THE INTERACTION OF COLLINEAR GAPS OF ARBITRARY CHARGE IN A TWO DIMENSIONAL DIMER SYSTEM

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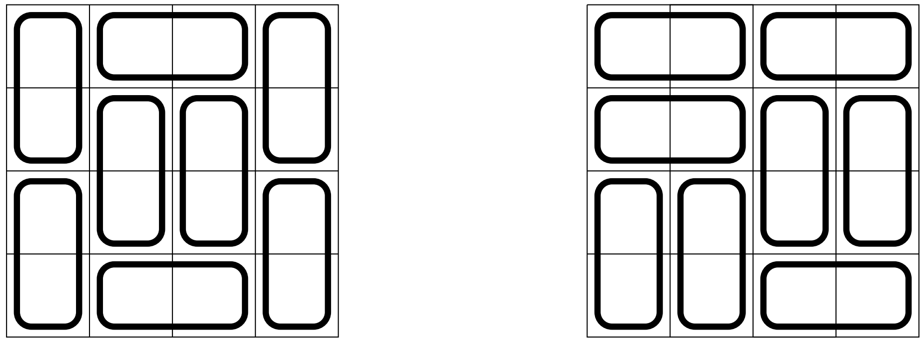


Domino tilings $\longleftrightarrow$ perfect matchings (dimer coverings)

$\mathrm{M}(G) :=$ # perfect matchings of graph G.

Let G_n:=$n \times n$ grid graph.

Dimer problem on square lattice (1936): $\mathrm{M}(G_{2n}) =?$

Theorem (Kasteleyn, and independently Temperley and Fisher, 1961).

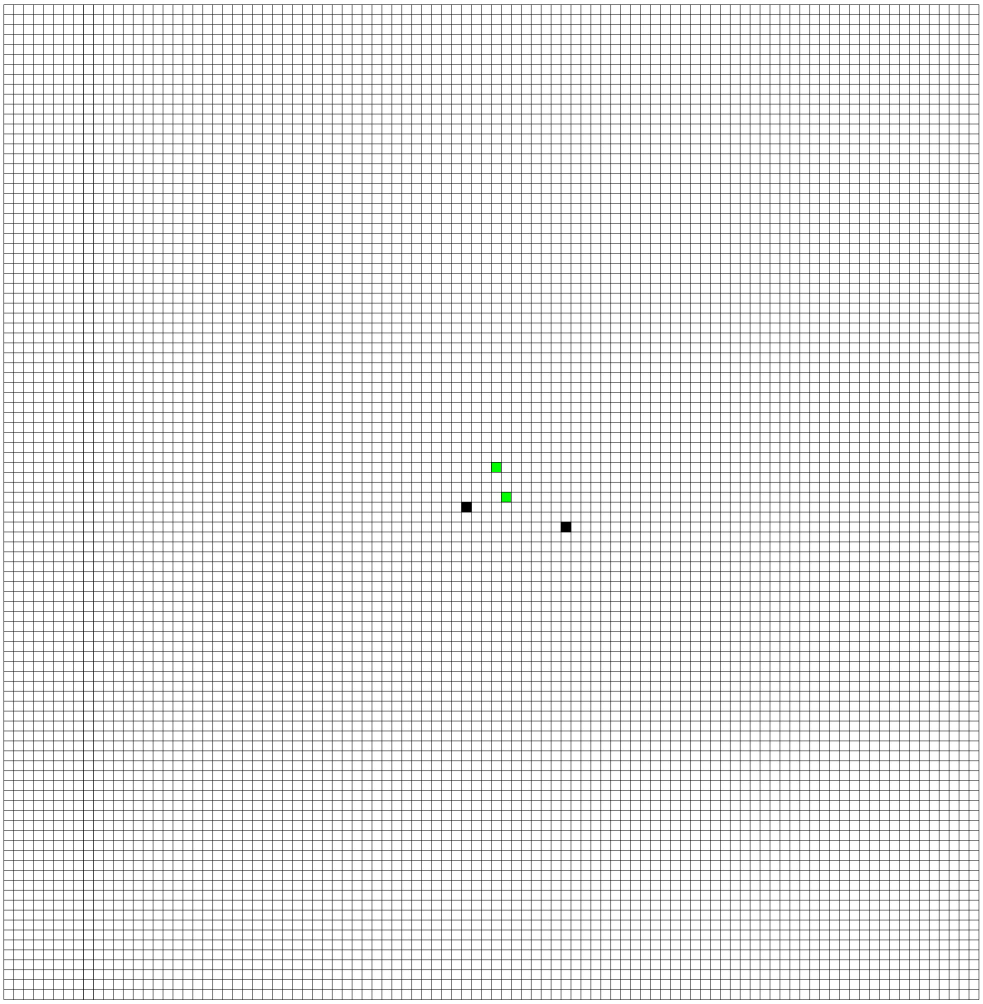
$$\mathrm{M}(G_{2n}) = 2^{2n^2} \prod_{j=1}^{n} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2n+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

- Pfaffian method
- permanent-determinant method

For any planar graph G:

$$\mathrm{M}(G) = \sqrt{|\det(A)|},$$

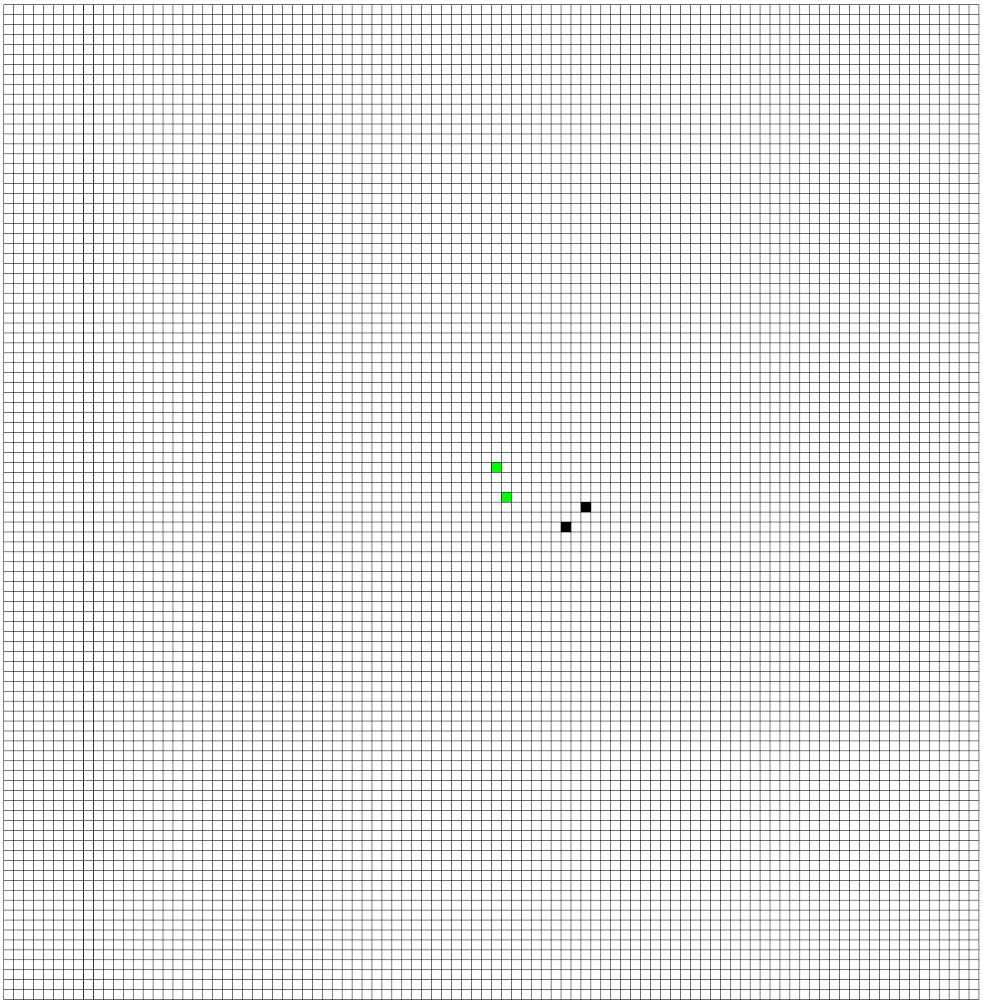
where A is a certain signed adjacency matrix of G.



A different direction: What if there are some defects in the tiling?

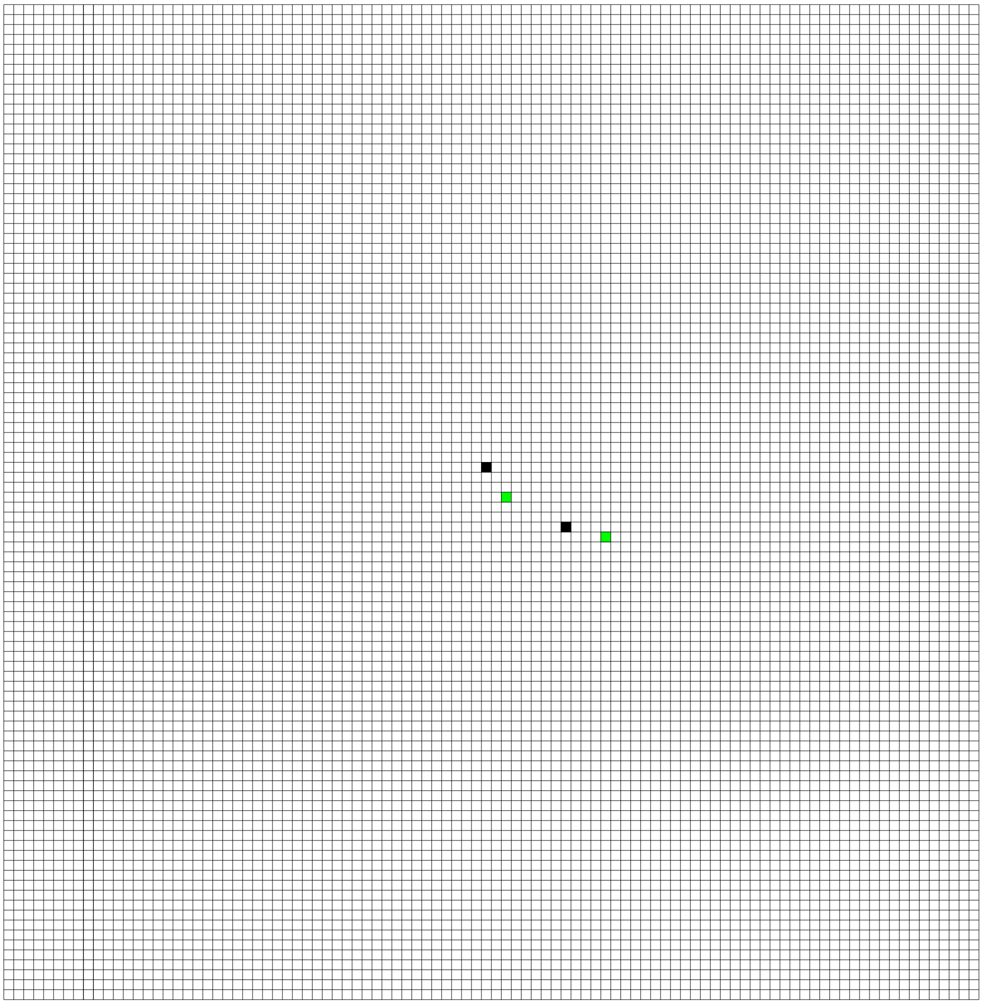
n_1=19950199996272315661984770563327145976410678066980942975683931965311961844841748608
894523528938454080130179249052794976495010594583960473633378239630712527177835744
189482141778269168228543141942319558891717399078701723213708196928952513374263998
117164861452452801924382982957169654674316949637751609391392621004772288679792
312792066862139707477334804155334433959899514513833374468070652065302156579613630
562991061618654620909318948952422534777606469524348070708512964772733035704226247
123222606549171050897000918136187456639991274537647923507774745890228369644547318
073077792539151059864111745754731999865374258377920247412989199782024592842468
937002277850500852009559446559779474711160176031826435896998510098408151547481356
373444675681673860309049201465508347137727850574825609568254005530036540708685967
927911922863297496751149243683692618324319701324003318547203471906199436264665
438277561701005395075267947181216251536248670917247614664641917728407750249657909
052234890474509154621482581697747299981147519365222430355219805987434582597821494
905295563747423924178694950691214936593214465047356148726662618683461355988814
970887578122197145505105654378454095419217180040086740397224762806809233907947034
73089946643748412788914127462247339901742743552

domino tilings



The holes moved to a different position

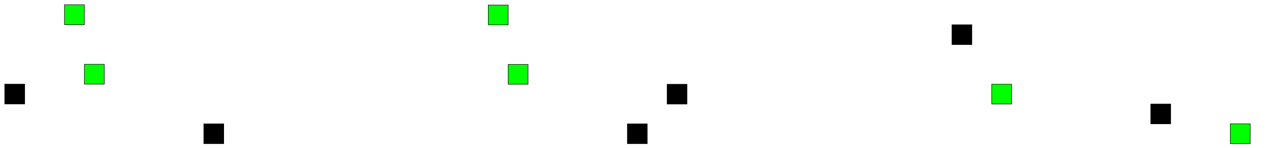
n_2=52054920244179804888541799941423481808863270728531286193262829564535357881268246259318756284145653836214143019241852759728510512638905530223465268593302400067861202138703972991739933634367597215642740974834020708804819939088282040613863071054353463094512887663380827873999043728067749170608781657796180627708598562946147666459938803059496477177093307934427459635509457428302106694800688576354394074425602753743952234939644932176375470497533180993158694050337001987944889368830610457424170662707482598964386600699594636395985771418383780307136973377625723386586291608813851346146364741448714050827831732464752388760580270876837899212933766361991198005523815751268153056750287154452238830536824397014364722396733197560882183254014864642846141562885801613303446289968013097575726607149675926264216190050056494143875687913625554688743767923064624093138575106295137100567061033552448441313892553844173848817581922732775960334370021369305500071001790293821764685666255290515695202816018470062663633526077692382350974462231097859919789947112969957203275875481974549167053629867342792738627485259443876430527323614878352885842337454431481767535117651040017165512226658899391803558920647629138916714494415345332722142240845634713997919976780848477275059493273 6 domino tilings



A third position of the holes

n_3=3582032307659598301991035747163652601751913832216783289928417897994251900018719048396778136606572013789985115631226239734994820050795335236574036448238094205389591670073902505355803749345136270326536701697857089990145143319113349802434637768617969996564809446932208084847341633135132080509823292047455096612696356799381307027801974698154622865337929646554245083198723052324323669571326167772739872971155411158924797111834992719284069258569098893857042302895310152128758309321273297680327465594098275583110946225161796130763415573026182599254152164996970809364185921423264134433207989683057552951664060698553658103926073750272093329764961785243647637133925483392211694802644221978636100310774728662362030650488654342950071317317975102753156978000169822449095400781839355672033344866308556877691313152058384988433262360963894302884247831373898682479299732211412258385618219651498816752184053499545624097379396586199158686128049762799144185450318939186951940318372200052448613515038787000254405126568030292921194665960995449237498051024456194251523884384673511776171946011169282404871012033534403660764781631771310057407794033999749871363943126354908406155636699537379894930293305164271339204090476609636885352684881353860297709583451595696176773673779200

domino tilings



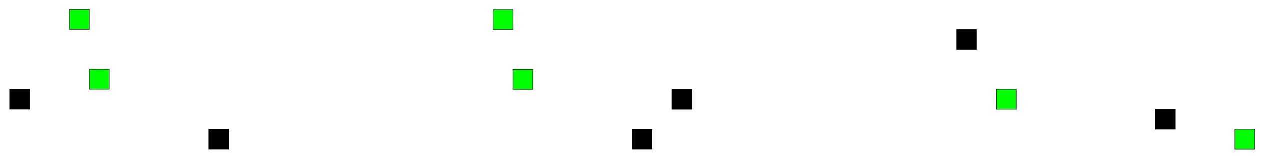
Relative sizes:

$$\frac{n_1}{n_2} = 3.832529164...$$

$$\frac{n_1}{n_3} = 0.556951983...$$

Can we understand *why* these ratios come out as they do?

Can we predict the ratios for other arrangements?



Positions of the green unit squares: $w_1, w_2 \in \mathbb{C}$

black unit squares: $b_1, b_2 \in \mathbb{C}$

Electrostatic Energy:

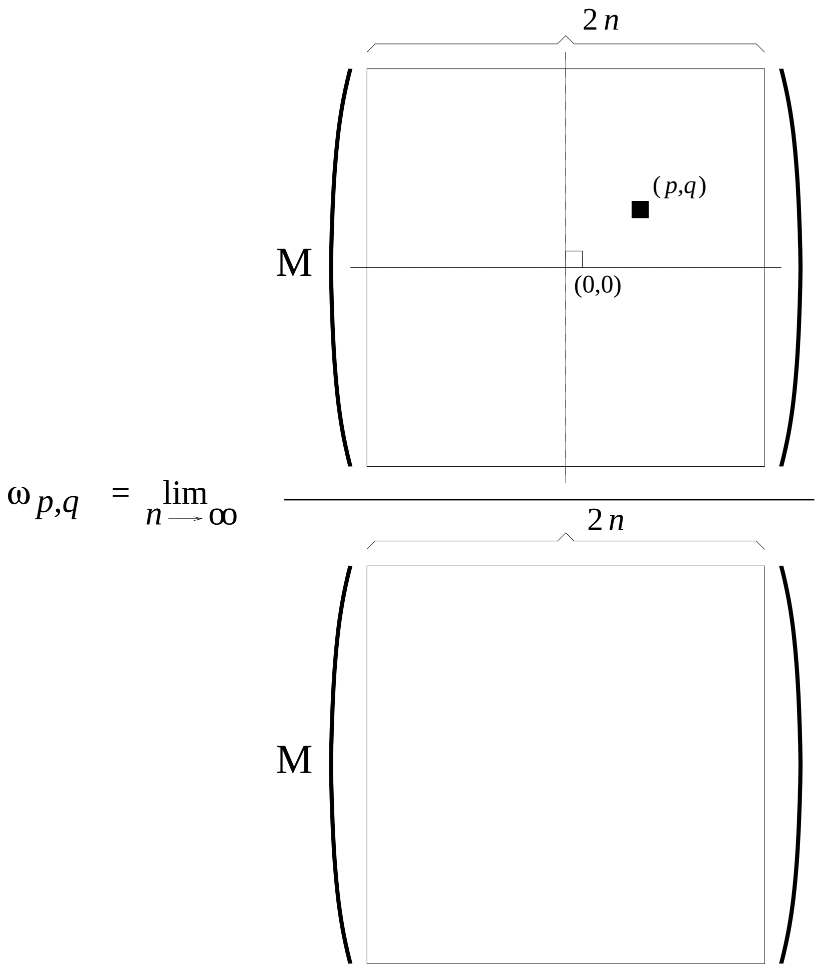
$$E(w_1, w_2, b_1, b_2) := \frac{\sqrt{|w_1 - w_2|}\sqrt{|b_1 - b_2|}}{\sqrt{|w_1 - b_1|}\sqrt{|w_1 - b_2|}\sqrt{|w_2 - b_1|}\sqrt{|w_2 - b_2|}}$$

Relative sizes of energies:

$$\frac{E_1}{E_2} = 3.726575104...$$

$$\frac{E_1}{E_3} = 0.570399674...$$

Fisher and Stephenson (1963): Correlation of monomers in a sea of dimers



Fisher and Stephenson conjectured (from exact data) that

$$\omega_{2p+1,0} \sim c \frac{1}{\sqrt{\mathrm{d}((2p+1,0),(0,0))}}$$

$$\omega_{p+1,p} \sim c \frac{1}{\sqrt{\mathrm{d}((p+1,p),(0,0))}}$$

as $p \to \infty$, with *same* c.

Based on this, they conjectured that $\omega_{p,q}$ is *rotationally invariant* for $p_k/q_k \to s$, $k \to \infty$, over all slopes s.

This still stands open.

Only proved direction is diagonal direction (Hartwig '66):

$$\omega(m_{0,0}, m_{d,d-1}) \sim \frac{\sqrt{e}}{2^{\frac{5}{6}} A^6} \frac{1}{\sqrt{d}} = \frac{\sqrt{e}}{2^{\frac{7}{12}} A^6} (d\sqrt{2})^{-\frac{1}{2}}, \quad d \to \infty,$$

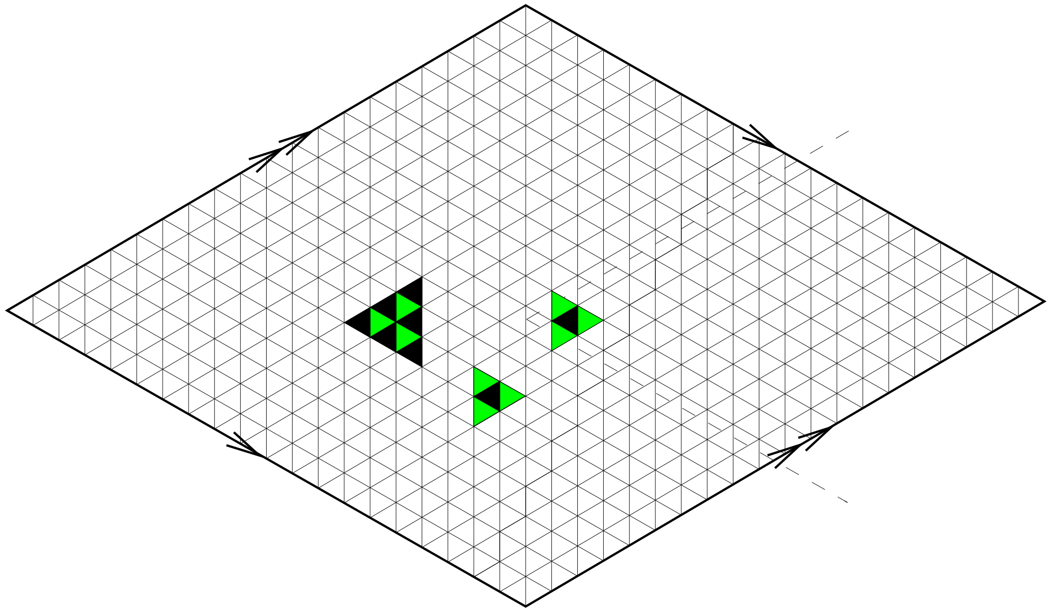
where

$$A = 1.28242712...$$

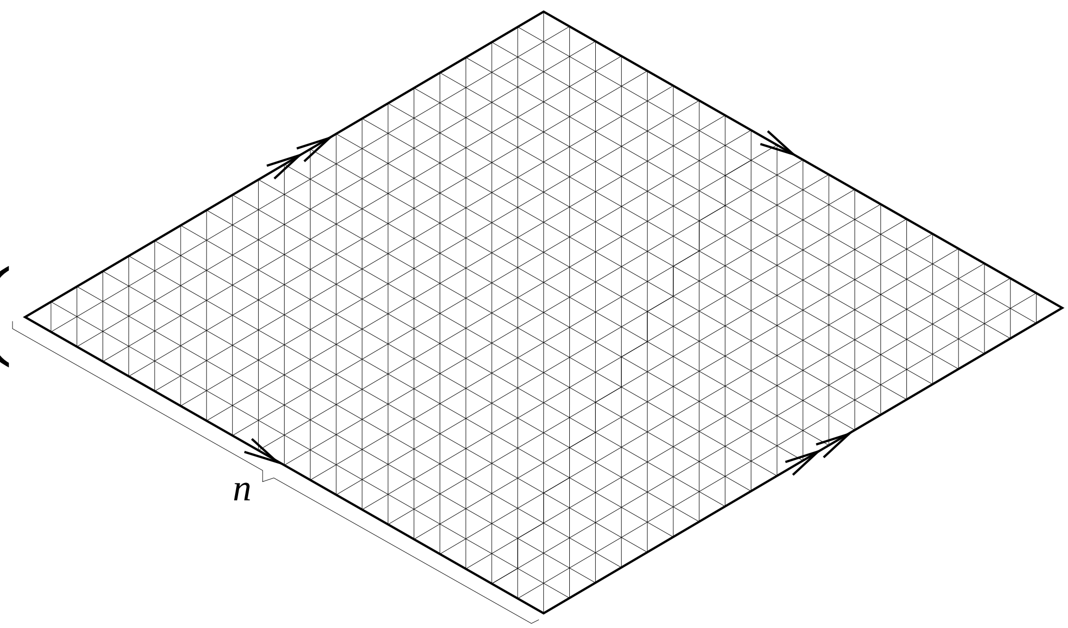
is the Glaisher-Kinkelin constant

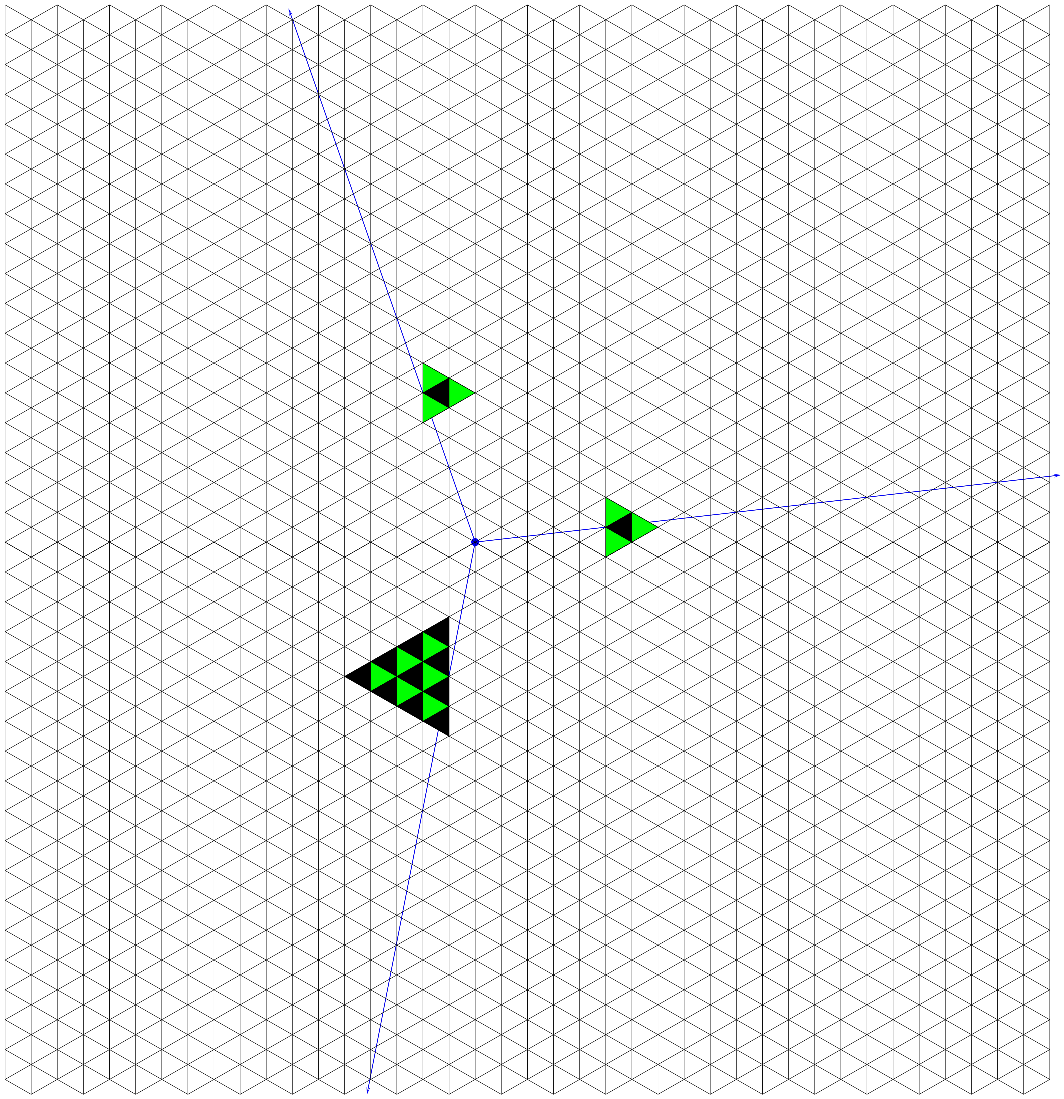
$$\lim_{n\to\infty} \frac{0!\, 1! \cdots (n-1)!}{n^{\frac{n^2}{2} - \frac{1}{12}} (2\pi)^{\frac{n}{2}} e^{-\frac{3n^2}{4}}} = \frac{e^{\frac{1}{12}}}{A}$$

Correlation defined via tori

$M\Big($  $\Big)$

$$\omega := \lim_{n\to\infty} \frac{}{}$$

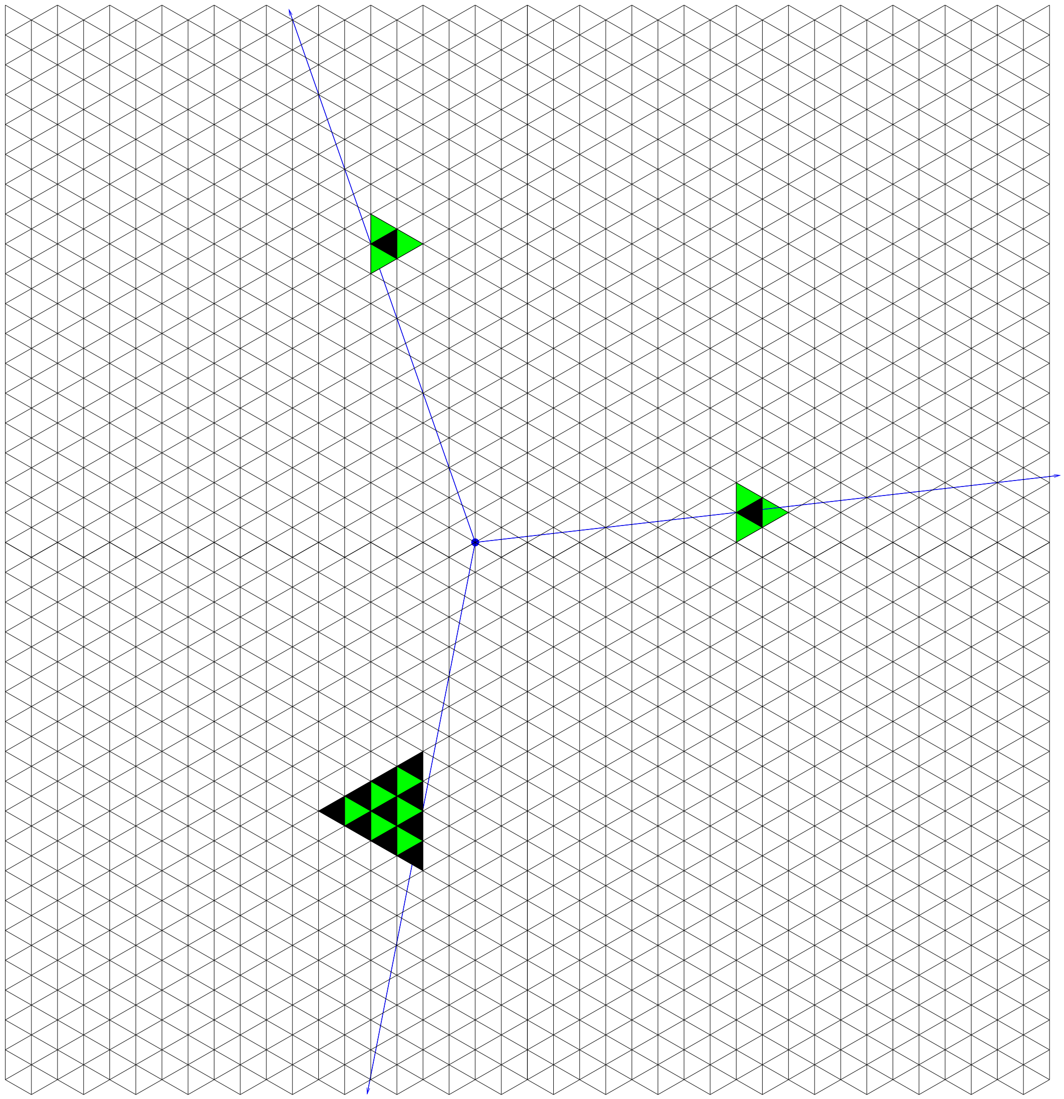
$M\Big($  $\Big)$



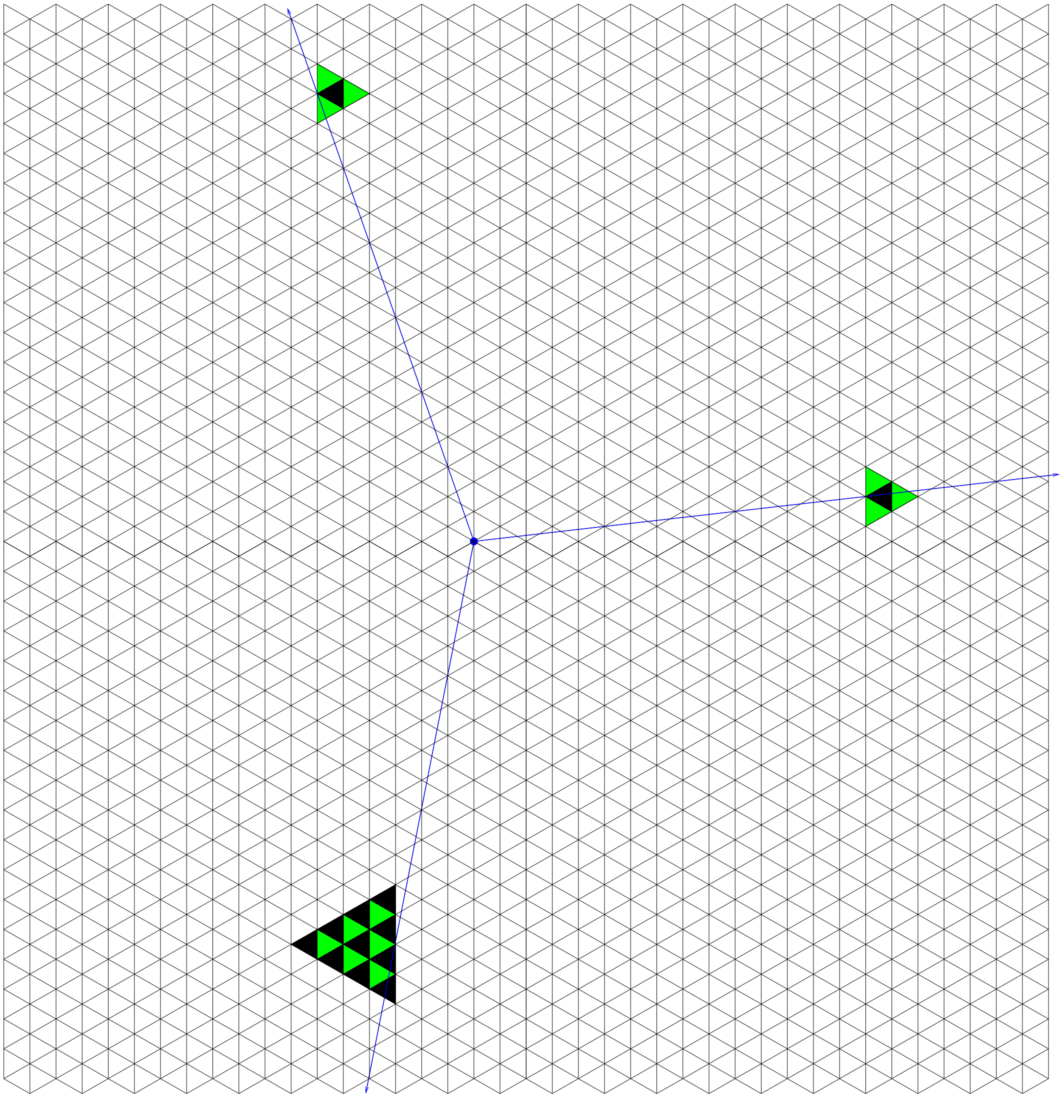
O_1, O_2, O_3

$\mathbf{r}$: position vector of O

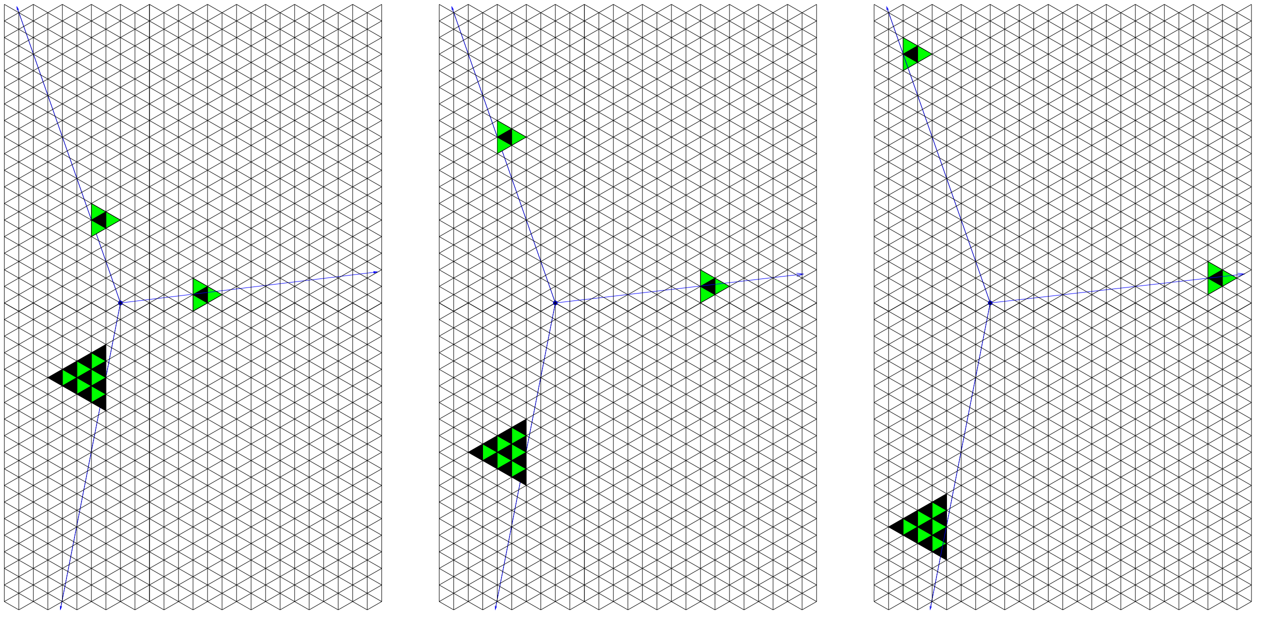
αO: translation of O whose position vector is $\alpha \mathbf{r}$



$2O_1$, $2O_2$, $2O_3$



$3O_1$, $3O_2$, $3O_3$



Charge of a hole:

$$\mathrm{q}(O) := \#\,(\text{green monomers in } O) - \#\,(\text{black monomers in } O)$$

Electrostatic Energy:

$$E(O_1, \ldots, O_n) := \prod_{1 \le i < j \le n} \mathrm{d}(O_i, O_j)^{\frac{1}{2}\,\mathrm{q}(O_i)\,\mathrm{q}(O_j)}$$

Theorem (C., 2009). *Suppose O_i is either of type $\triangleright_{k_i}$ or of type $\triangleleft_{k_i}$, with k_i even, for $i = 1, \ldots, n$. Then*

$$\omega(\alpha O_1 \ldots, \alpha O_n) \sim c E(\alpha O_1 \ldots, \alpha O_n), \qquad \alpha \to \infty.$$

Electrostatic Hypothesis

Conjecture. *For* **any** *holes* $O_1, \ldots, O_n$ *we have*

$$\omega(\alpha O_1 \ldots, \alpha O_n) \sim cE(\alpha O_1 \ldots, \alpha O_n), \qquad \alpha \to \infty.$$

Remark. The square lattice analog of the above conjecture specializes in the case of a white and a black monomer to the original Fisher-Stephenson conjecture (the charges are $+1$ and -1).

The difficulty of odd holes

Two main approaches:

(i) exact formulas

(ii) inverse Kasteleyn matrix (Fisher-Stephenson '63, Kenyon '97, Kenyon-Okounkov-Sheffield '03)

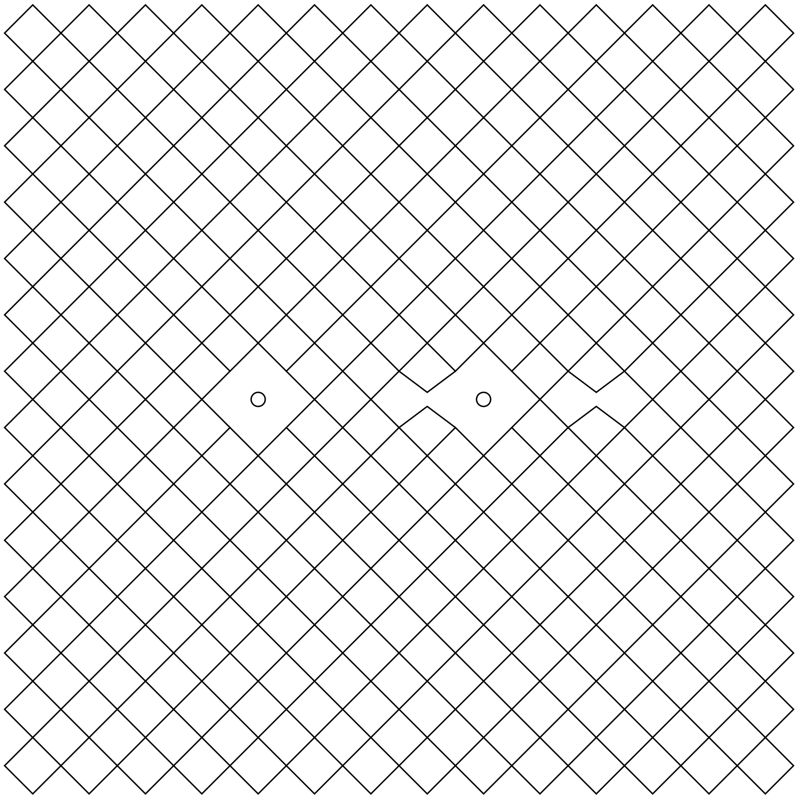
Odd holes pose difficulties in both:

- Gessel-Viennot-Lindström determinant formula fails in (i)
- Percus-Kenyon way to express $\dfrac{\det K(G')}{\det K(G)}$ as a minor of $K(G)^{-1}$ fails in (ii)

One exception

Defect clusters consisting of holes and separations

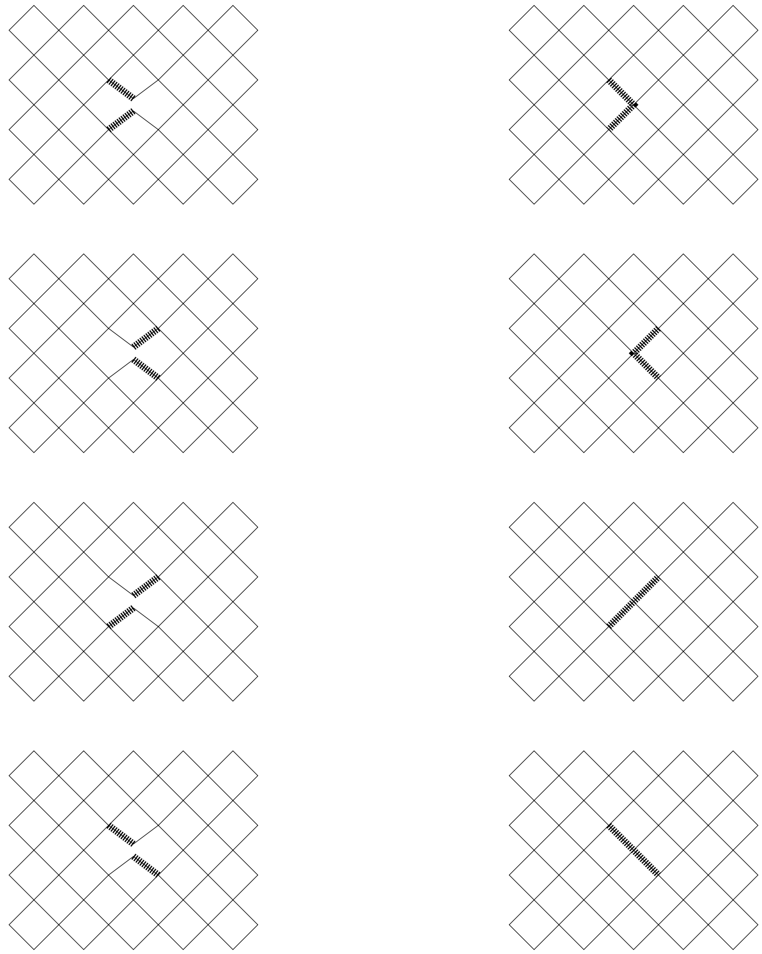
Aztec diamonds



$AD_{14}(\{5,9\};\{8,11\})$

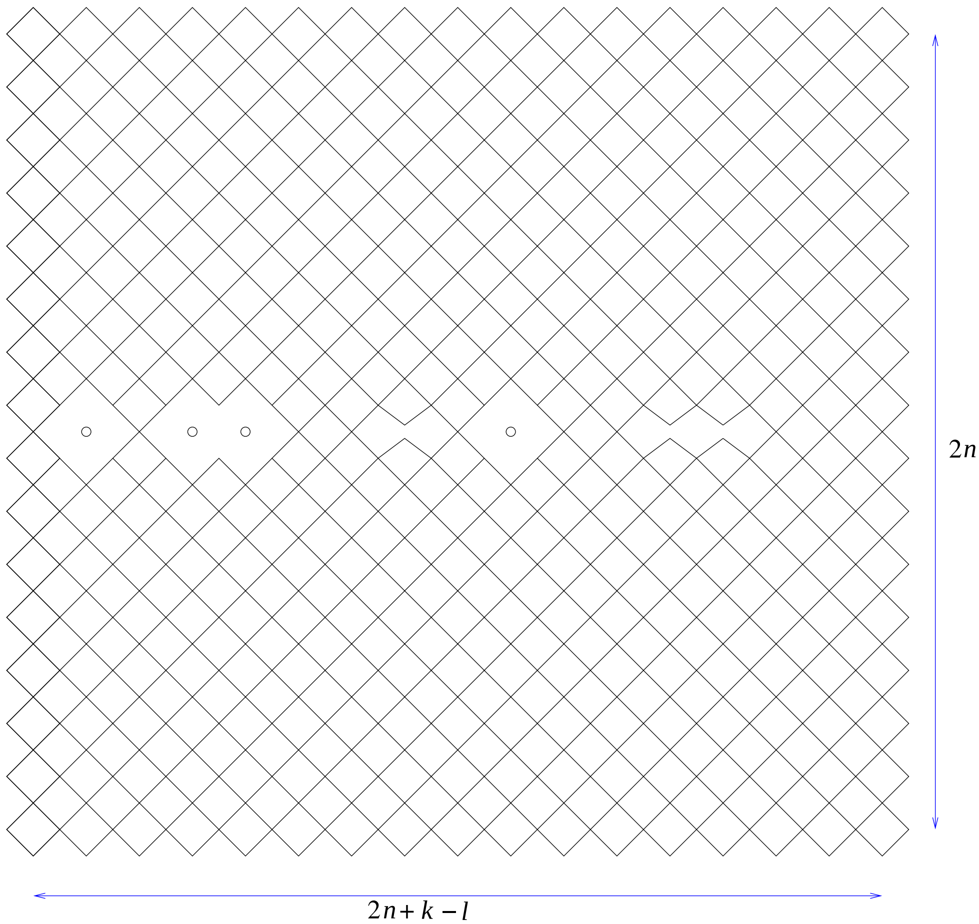
Define

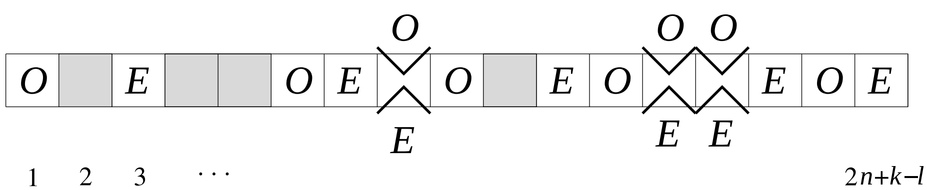
$$\omega(a_1,\dots,a_k;b_1,\dots,b_k) := \lim_{n\to\infty} \frac{\mathrm{M}(AD_{2n}(a_1,\dots,a_k;b_1,\dots,b_k))}{\mathrm{M}(AD_{2n})}$$



separation: “superposition” of four trimers each of charge -1

Product formula





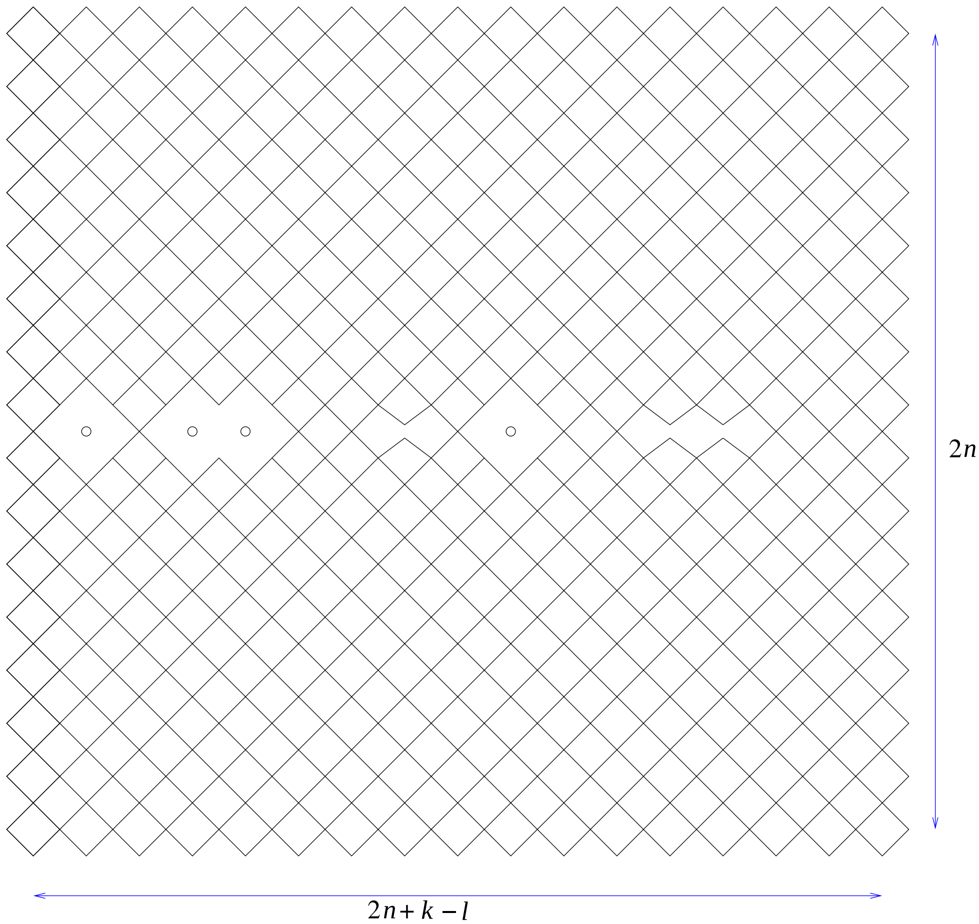
$\mathcal{O}$: Coordinates of O's

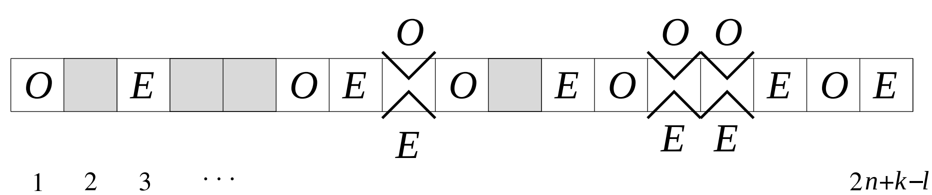
$\mathcal{E}$: Coordinates of E's

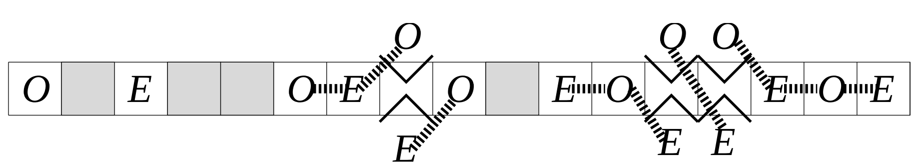
$\Delta(\{x_1,\ldots,x_m\}) := \prod_{1\le i<j\le m}(x_j - x_i)$

Theorem. $\dfrac{2^{n^2+2n-l}}{(0!\,1!\cdots(n-1)!)^2}\Delta(\mathcal{O})\Delta(\mathcal{E})$ *perfect matchings*

O-E labeled strings mirror holes and separations







monomers $\longleftrightarrow$ gaps

separations $\longleftrightarrow$ laps

Extending ω to $\bar{\omega}$

- $k = l$: $\quad \bar{\omega}(a_1, \ldots, a_k; b_1, \ldots, b_k) = \omega(a_1, \ldots, a_k; b_1, \ldots, b_k)$
- $k > l$:

$$\bar{\omega}(a_1, \ldots, a_k; b_1, \ldots, b_l) = \frac{1}{\bar{\omega}\binom{\vee}{\wedge}} \lim_{d\to\infty} (d\sqrt{2})^{\frac{k-l}{2}}\, \bar{\omega}(a_1, \ldots, a_k; b_1, \ldots, b_l, d)$$

- $k < l$:

$$\bar{\omega}(a_1, \ldots, a_k; b_1, \ldots, b_l) = \frac{1}{\bar{\omega}(\circ)} \lim_{d\to\infty} (d\sqrt{2})^{\frac{l-k}{2}}\, \bar{\omega}(a_1, \ldots, a_k, d; b_1, \ldots, b_l),$$

where

$$\bar{\omega}(\circ) := \frac{e^{\frac{1}{4}}}{2^{\frac{7}{24}} A^3}$$

$$\bar{\omega}\binom{\vee}{\wedge} := \frac{2^{\frac{5}{24}} e^{\frac{1}{4}}}{A^3}$$

Strong Superposition Principle

Fix a lattice diagonal ℓ

Defect cluster: Union of unit holes and separations on ℓ

Charge: $\mathrm{q}(O) := \#\,(\text{unit holes in } O) - \#\,(\text{separations in } O)$

Energy: $E(O_1, \ldots, O_n) := \prod_{1 \le i < j \le n} \mathrm{d}(O_i, O_j)^{\frac{1}{2}\,\mathrm{q}(O_i)\,\mathrm{q}(O_j)}$

(d: Euclidean distance)

Theorem (C., 2011). *Let* $O_1, \ldots, O_n$ *be* **arbitrary** *defect clusters on* ℓ. *Then*

$$\bar{\omega}(\alpha O_1, \ldots, \alpha O_n) \sim \bar{\omega}(O_1) \cdots \bar{\omega}(O_n) E(\alpha O_1, \ldots, \alpha O_n), \quad \alpha \to \infty.$$

Ingredients of proof

Exactness

- holes at $a_1, \ldots, a_k$
- separations at $b_1, \ldots, b_l$
- $E(a_1, \ldots, a_k; b_1, \ldots, b_l) := \dfrac{\prod_{1\le i<j\le k} |a_i - a_j|^{\frac{1}{2}} \prod_{1\le i<j\le l} |b_i - b_j|^{\frac{1}{2}}}{\prod_{i=1}^{k} \prod_{j=1}^{l} |a_i - b_j|^{\frac{1}{2}}}$

Then

$$\frac{\bar{\omega}(a_1, a_2 \ldots, a_k; b_1, b_2 \ldots, b_l)}{\bar{\omega}(b_1, a_2 \ldots, a_k; a_1, b_2, \ldots, b_l)} = \frac{E(a_1, a_2 \ldots, a_k; b_1, b_2 \ldots, b_l)}{E(b_1, a_2 \ldots, a_k; a_1, b_2, \ldots, b_l)}$$

Elementary move

- $L(d) := \dfrac{\Gamma^2\left(\frac{d-1}{2}\right)\Gamma^2\left(\frac{d+1}{2}\right)}{\Gamma^4\left(\frac{d}{2}\right)}$
- $U(d) := \dfrac{\Gamma^2\left(\frac{d-1}{2}\right)\Gamma^2\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}-1\right)\Gamma^2\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+1\right)}$

If all consecutive distances in $\{a_1, \dots, b_l\}_<$ are even, then

$$\frac{\bar{\omega}(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_k; b_1, \dots, b_l)}{\bar{\omega}(a_1, \dots, a_{i-1}, a_i - 2, a_{i+1}, \dots, a_k; b_1, \dots, b_l)} =$$

$$\frac{\prod_{j:a_j<a_i} L(|a_i - a_j|) \prod_{j:b_j<a_i} U(|a_i - b_j|)}{\prod_{j:a_j>a_i} L(|a_i - a_j|) \prod_{j:b_j>a_i} U(|a_i - b_j|)}$$

$$\frac{\bar{\omega}(a_1, \dots, a_k; b_1, \dots, b_{i-1}, b_i, b_{i+1}, \dots, b_l)}{\bar{\omega}(a_1, \dots, a_k; b_1, \dots, b_{i-1}, b_i - 2, b_{i+1}, \dots, b_l)} =$$

$$\frac{\prod_{j:b_j<b_i} L(|b_i - b_j|) \prod_{j:a_j<b_i} U(|b_i - a_j|)}{\prod_{j:b_j>b_i} L(|b_i - b_j|) \prod_{j:a_j>b_i} U(|b_i - a_j|)}$$

A special case: Two white holes

$$\bar{\omega}(\underbrace{\diamond \qquad \diamond}_{d}) \sim \frac{\sqrt{e}}{2^{\frac{7}{12}} A^6} (d\sqrt{2})^{\frac{1}{2}}, \quad d \to \infty.$$

Form conjectured by physicists Moessner and Sondhi in 2002.

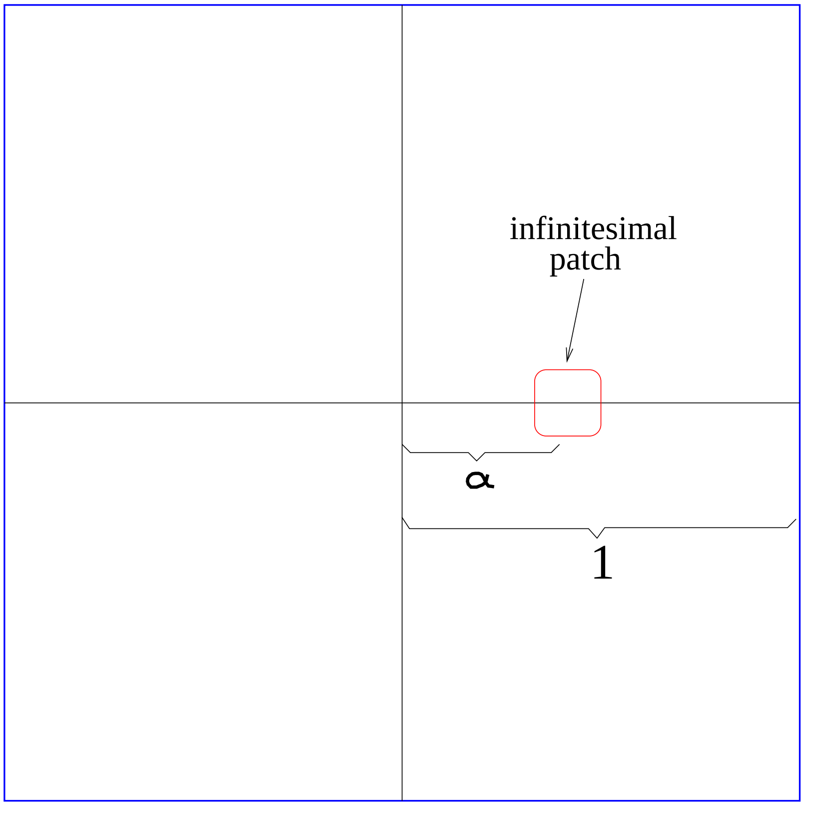
Counterpart of

$$\bar{\omega}_{\mathrm{sq}}(\underbrace{\diamond \qquad \blacklozenge}_{d}) \sim \frac{\sqrt{e}}{2^{\frac{7}{12}} A^6} (d\sqrt{2})^{-\frac{1}{2}}, \quad d \to \infty.$$

(Hartwig, 1966)

How about other locations?

How about other locations?



Scaled $AR_{2n,2n+k-l}$, as $n \to \infty$

$$-1 < \alpha < 1$$

Elementary moves for $\alpha \neq 0$:

$$\frac{\bar{\omega}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k; b_1, \ldots, b_l)}{\bar{\omega}(a_1, \ldots, a_{i-1}, a_i - 2, a_{i+1}, \ldots, a_k; b_1, \ldots, b_l)} =$$

$$\frac{1-\alpha}{1+\alpha} \frac{\prod_{j:a_j<a_i} L(|a_i - a_j|) \prod_{j:b_j<a_i} U(|a_i - b_j|)}{\prod_{j:a_j>a_i} L(|a_i - a_j|) \prod_{j:b_j>a_i} U(|a_i - b_j|)}$$

$$\frac{\bar{\omega}(a_1, \ldots, a_k; b_1, \ldots, b_{i-1}, b_i, b_{i+1}, \ldots, b_l)}{\bar{\omega}(a_1, \ldots, a_k; b_1, \ldots, b_{i-1}, b_i - 2, b_{i+1}, \ldots, b_l)} =$$

$$\frac{1+\alpha}{1-\alpha} \frac{\prod_{j:b_j<b_i} L(|b_i - b_j|) \prod_{j:a_j<b_i} U(|b_i - a_j|)}{\prod_{j:b_j>b_i} L(|b_i - b_j|) \prod_{j:a_j>b_i} U(|b_i - a_j|)}$$

So get **exponential interaction** between any two clusters, *unless both have charge zero.*

Boundary effects

- Aztec diamond AR_{2n}
- unit holes at $a_1, \ldots, a_k$
- separations at $b_1, \ldots, b_k$
- $L(a) := \#(\text{sites left of } a) + \frac{1}{2}$
- $R(a) := \#(\text{sites right of } a) + \frac{1}{2}$

Theorem (C, 2011). *For large separations between the defects we have*

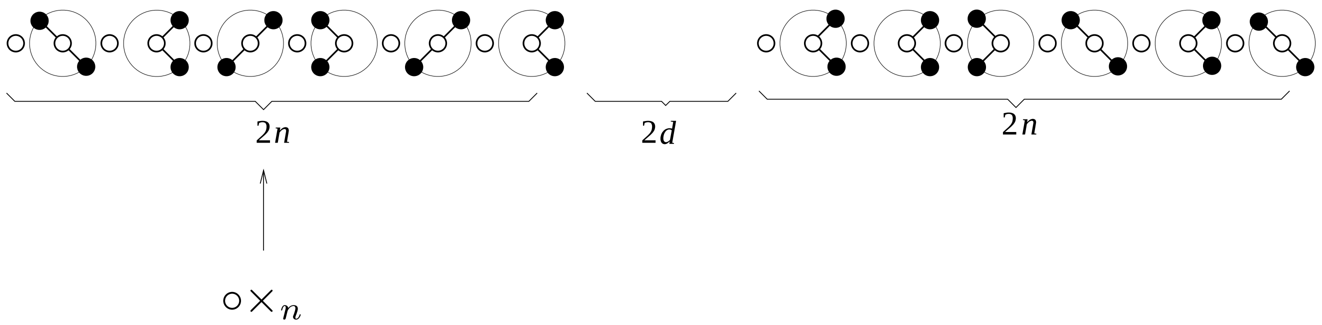
$$\frac{\mathrm{M}(AR_{2n} \setminus \{\circ_{a_1}, \ldots, \circ_{a_k}, \times_{b_1}, \ldots, \times_{b_k}\})}{\mathrm{M}(AR_{2n})} \sim$$

$$C_k \sqrt{\prod_{i=1}^{k} \frac{L(b_i)^{L(b_i)} R(b_i)^{R(b_i)}}{L(a_i)^{L(a_i)} R(a_i)^{R(a_i)}}}$$

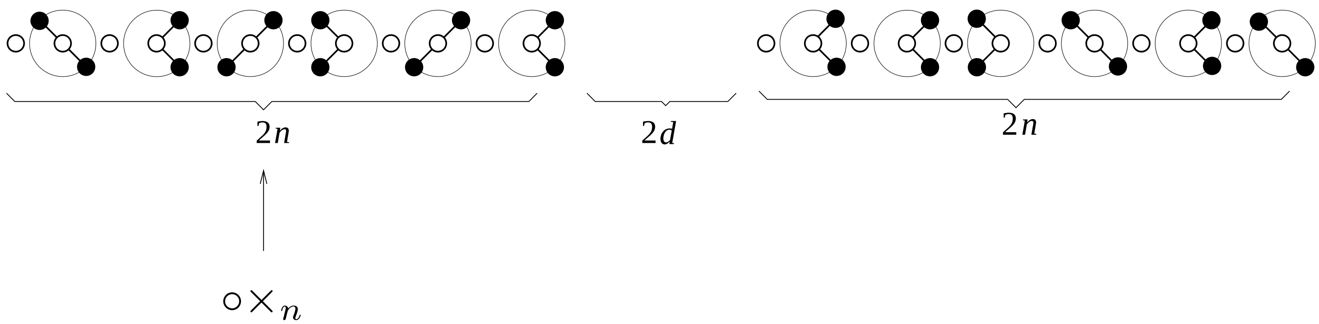
$$\times \frac{\prod_{1 \le i < j \le k} \sqrt{|a_i - a_j|} \prod_{1 \le i < j \le k} \sqrt{|b_i - b_j|}}{\prod_{i=1}^{k} \prod_{j=1}^{k} \sqrt{|a_i - b_j|}},$$

$n \to \infty$

Long neutral slits



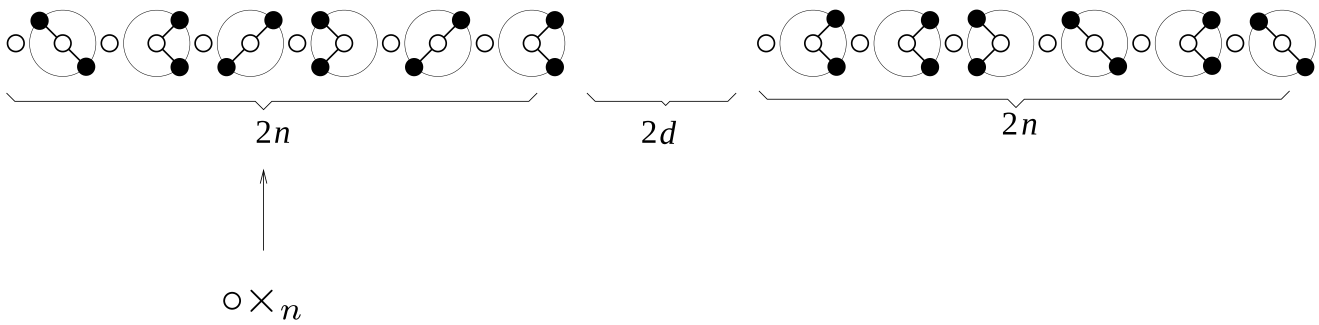
Long neutral slits



Theorem (C., 2011). *As* $n, d \to \infty$ *with* $\frac{d}{n} \to \alpha > 0$,

$$\frac{\omega(\circ \times_n \underbrace{\qquad\quad}_{2d} \circ \times_n)}{\omega(\circ \times_n \circ \times_n)} \sim \frac{2^{\frac{1}{3}} e^{\frac{1}{4}}}{A^3} \left[\frac{(1+\alpha)^2}{\alpha(2+\alpha)} \right]^{\frac{1}{4}} \frac{1}{n^{\frac{1}{4}}}$$

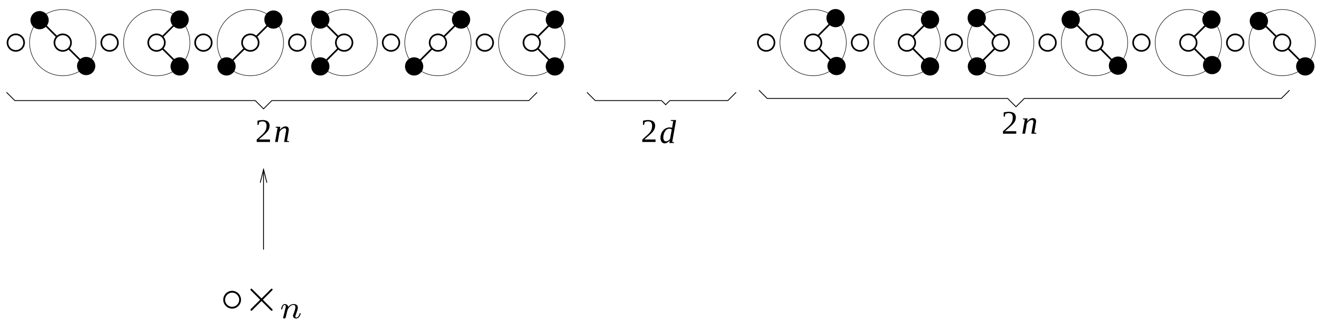
Long neutral slits



Theorem (C., 2011). *As* $n, d \to \infty$ *with* $\frac{d}{n} \to \alpha > 0$,

$$\frac{\omega(\circ \times_n \underbrace{\hspace{5em}}_{2d} \circ\times_n)}{\omega(\circ \times_n \circ\times_n)} \sim \frac{2^{\frac{1}{3}} e^{\frac{1}{4}}}{A^3} \left[\frac{(1+\alpha)^2}{\alpha(2+\alpha)}\right]^{\frac{1}{4}} \frac{1}{n^{\frac{1}{4}}}$$

$$\frac{\omega(\circ \times_n \underbrace{\hspace{5em}}_{2d-1} \times\circ_n)}{\omega(\circ \times_n \circ\times_n)} \sim \frac{2^{\frac{1}{3}} e^{\frac{1}{4}}}{A^3} \left[\frac{\alpha(2+\alpha)}{(1+\alpha)^2}\right]^{\frac{1}{4}} \frac{1}{n^{\frac{1}{4}}}$$



Theorem (C., 2011). (a). *As $n, d \to \infty$ with $\frac{d}{n} \to \alpha > 0$,*

$$\frac{\omega(\circ \times_n \underbrace{\qquad\qquad}_{2d} \circ\times_n)}{\omega(\circ \times_n \circ\times_n)} \sim \frac{2^{\frac{1}{3}} e^{\frac{1}{4}}}{A^3} \left[\frac{(1+\alpha)^2}{\alpha(2+\alpha)}\right]^{\frac{1}{4}} \frac{1}{n^{\frac{1}{4}}}$$

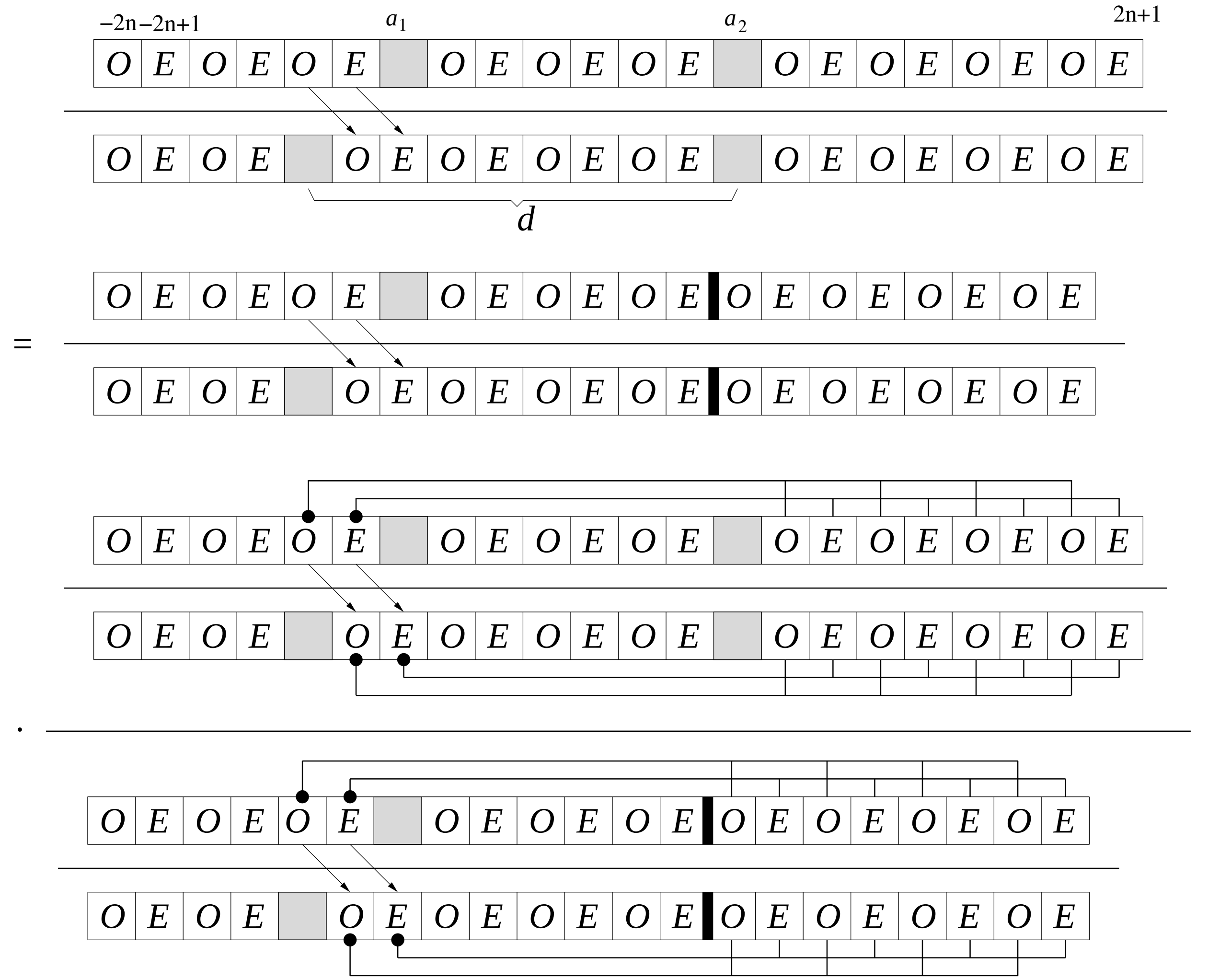
$$\frac{\omega(\circ \times_n \underbrace{\qquad\qquad}_{2d-1} \times\circ_n)}{\omega(\circ \times_n \circ\times_n)} \sim \frac{2^{\frac{1}{3}} e^{\frac{1}{4}}}{A^3} \left[\frac{\alpha(2+\alpha)}{(1+\alpha)^2}\right]^{\frac{1}{4}} \frac{1}{n^{\frac{1}{4}}}$$

(b). *For any fixed n,*

$$\frac{\omega(\circ \times_n \underbrace{\qquad\qquad}_{2d-1} \circ\times_n)}{\omega(\circ \times_n \circ\times_n)} = \textit{independent of } d$$

$$\frac{\omega(\circ \times_n \underbrace{\qquad\qquad}_{2d} \times\circ_n)}{\omega(\circ \times_n \circ\times_n)} = \textit{independent of } d$$

Proof of elementary move formula



O-E string: $\Delta(\mathcal{O})\Delta(\mathcal{E})$

$$\textbf{O-part 1st factor} \sim \frac{2\cdot 4\cdot \ldots \cdot (2n)}{3\cdot 5\cdot \ldots \cdot (2n+1)} \cdot \frac{3\cdot 5\cdot \ldots \cdot (2n+1)}{2\cdot 4\cdot \ldots \cdot (2n)} = 1$$

$$\textbf{O-part 2nd factor} \sim \frac{\frac{(d+1)(d+3)\cdots(d+2n-1)}{d(d+2)\cdots(d+2n-2)}}{\frac{(d)(d+2)\cdots(d+2n-2)}{(d-1)(d+1)\cdots(d+2n-3)}} = \frac{\left(\frac{d-1}{2}\right)_n \left(\frac{d+1}{2}\right)_n}{\left(\frac{d}{2}\right)_n^2}$$

$$(a)_k := a(a+1)\cdots(a+k-1)$$

Expressing the Pochhammer symbols in terms of Gamma functions by $(a)_k = \Gamma(a+k)/\Gamma(a)$ and using Stirling's formula, one readily sees that

$$\lim_{n\to\infty} \frac{\left(\frac{d-1}{2}\right)_n \left(\frac{d+1}{2}\right)_n}{\left(\frac{d}{2}\right)_n^2} = \frac{\Gamma^2\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}, \qquad n \to \infty.$$

This argument can be extended to be an induction step, proving the elementary move formula.