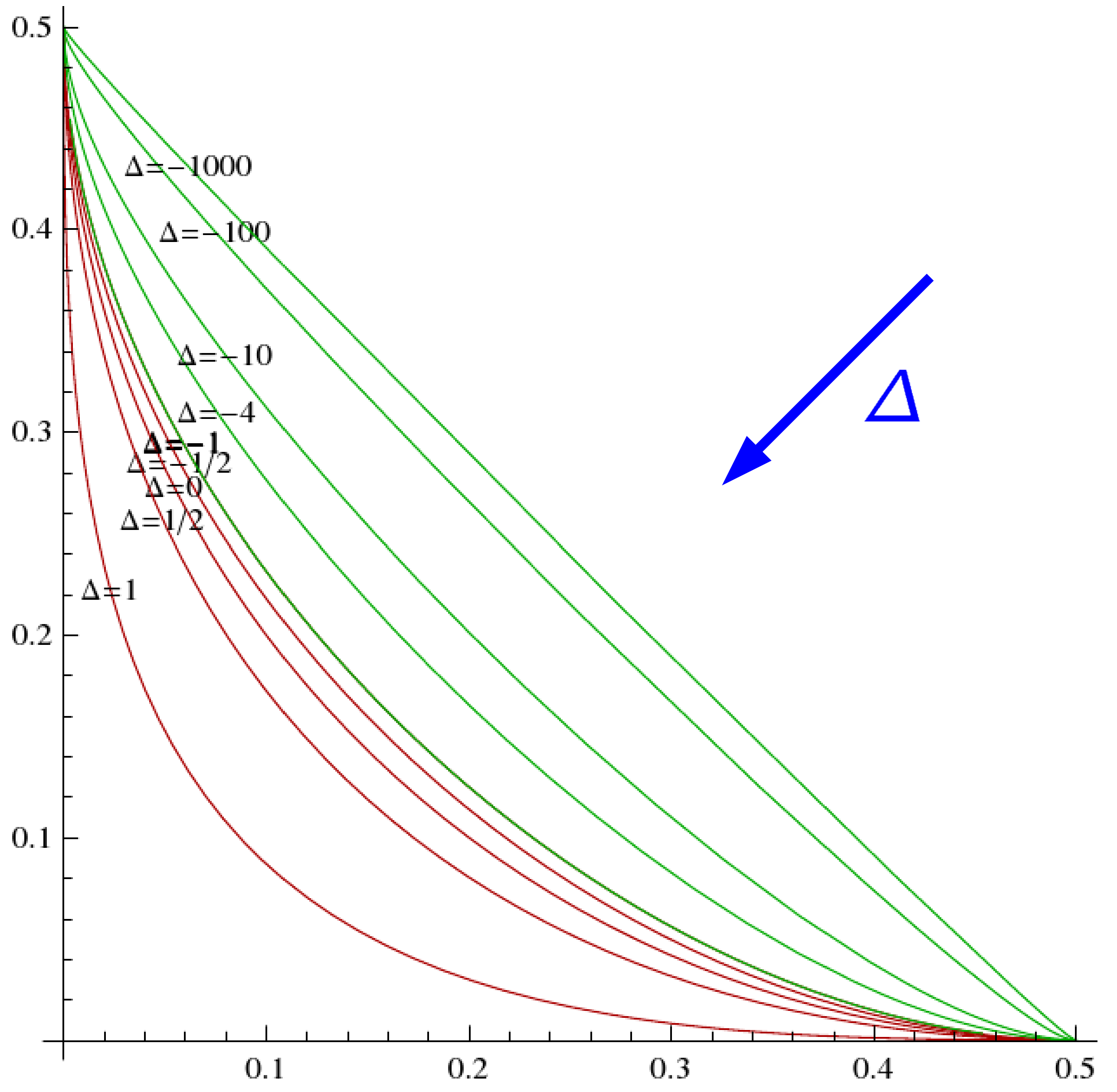
$$
\Psi(\zeta) := \cot\zeta-\coth(\eta-\lambda-\zeta)-\gamma\frac{\vartheta_1'(\gamma\zeta)}{\vartheta_1(\gamma\zeta)}+\gamma\frac{\vartheta_1'(\gamma(\zeta+\lambda-\eta))}{\vartheta_1(\gamma(\zeta+\lambda-\eta))} ,
$$

(Anti-ferroelectric regime, $-\infty<\Delta<-1$)

NB: $(\Delta, t, z) \longrightarrow (\eta, \lambda, \zeta)$

Red curves: disordered regime [FC-Pronko'10]

Green curves: anti-ferroelectric regime [FC-Pronko-Zinn-Justin'10]



NB: $\Delta = 0 \Longrightarrow$ Arctic Circle

$\Delta = -\infty \Longrightarrow$ straight line

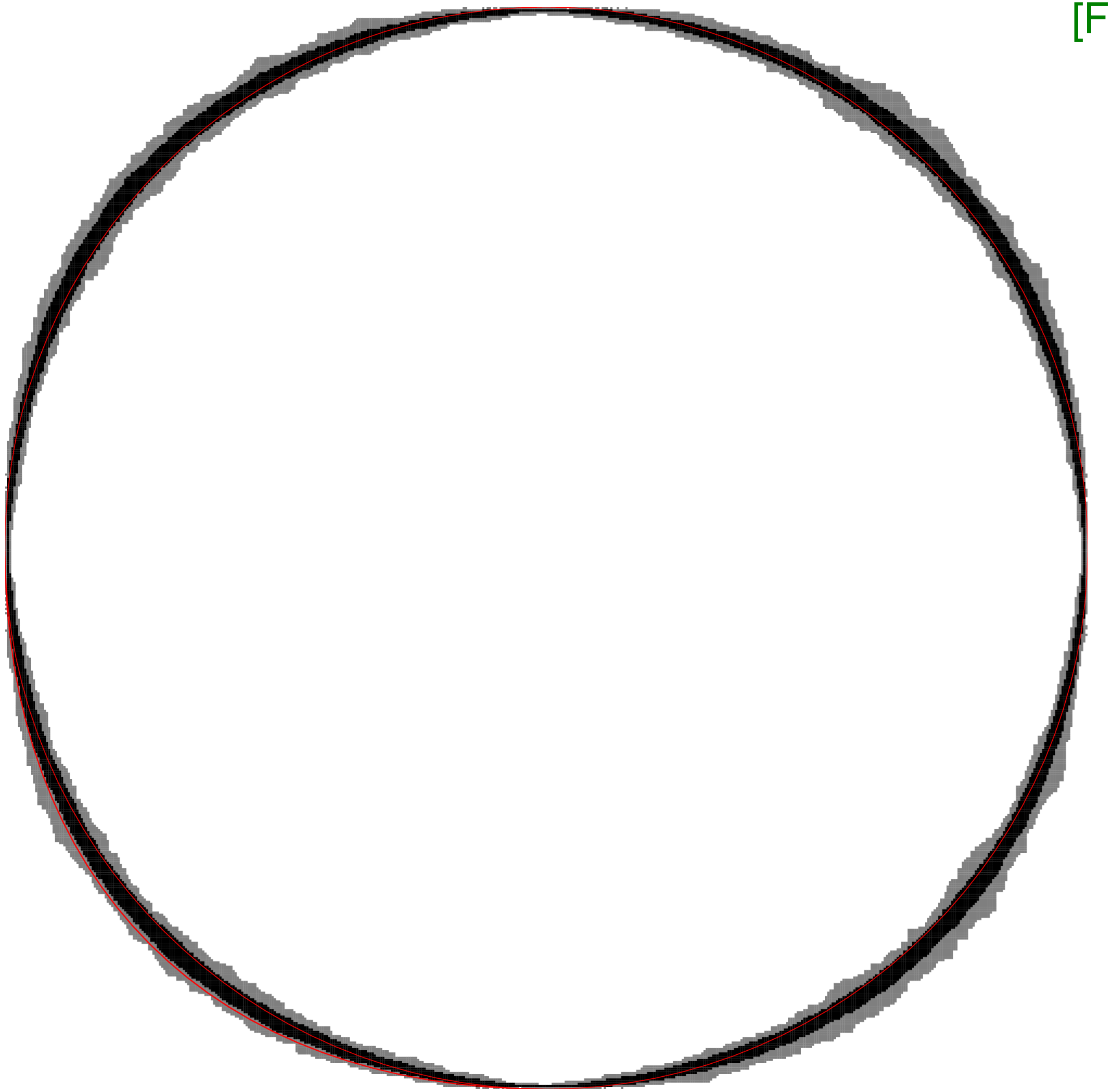
$$(2x-1)^2 + (2y-1)^2 - 4xy = 1, \qquad x, y \in \left[0, \tfrac{1}{2}\right].$$

ASMs: N=500

199 samples

$\Delta = 1/2$

[FC-Pronko'10]



Ben Wieland (January 2008)

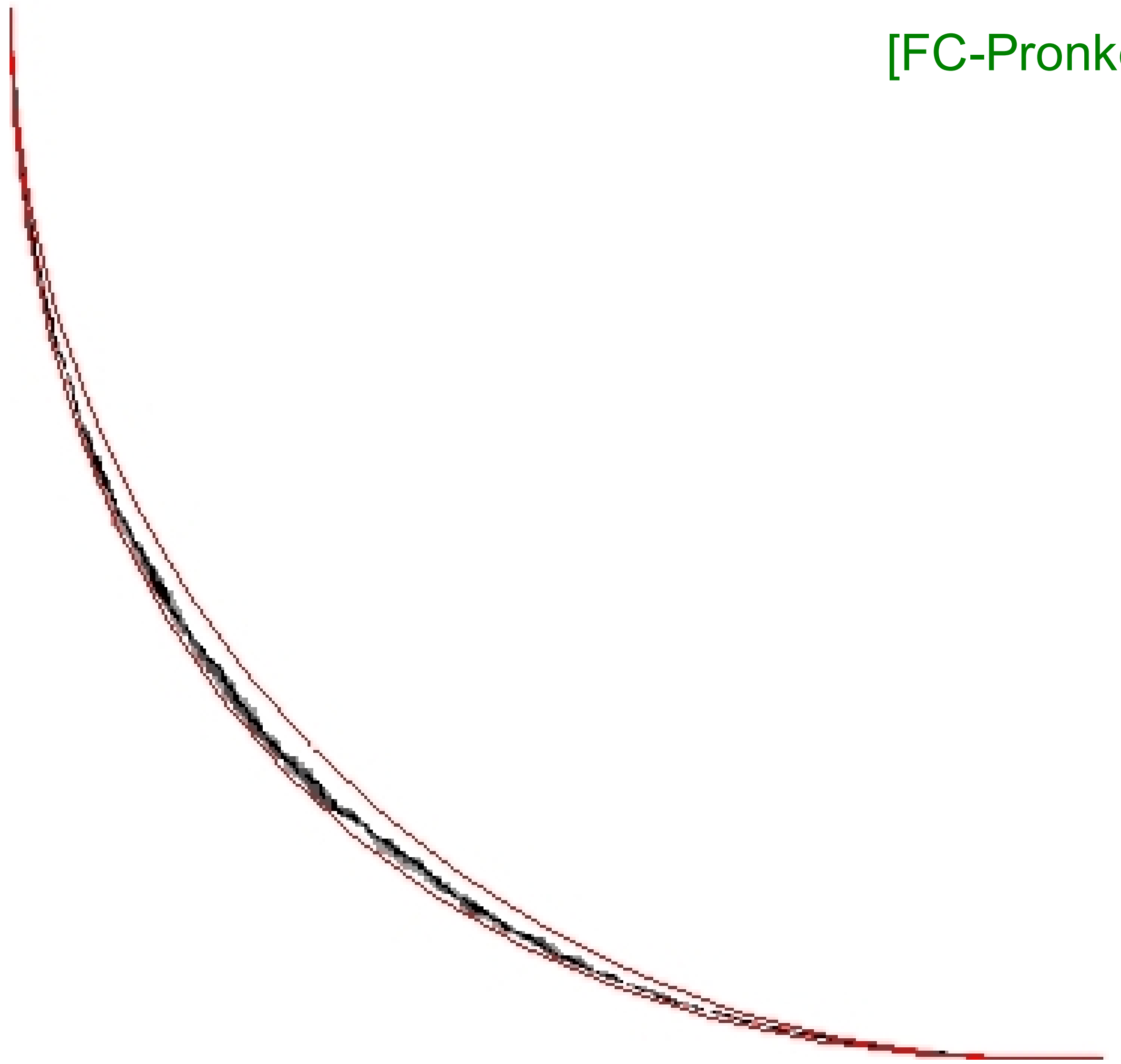
http://www.math.brown.edu/~wieland

ASMs: N=1500

10 samples

$\Delta = 1/2$

[FC-Pronko'10]



Ben Wieland

http://www.math.brown.edu/~wieland

Algebraic curves ($-1 < \Delta < 1$)

- When: $\gamma := \dfrac{\pi}{\pi - 2\eta} = \dfrac{n}{d}$ (n, d coprime)

  the Arctic curve is algebraic, with degree $m \leq 4d + 4n - 10$.

- When $\gamma := \dfrac{\pi}{\pi - 2\eta} = \dfrac{n}{2}$ and $\lambda = \dfrac{\pi}{2}$

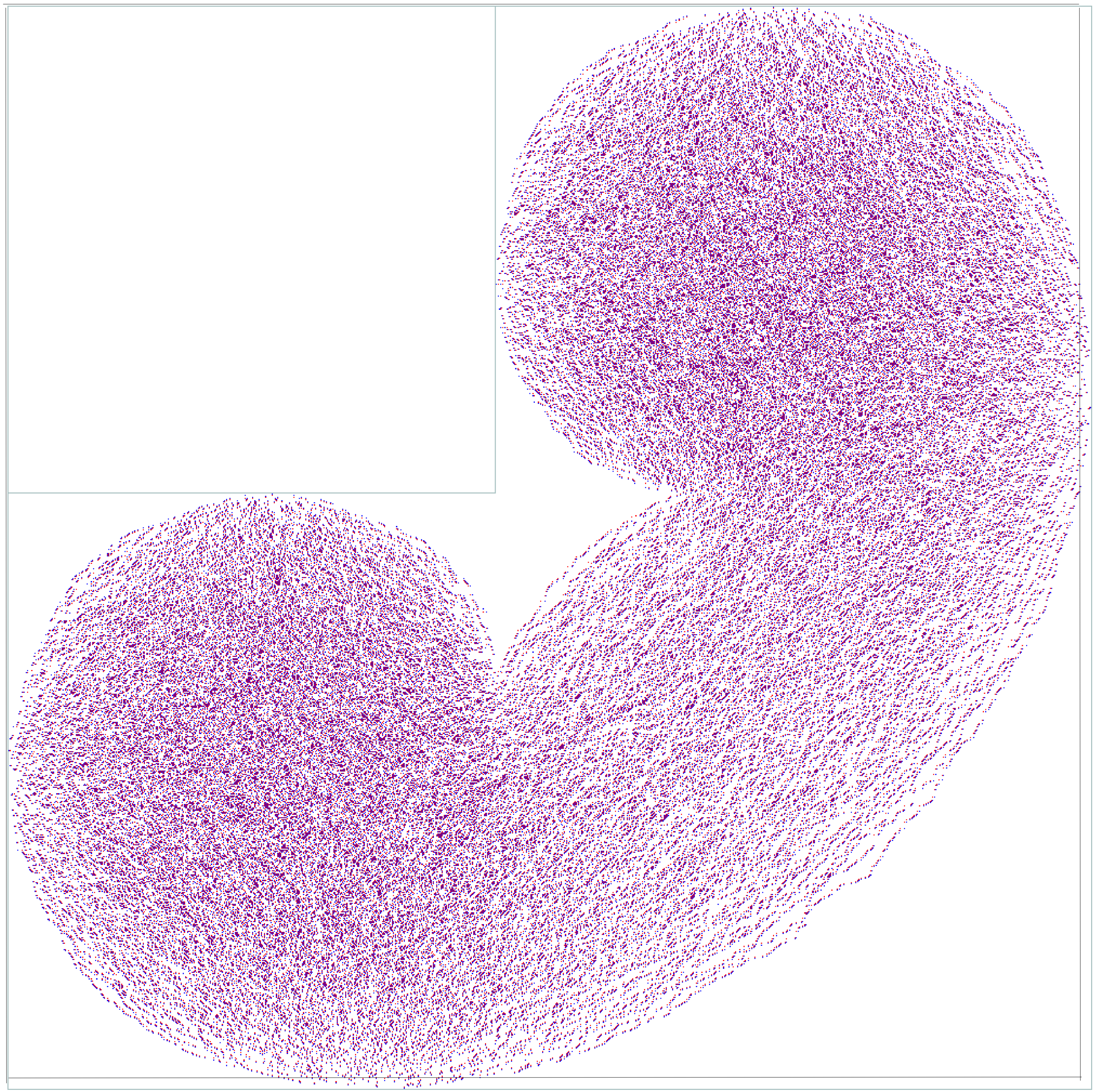
  the Arctic curve is algebraic, with degree $m \leq 4n - 10$.

Criticisms

- The present derivation of Arctic curves is based on an assumption (the `condensation hypothesis') which is rather bold and probably hard to prove.

6VM

$\Delta = \frac{1}{2}$



$N = 1000$
$r, s = 450$

Criticisms

- The present derivation of Arctic curves is based on an assumption (the `condensation hypothesis') which is rather bold and probably hard to prove.

- Moreover the whole procedure is rather `ad hoc' and probably it can not be extended to more general situations (e.g., generic regions of the lattice).

Six-vertex model with generic (fixed) BC?

[FC-Sportiello, *in prep.*]

Our previous result on the Arctic curve in a square domain can be rephrased as follows:

The arctic curve is the geometric caustic (envelope) of the family of straight lines:

$$-x\frac{1}{z}+y\frac{(t^2-2\Delta t+1)}{(z-1)(t^2 z-2\Delta t+1)}+\lim_{N\to\infty}\frac{1}{N}\partial_z \ln h_N(z)=0\,, \qquad z\in[1,+\infty)$$

<u>Questions:</u>

- What is the geometrical meaning of this family of straight line?
    - why the constant term is determined by the boundary correlator $h_N(z)$?
    - what determines the angular coefficient of these lines?

Understanding this would provide:

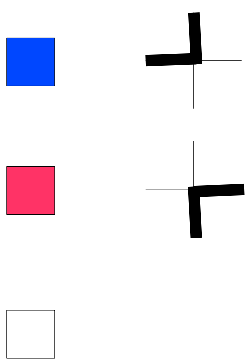
- an alternative (geometrical) derivation of the Arctic curve;
- a geometrical strategy to attack the problem of Arctic curves in generic domains.

Some numerical results

[FC-Sportiello, *in prep.*]

- From now on we restrict to the case of $\Delta = \frac{1}{2}$ and $t = 1$.
- Pictures are produced with a C code based on a version kindly provided by Ben Wieland, exploiting the `Coupling From The Past' algorithm [Propp-Wilson '96].
- We focus on ASMs restricted by the condition that they should have only 0's in a top-left rectangular region of size $r \times s$

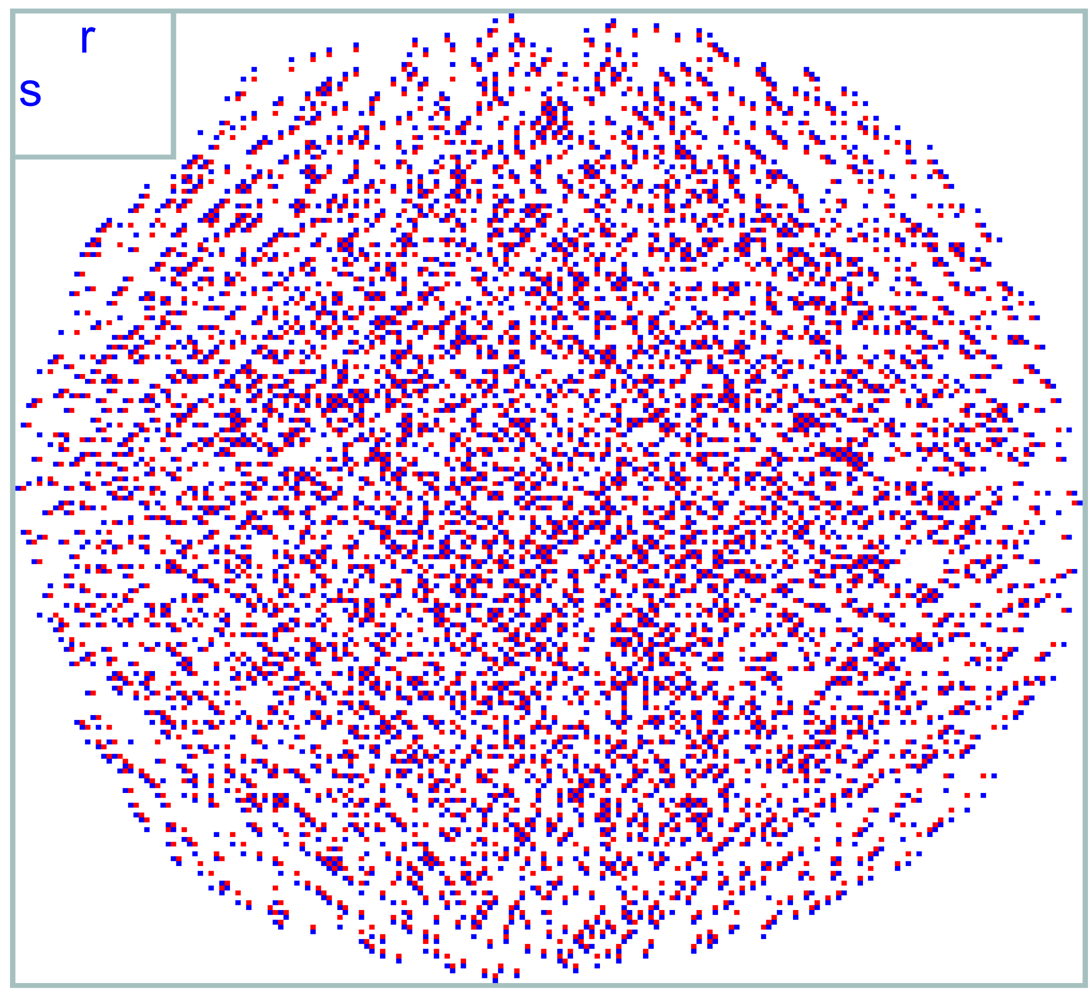
$$\Delta = \frac{1}{2}$$



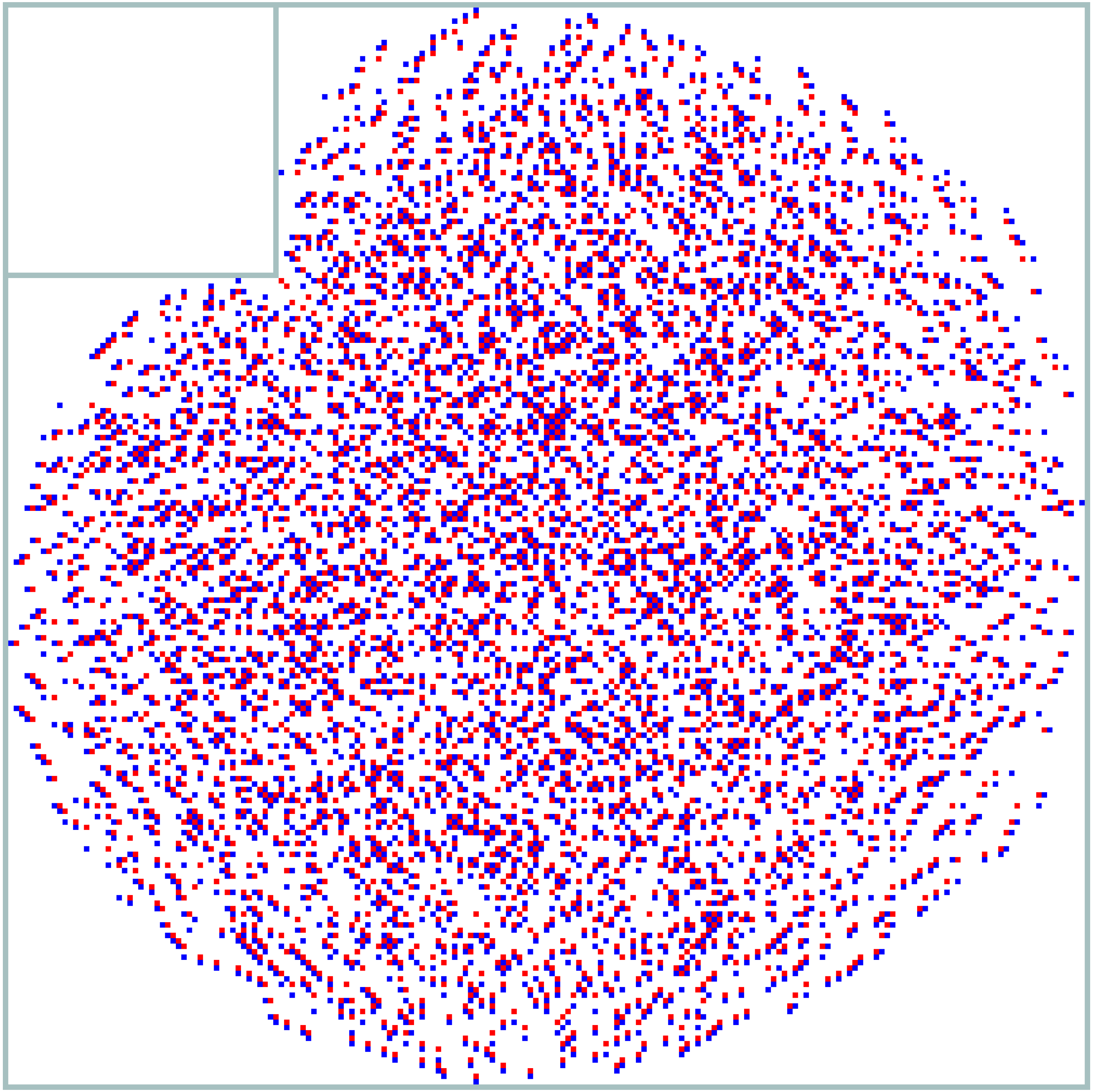
= c

= a, b

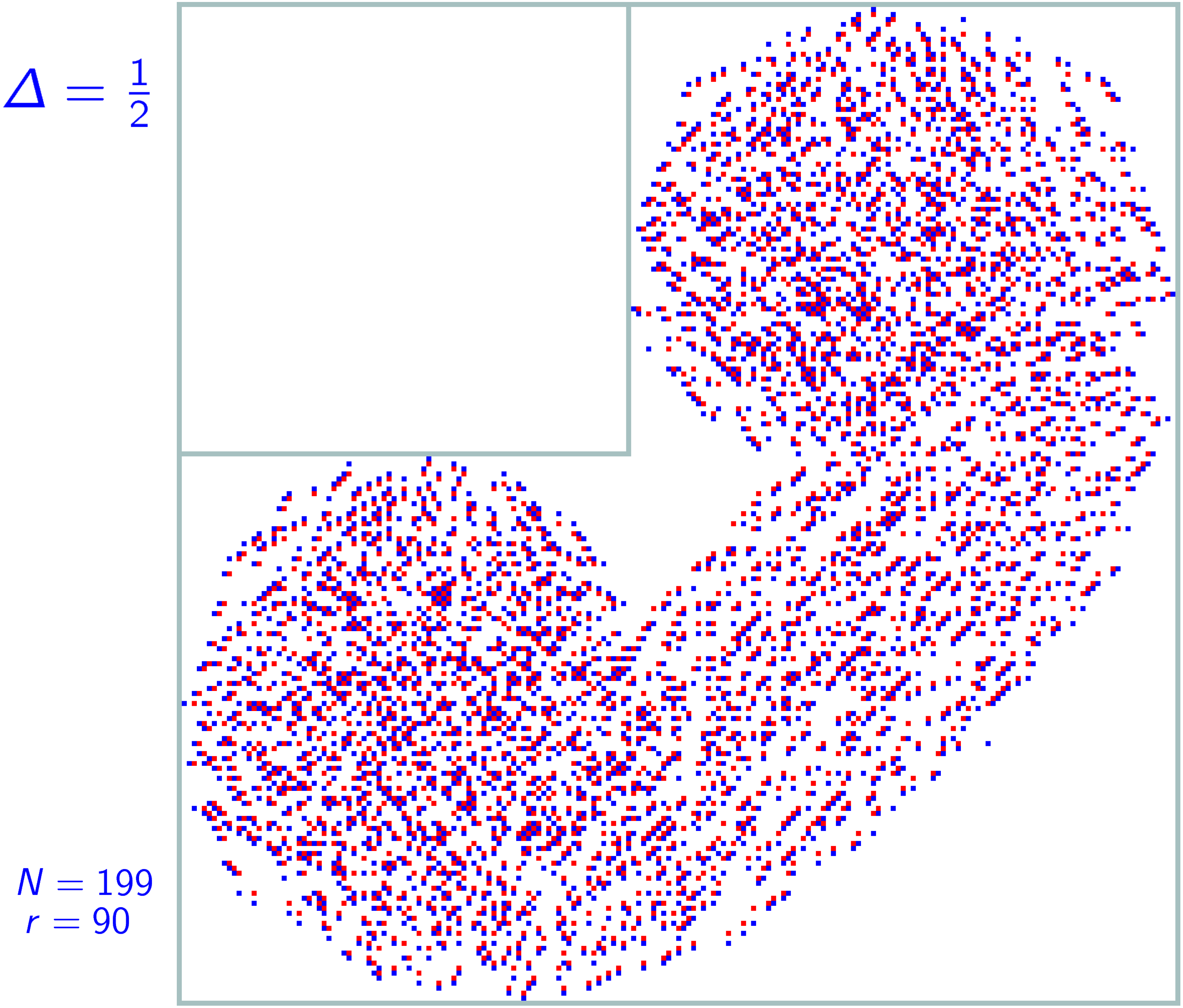
$$N = 199$$
$$r, s = 30$$

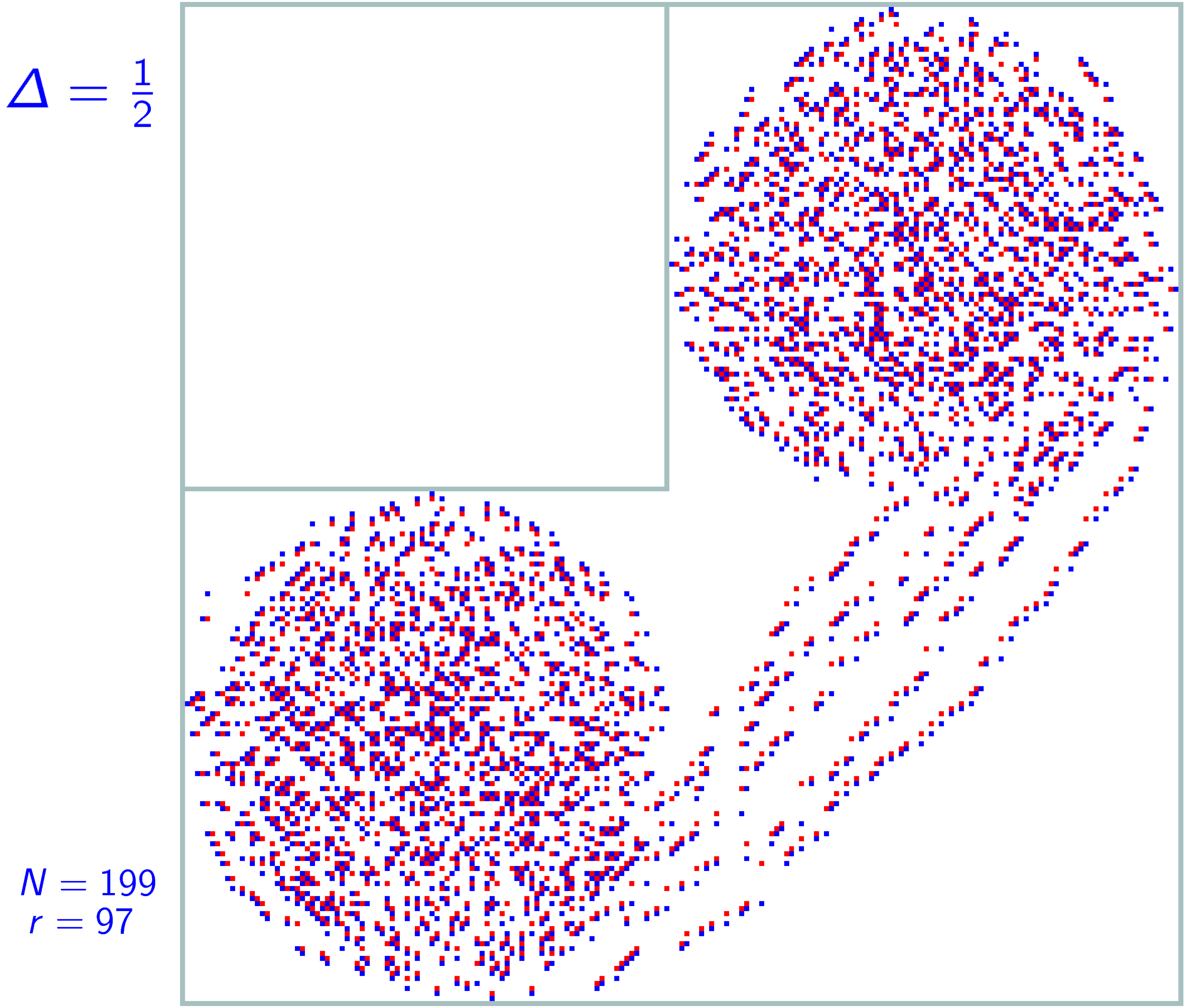


$\Delta = \frac{1}{2}$



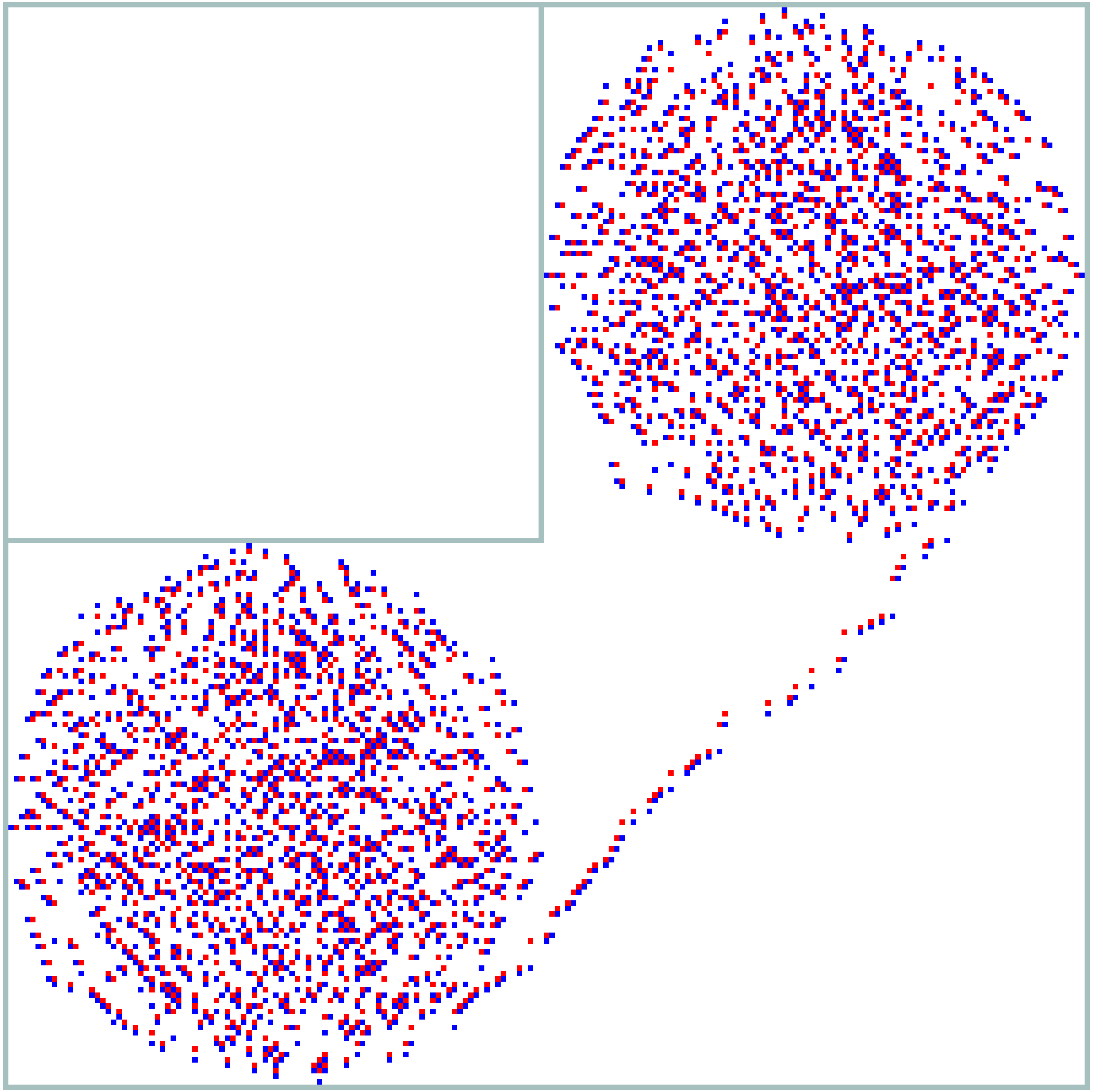
$N = 199$
$r = 50$


$\Delta = \frac{1}{2}$
$N = 199$
$r = 90$

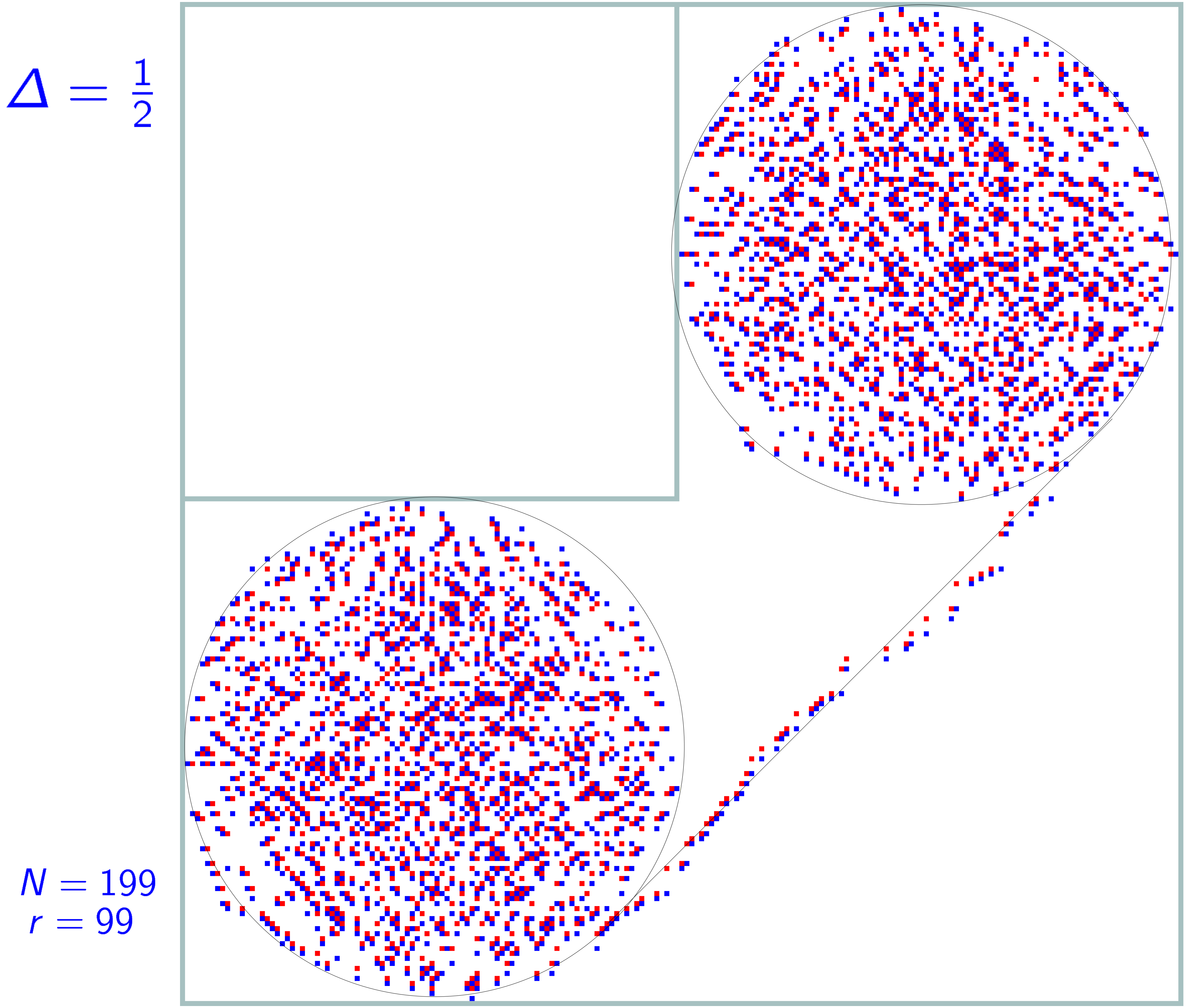


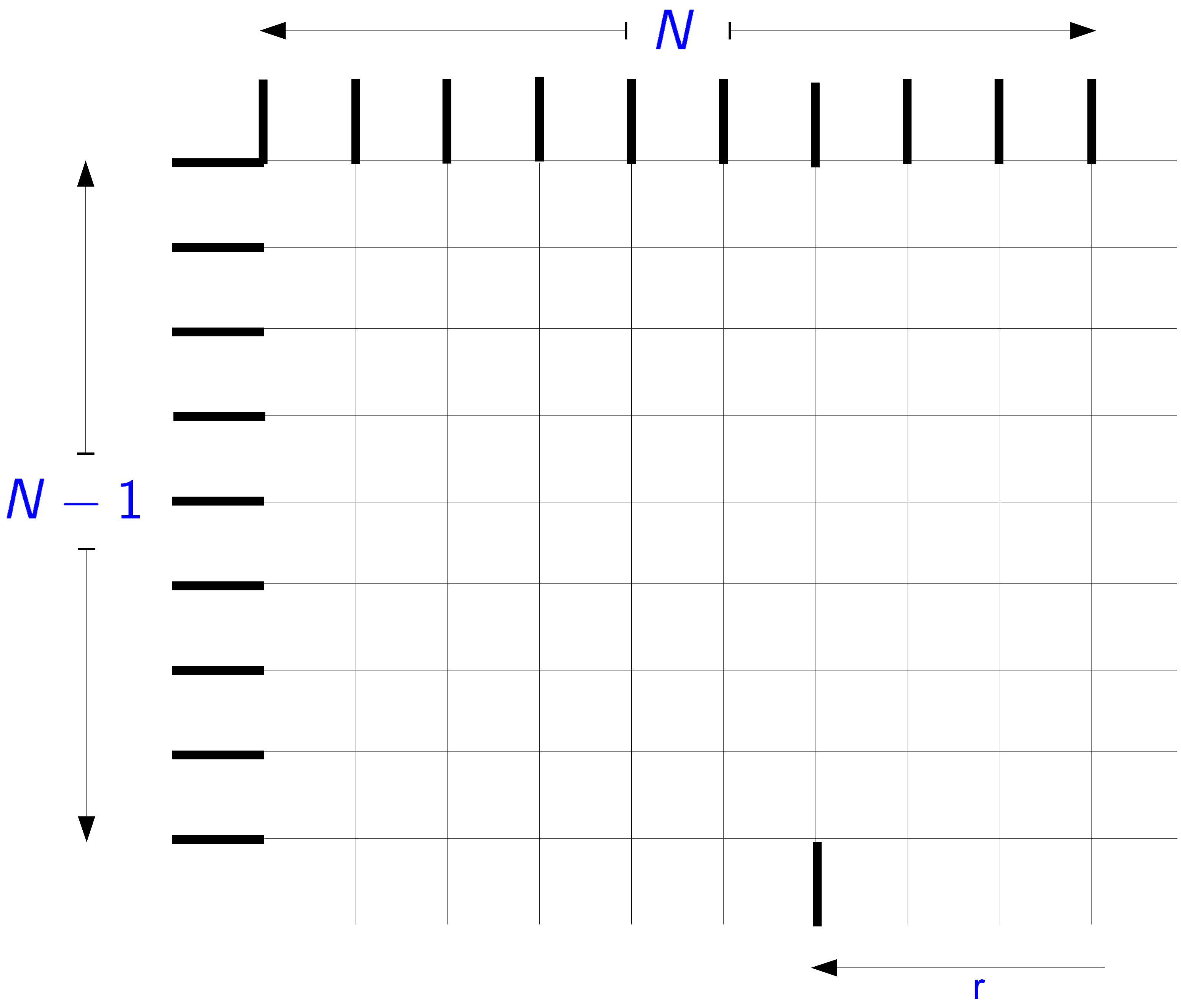
$\Delta = \frac{1}{2}$
$N = 199$
$r = 97$

$\Delta = \frac{1}{2}$



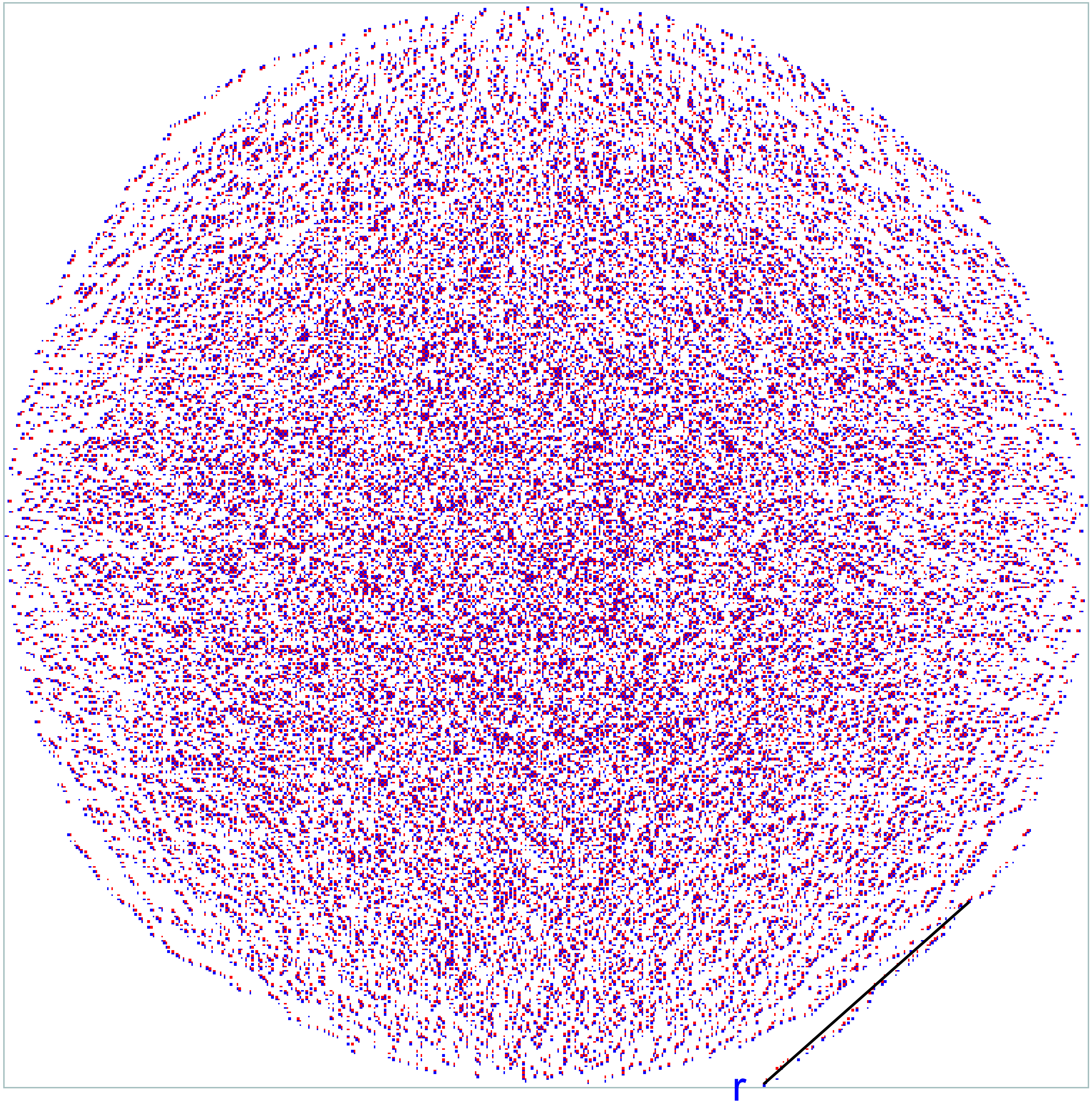
$N = 199$
$r = 99$


$\Delta = \frac{1}{2}$
$N = 199$
$r = 99$



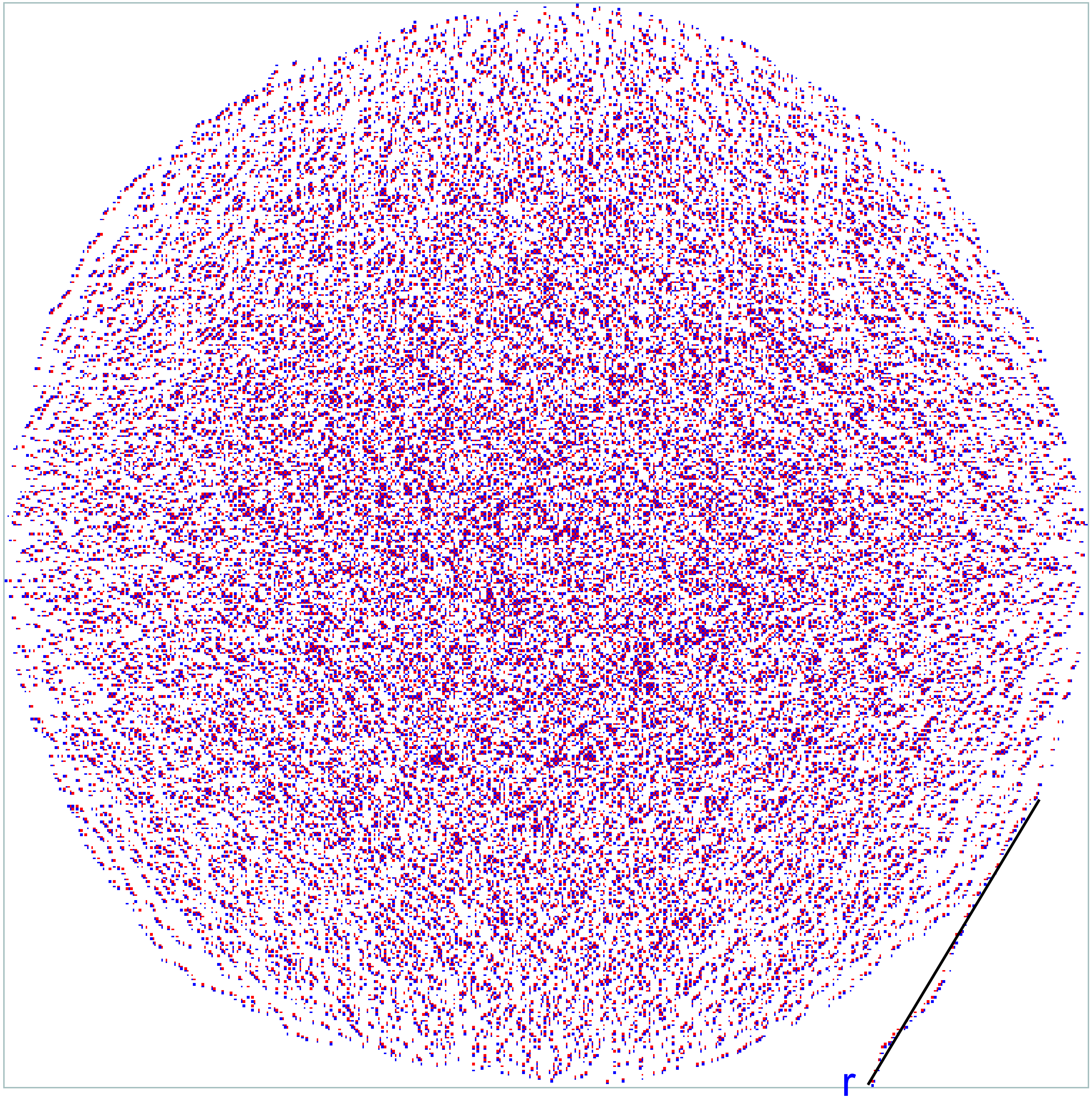
N
$N-1$
r

$\Delta = \frac{1}{2}$

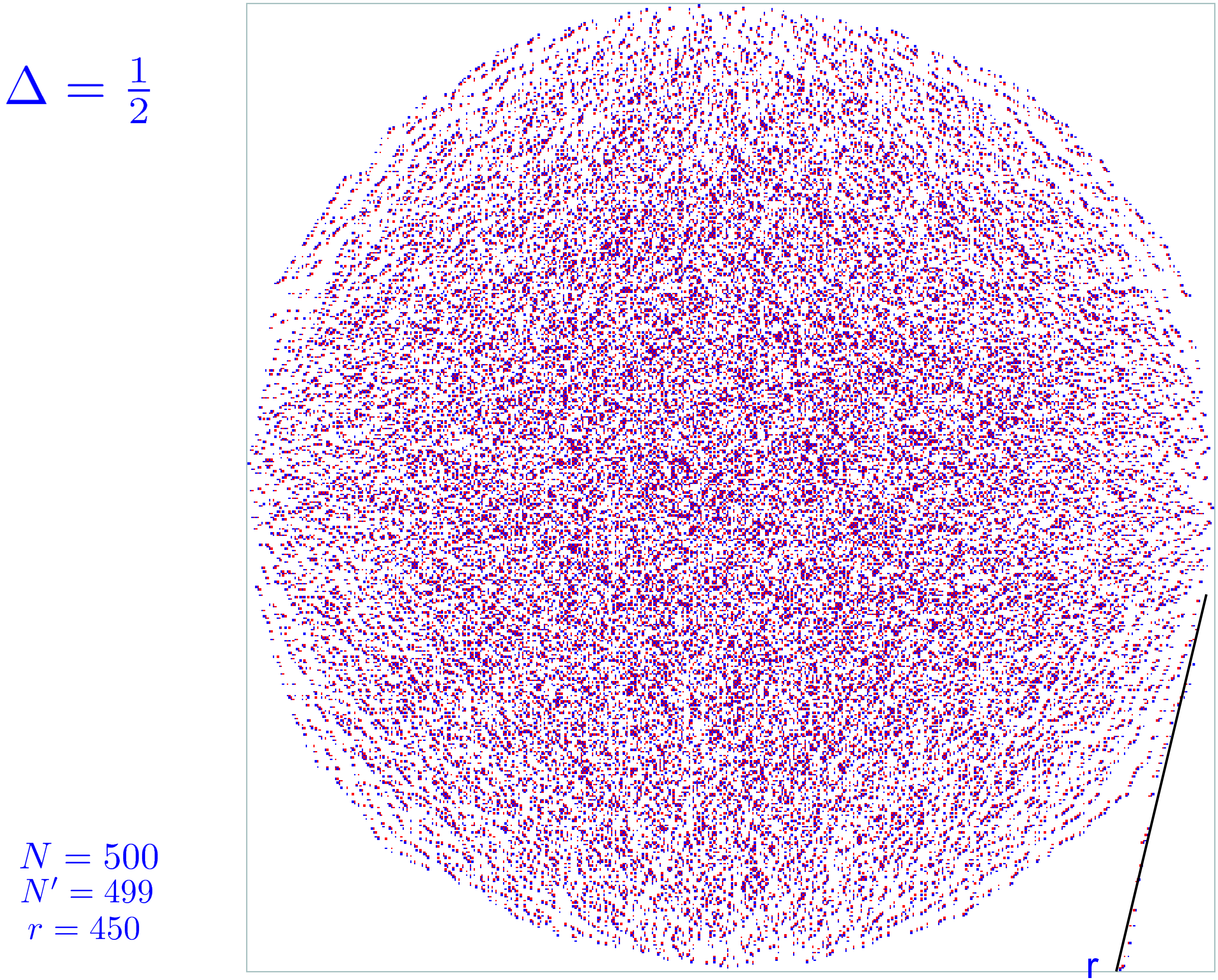


$N = 500$
$N' = 499$
$r = 350$

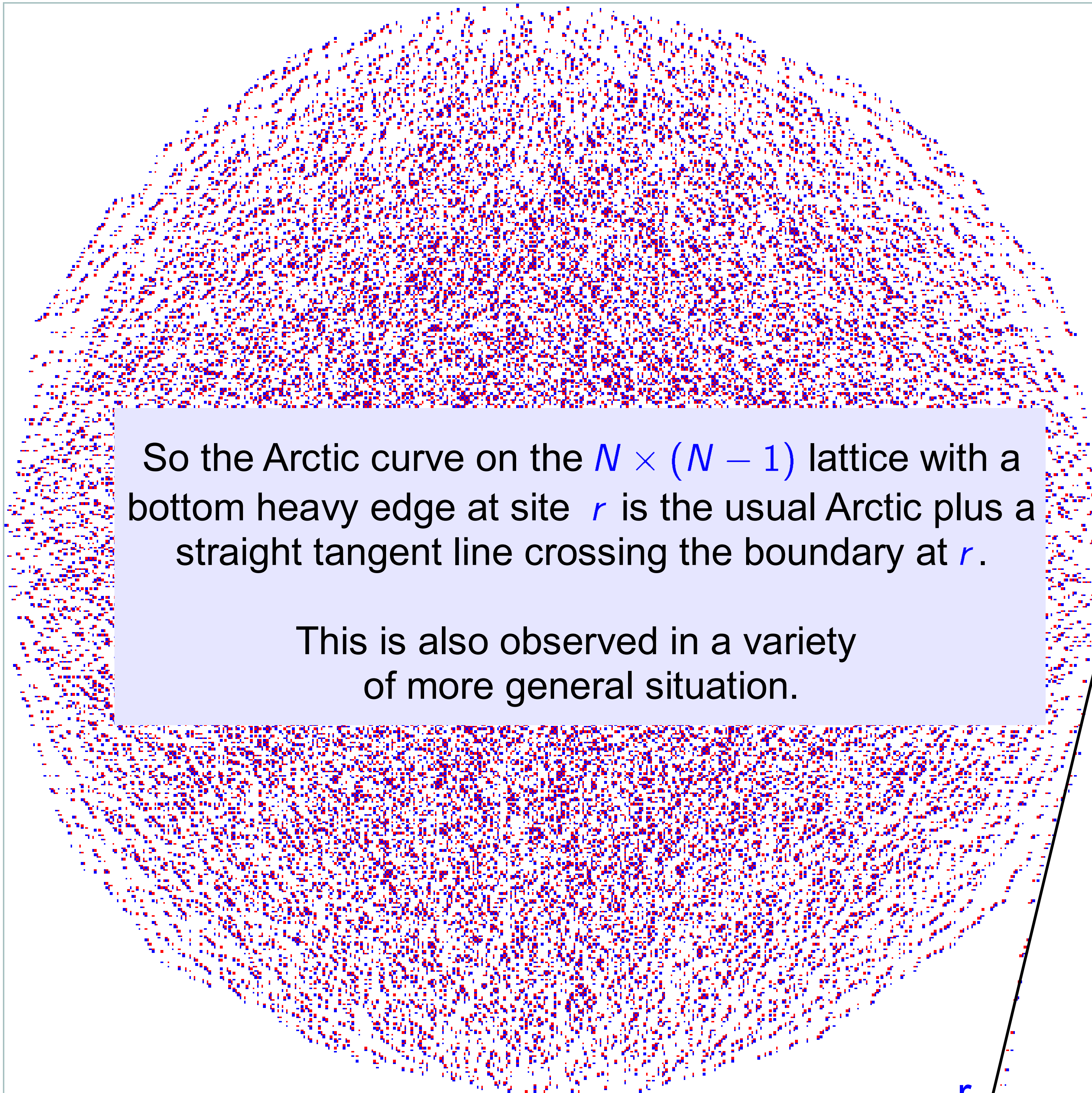
$\Delta = \frac{1}{2}$



$N = 500$
$N' = 499$
$r = 400$


$\Delta = \frac{1}{2}$
$N = 500$
$N' = 499$
$r = 450$
r

$$\Delta = \frac{1}{2}$$

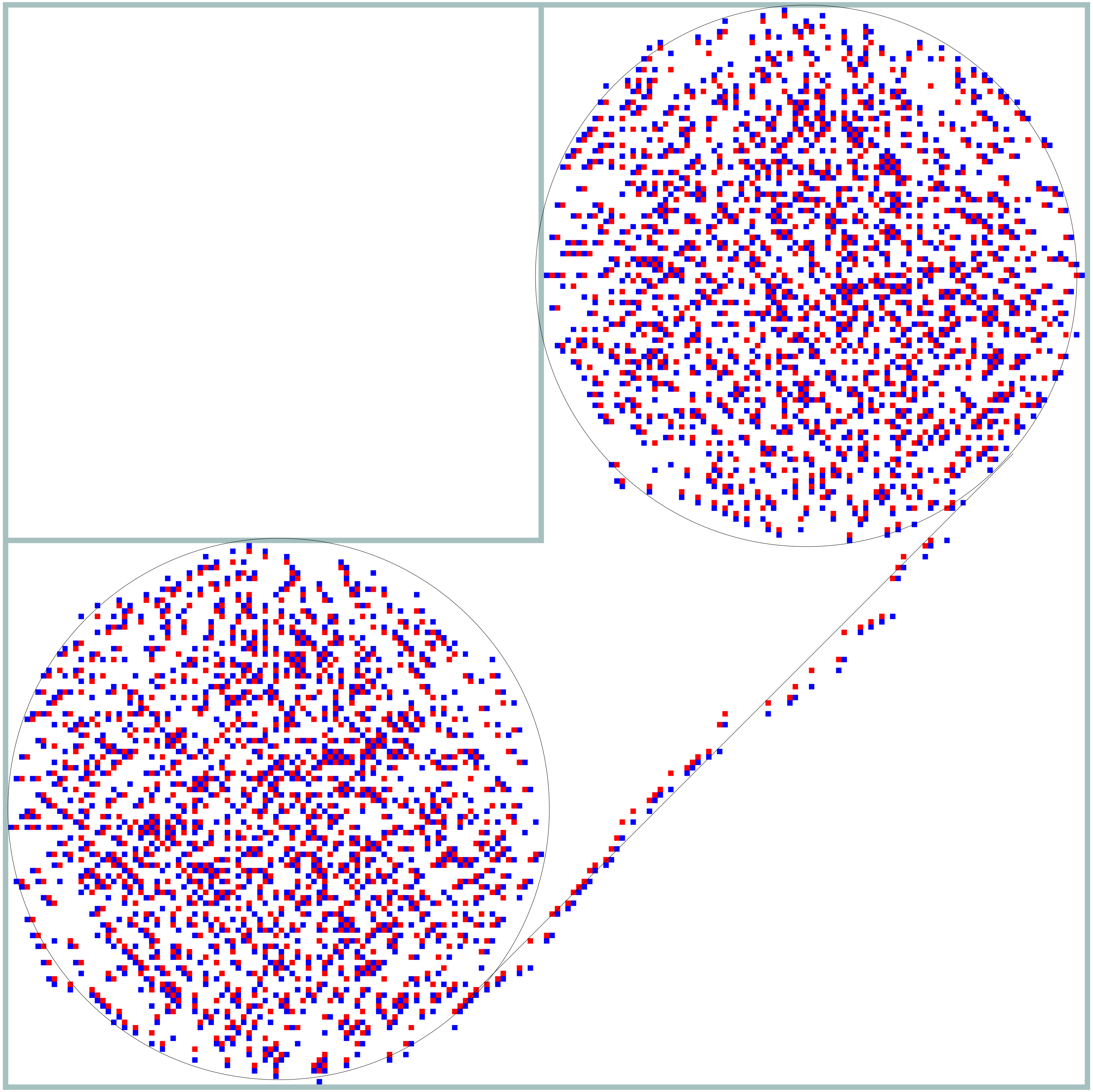


So the Arctic curve on the $N \times (N-1)$ lattice with a bottom heavy edge at site r is the usual Arctic plus a straight tangent line crossing the boundary at r.

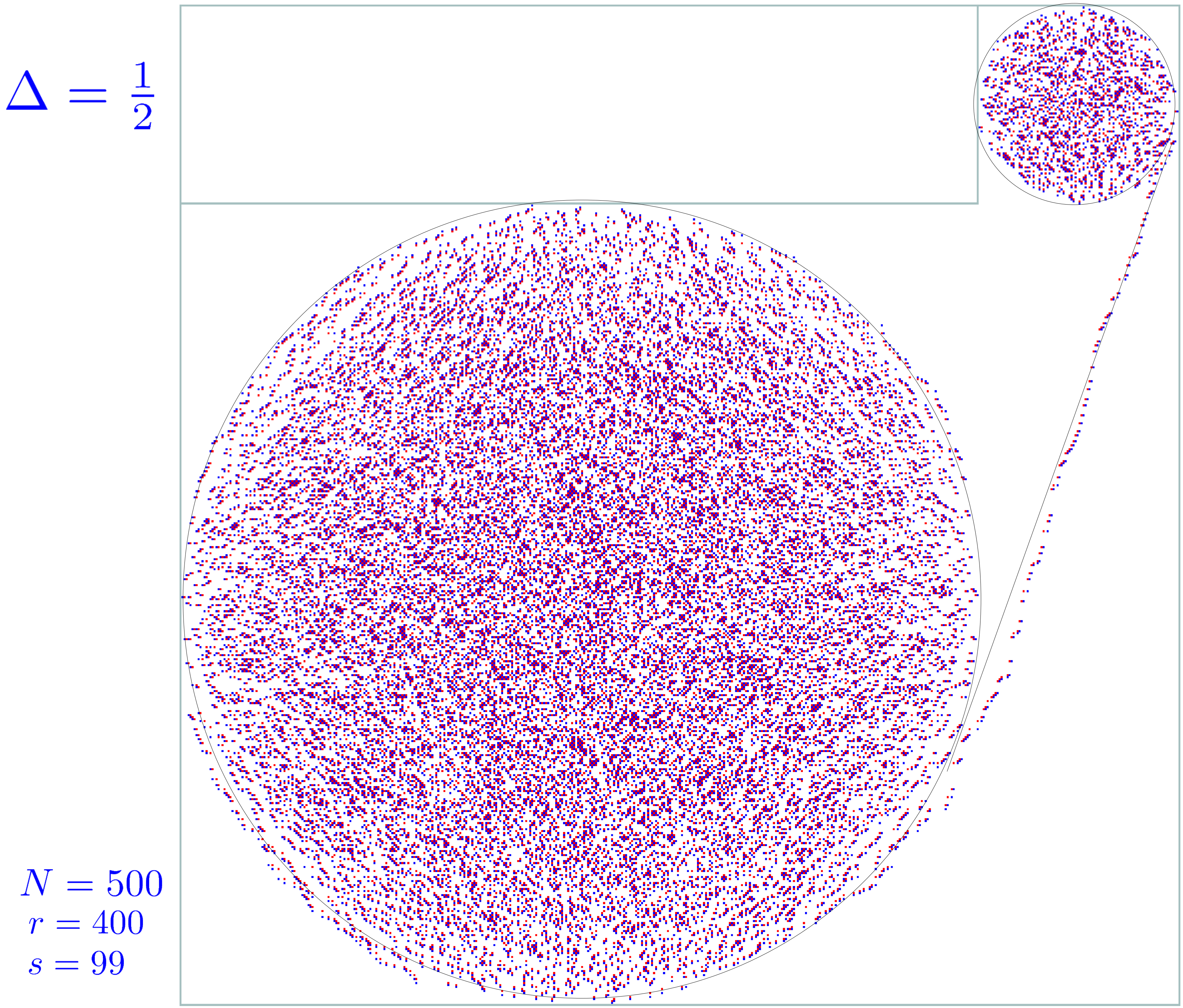
This is also observed in a variety of more general situation.

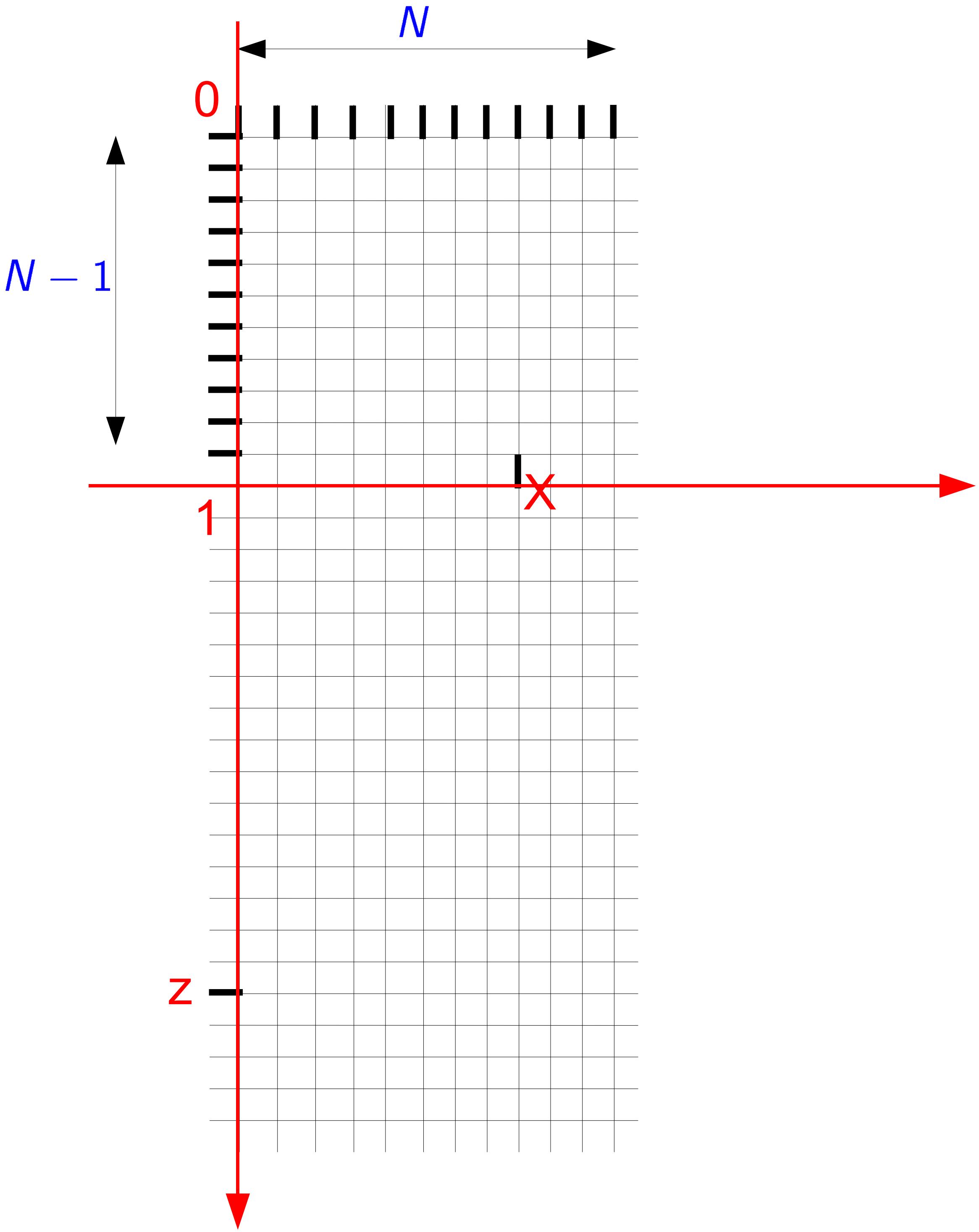
$N = 500$
$N' = 499$
$r = 450$

$\Delta = \frac{1}{2}$

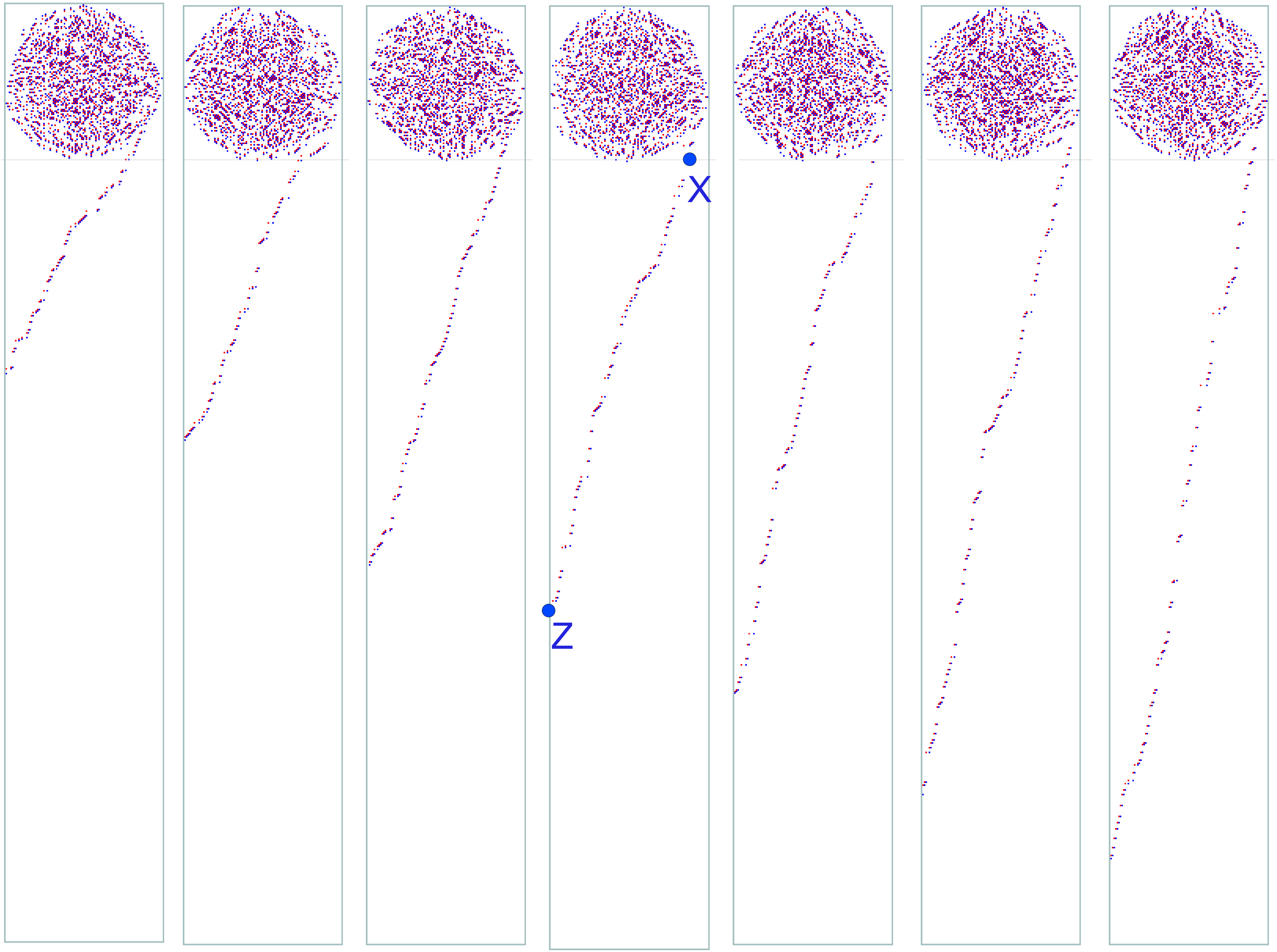


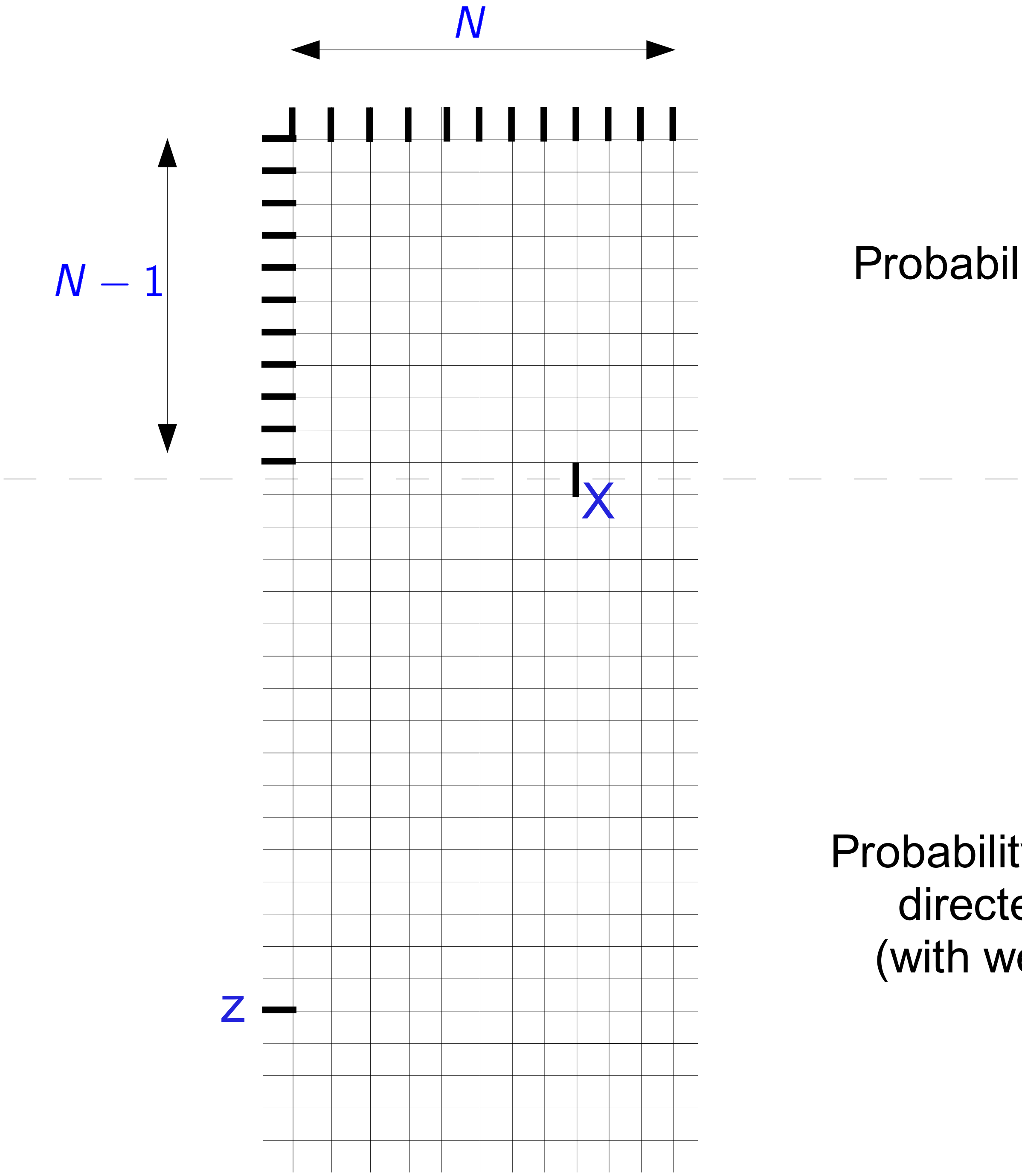
$N = 199$
$r = 99$


$\Delta = \frac{1}{2}$
$N = 500$
$r = 400$
$s = 99$



N
0
N − 1
1
x
z


X
Z



Probability $\propto$ $H_N^{(x)}$

Probability of having a weighted directed path from X to z (with weights given by 6VM)

Maximizing the above probability with respect to X , one obtains a family of straight lines, parameterized by z :

$$-x\frac{1}{z}+y\frac{(t^2-2\Delta t+1)}{(z-1)(t^2z-2\Delta t+1)}+\lim_{N\to\infty}\frac{1}{N}\partial_z \ln h_N(z)=0\,, \qquad z\in[1,+\infty)$$

which we immediately recognize! The point is that this `geometrical' construction interpretation holds for generic domains!

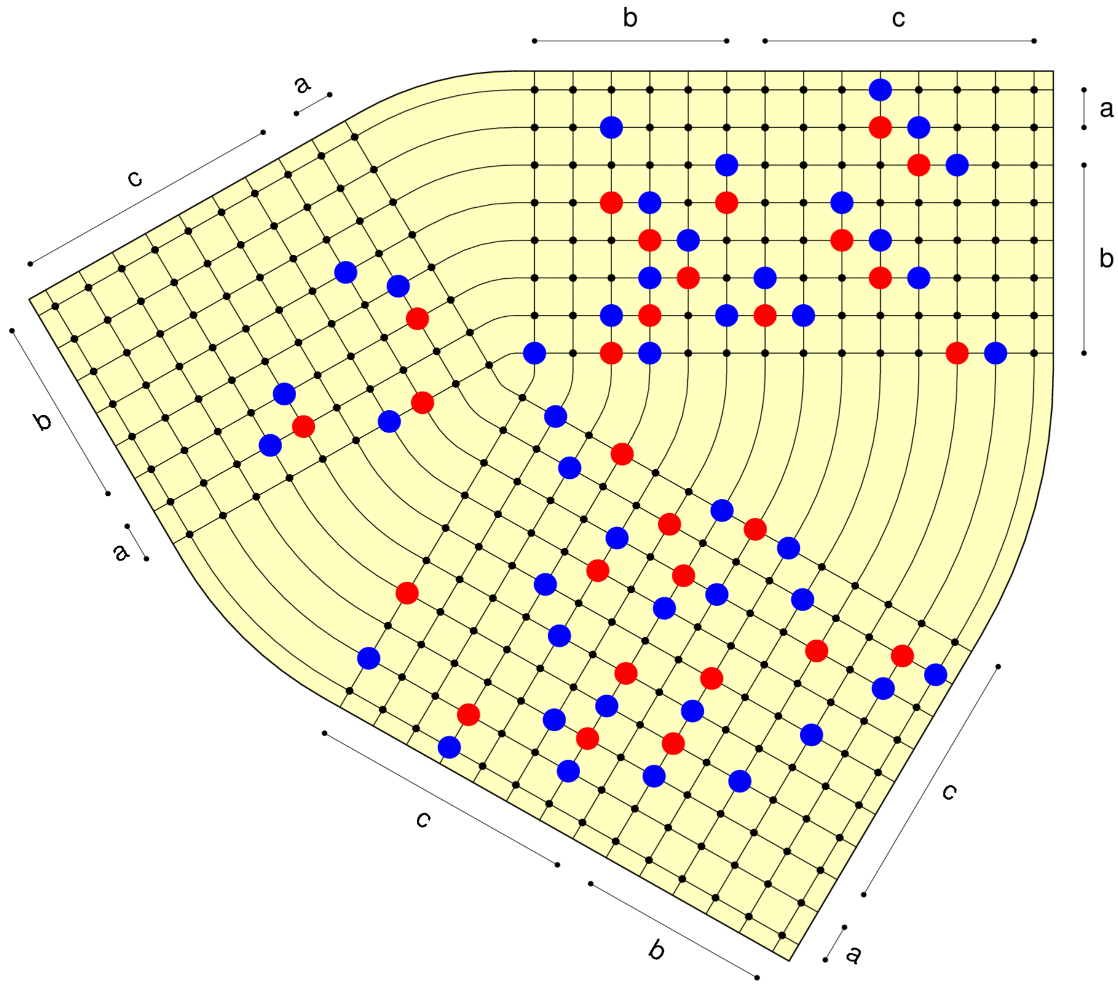
Thus on generic domains the problem of computing the Arctic curve is reduced to the evaluation of the (generating function of the) boundary correlation function, $h_N(z)$.

Does this really work?

- Checking our recipe in two cases where the boundary correlation function $h_N(z)$ is available, we have reproduced:
  - the Arctic curve of the DW 6VM for generic values of Δ and t
  - the Arctic circle of the rhombus tiling of an hexagon (use the formula for Semi-strict Gelfand patterns to evaluate the refined enumeration you need, see [Cohn-Larsen-Propp '98])

What about new results?
You need to know the boundary correlator!

Consider the six-vertex model at ASM point, on three bundles crossing each other:
[Cantini-Sportiello '11]



A corollary of the generalized R-S correspondence is that

$$A_{a,b,c} = A_{a+b+c} M_{a,b,c} \ ,$$

where $M_{a,b,c}$ counts rhombi tilings of the $a \times b \times c$ hexagon.

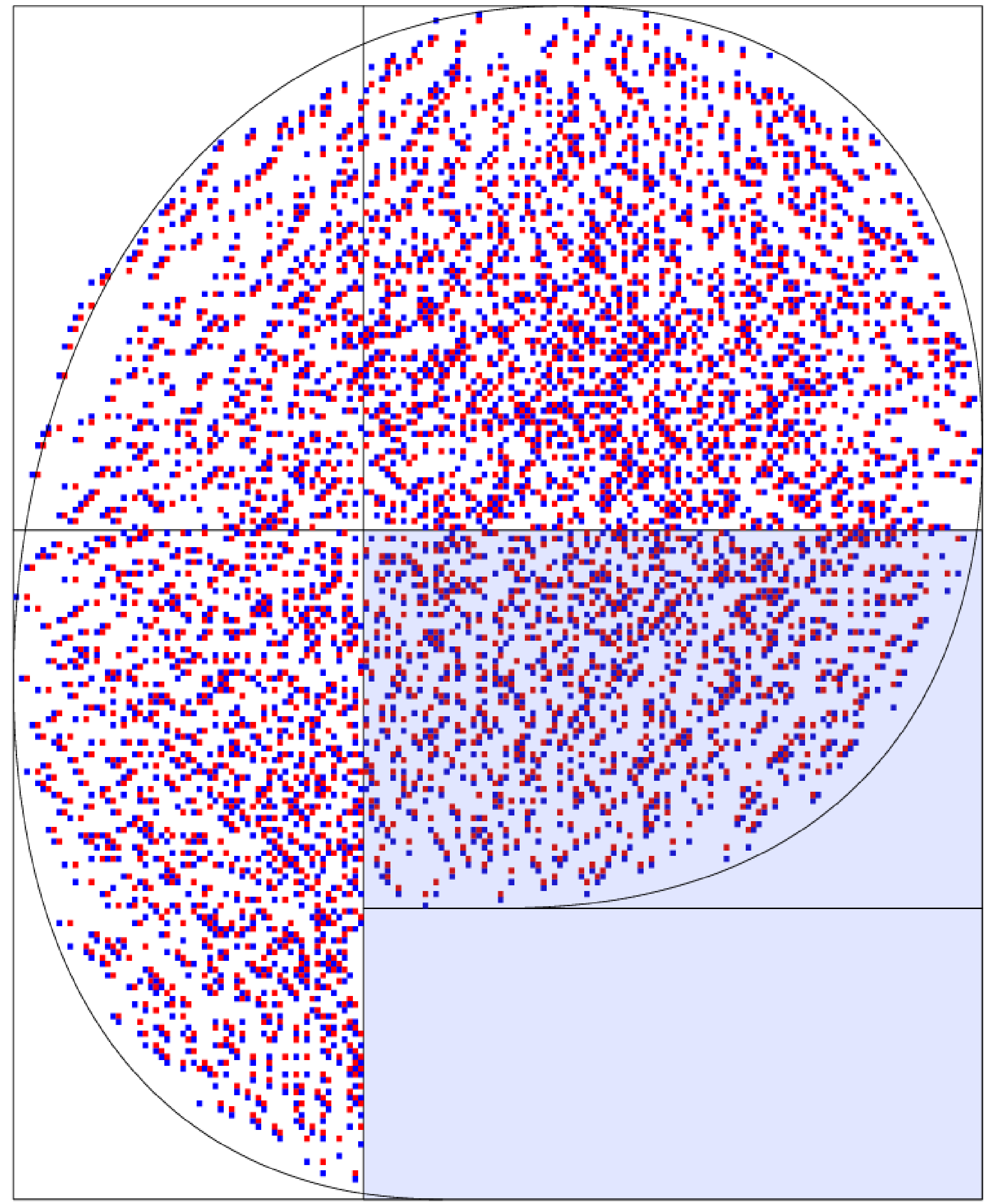
But more is true: $$A_{[a,b,c]}(r) = \sum_{r'} A_{a+b+c}(r - r') M_{a,b,c}(r')$$

[Cantini-Sportiello '12]:

$$x(b,c;p) = \frac{3-c}{2} - \frac{2-p}{2\sqrt{1-p+p^2}} - \frac{(1-c)(1-pb-pc+c)-2bpc}{2\sqrt{pb+pc-c)^2-2(pb-qc+c)+1}}$$

$$y(b,c;p) = x(c,b;1-p)$$

where $p \in [0,1]$ parametrizes the Arctic curve, and $a+b+c=1$



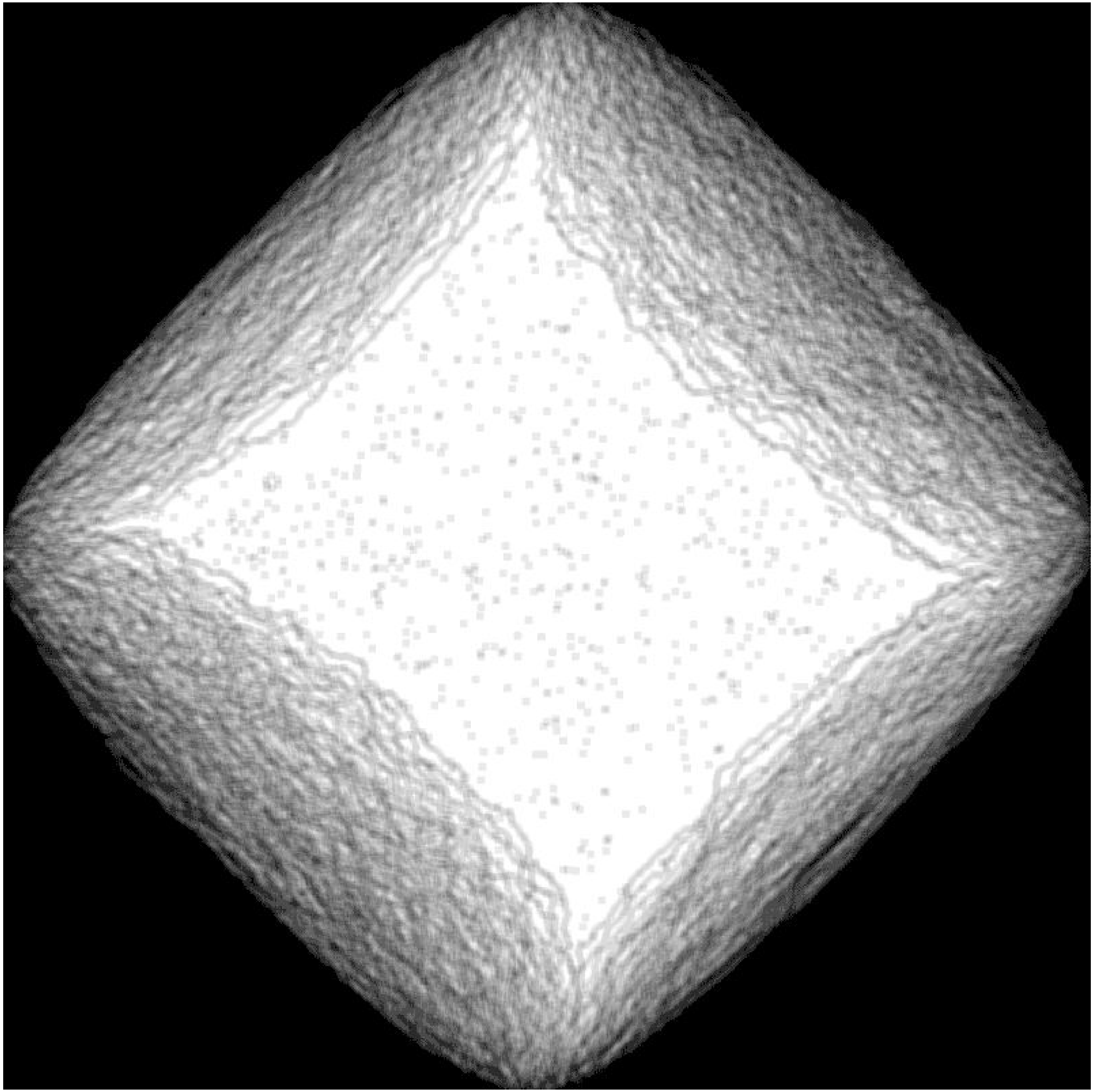
($a = 20,\ b = 45,\ c = 70$)

(Anti-ferroelectric regime)

$N = 1000$

$\Delta = -3$

$t = 0.5$



White pixels represents c-vertices

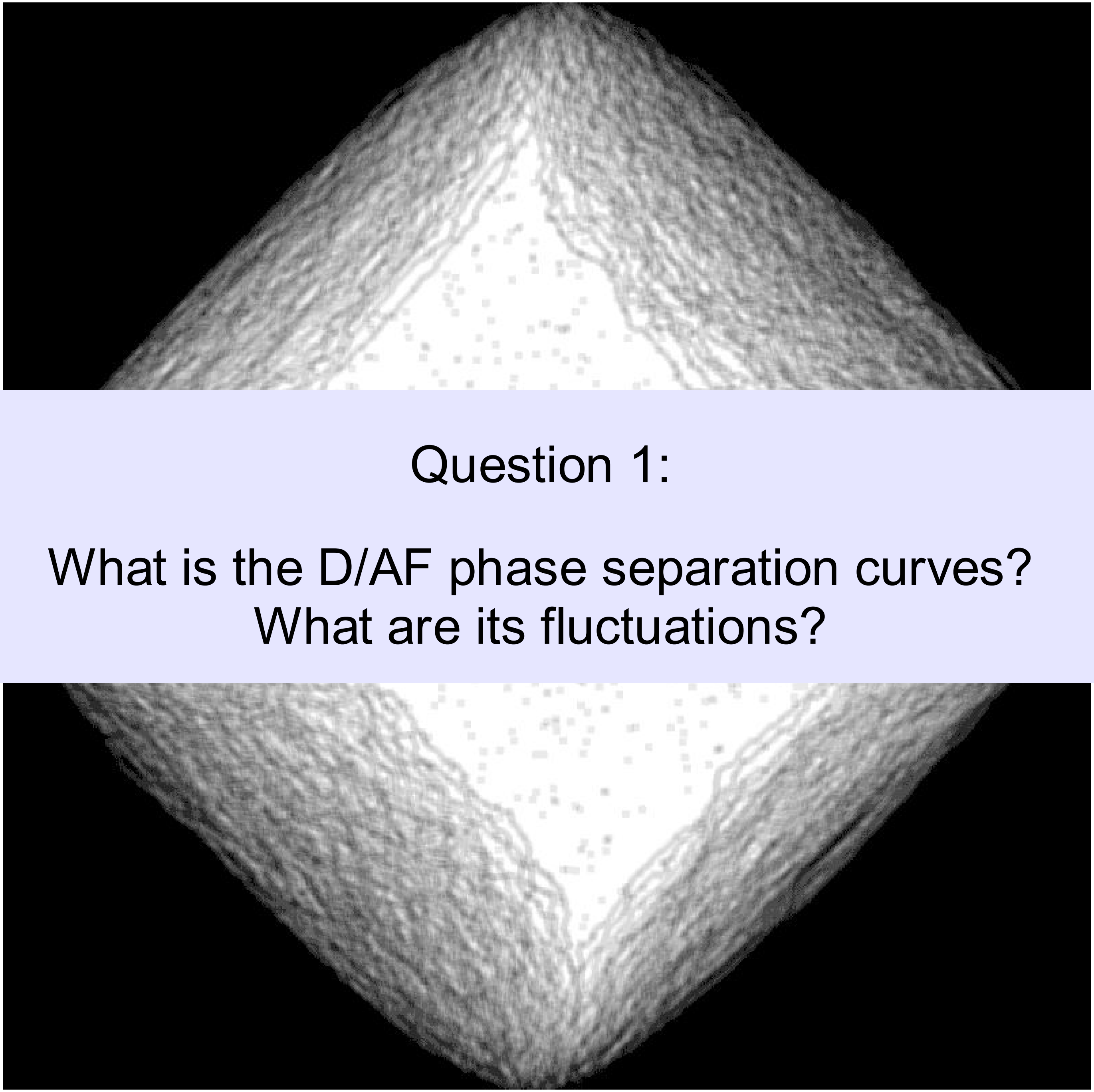
[Allison-Reshetikhin'05]

(Anti-ferroelectric regime)

$N = 1000$

$\Delta = -3$

$t = 0.5$



Question 1:

What is the D/AF phase separation curves?
What are its fluctuations?

White pixels represents c-vertices

[Allison-Reshetikhin'05]

Question 2: polarization

(one-point correlation function)