The critical Z-invariant Ising model via dimers

Béatrice de Tilière
University of Neuchâtel

joint work with Cédric Boutillier, University Paris 6

Ascona, 27 May 2010

The 2-dimensional Ising model

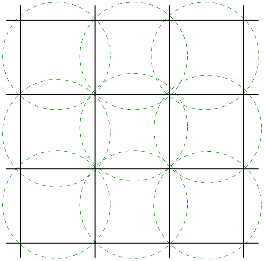
- Planar graph, $G = (V(G), E(G))$.
- Spin Configurations, $\sigma : V(G) \to \{-1, 1\}$.
- Every edge $e \in E(G)$ has a coupling constant, $J_e > 0$.
- Ising Boltzmann measure (finite graph):

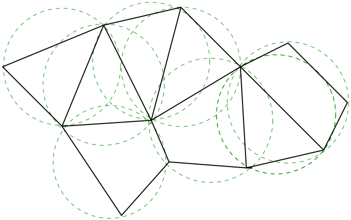
$$P_{\text{Ising}}(\sigma) = \frac{1}{Z_{\text{Ising}}} \exp\left(\sum_{e=uv \in E(G)} J_e \sigma_u \sigma_v \right),$$

where Z_{Ising} is the partition function.

Isoradial graphs

- Graph G isoradial : can be embedded in the plane so that every face is inscribable in a circle of radius 1.

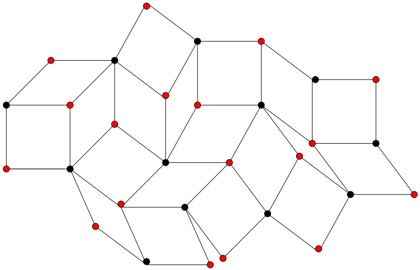




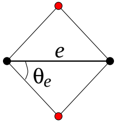
- G isoradial $\Rightarrow G^*$ isoradial : vertices of G^* = circumcenters.

Isoradial graphs

- $G^\diamond$: corresponding diamond graph $\begin{cases} \text{vertices} : V(G) \cup V(G^*) \\ \text{edges} : \text{radii of circles} \end{cases}$



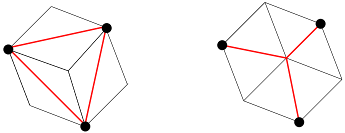
- Edge $e \rightarrow \begin{cases} \text{rhombus} \\ \text{angle } \theta_e \end{cases}$



$\Rightarrow$ Coupling constants: $J_e(\theta_e)$.

The Z-invariant Ising model [Baxter]

- Star-triangle transformation on G: preserves isoradiality.



- Z-invariant Ising model : satisfies $\Delta - Y$

$$\Rightarrow \quad \sinh(2J_e(\theta_e)) = \frac{\mathrm{sn}\left(\frac{2K(k)}{\pi}\theta_e|k\right)}{\mathrm{cn}\left(\frac{2K(k)}{\pi}\theta_e|k\right)}, \quad k^2 \in \mathbb{R}.$$

sn, cn : Jacobi elliptic trigonometric functions.
K : complete elliptic integral of the first kind.

The critical Z-invariant Ising model [Baxter]

- High and low temperature expansion of the partition function $\Rightarrow$ measure on contour configurations of G and G^*.
- Generalized form of self-duality $\Rightarrow$ $k = 0$, and

$$J_e(\theta_e) = \frac{1}{2}\log\left(\frac{1+\sin\theta_e}{\cos\theta_e}\right). \tag{1}$$

- Examples.
  - $G = \mathbb{Z}^2$: $\theta_e = \pi/4$, $J_e(\theta_e) = \log\sqrt{1+\sqrt{2}}$.
  - G triangular lattice : $\theta_e = \pi/6$, $J_e(\theta_e) = \log(3^{\frac{1}{4}})$.
  - G hexagonal lattice : $\theta_e = \pi/3$, $J_e(\theta_e) = \log\sqrt{2+\sqrt{3}}$.

Critical temperatures (Kramers, Kramers-Wannier)

(1) are called critical coupling constants.

Statistical mechanics on isoradial graphs

- Critical Ising model: [Baxter, Costa-Santos, Mercat, Smirnov, Chelkak & Smirnov].
- Explicit expressions for:
  - the Green's function with weights "$\tan(\theta_e)$" [Kenyon],
  - the inverse of the Dirac operator $\bar{\partial}$ for bipartite graphs, with weights "$2\sin\theta_e$" [Kenyon],

  which only depend on the *local geometry* of the graph.
- What is special about this setting ?
  - Z-invariance (integrability).
  - Natural setting for discrete complex analysis [Duffin, Mercat, Chelkak & Smirnov].

The dimer model

- Graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ (planar).
- Dimer configuration: perfect matching M.
- Weight function, $\nu : E(\mathcal{G}) \to \mathbb{R}^+$
- Dimer Boltzmann measure (finite graph) :

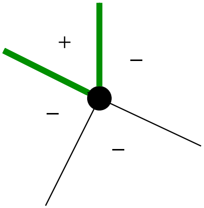
$$P_{\text{dimer}}(M) = \frac{1}{Z_{\text{dimer}}} \prod_{e \in M} \nu_e,$$

where Z_{dimer} is the dimer partition function.

Ising-dimer correspondence [Fisher]

Ising model on toroidal graph G, coupling constants J.

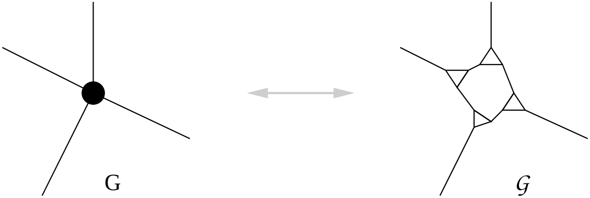
- Low temperature expansion on G^* $\rightarrow$ measure on polygonal contours of G.



Ising-dimer correspondence [Fisher]

Ising model on toroidal graph G, coupling constants J.

- Low temperature expansion on G^* → measure on polygonal contours of G.

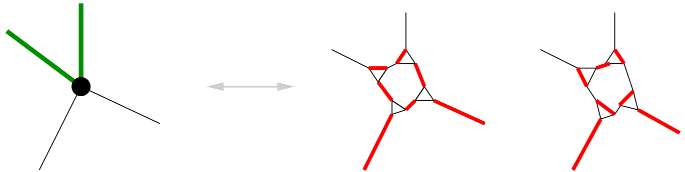


- $\mathcal{G}$: Fisher graph of G.

Ising-dimer correspondence [Fisher]

Ising model on toroidal graph G, coupling constants J.

- Low temperature expansion on G^* → measure on polygonal contours of G.



- $\mathcal{G}$: Fisher graph of G.
- Correspondence : Contour conf. → $2^{|V(G)|}$ dimer configurations.

- Critical Ising model on $G \leftrightarrow$ critical dimer model on $\mathcal{G}$.

$$\nu_e = \begin{cases} \cot \frac{\theta_e}{2} & \text{original edges} \\ 1 & \text{edges of the decorations.} \end{cases}$$

Critical dimer model on infinite Fisher graph: periodic case

- G: infinite $\mathbb{Z}^2$-periodic isoradial graph, $\mathcal{G}$: corresponding Fisher graph.
- Toroidal exhaustion $\{\mathcal{G}_n\} = \{\mathcal{G}/n\mathbb{Z}^2\}$. $\mathcal{G}_1$: fundamental domain. Z_n: partition function of $\mathcal{G}_n$, P_n: Boltzmann measure of $\mathcal{G}_n$.

Critical dimer model on infinite Fisher graph: periodic case

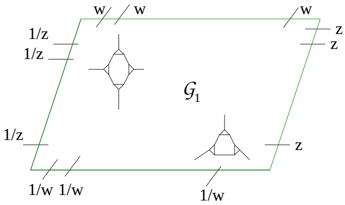
- G: infinite $\mathbb{Z}^2$-periodic isoradial graph,
  $\mathcal{G}$: corresponding Fisher graph.
- Toroidal exhaustion $\{\mathcal{G}_n\} = \{\mathcal{G}/n\mathbb{Z}^2\}$. $\mathcal{G}_1$: fundamental domain.
  Z_n: partition function of $\mathcal{G}_n$, P_n: Boltzmann measure of $\mathcal{G}_n$.
- Our goal is to:
  - compute the free energy:
    $$f = -\lim_{n\to\infty} \frac{1}{n^2} \log Z_n.$$
  - obtain an explicit expression for a natural Gibbs measure: probability measure P such that if one fixes a perfect matching in an annular region, matchings inside and outside of the annulus are independent, and
    $$P(M)\alpha \prod_{e\in M} \nu_e,$$
    when M is a matching inside of the annulus (DLR conditions).

Kasteleyn matrix K of the graph $\mathcal{G}$

- Kasteleyn orientation of the graph: all elementary cycles are clockwise odd.
- K: corresponding weighted oriented adjacency matrix,

$$K_{u,v} = \begin{cases} \nu_{uv} & \text{if } u \sim v,\ u \to v \\ -\nu_{uv} & \text{if } u \sim v,\ u \leftarrow v \\ 0 & \text{else.} \end{cases}$$

- $K_1(z,w)$: is the Kasteleyn matrix of $\mathcal{G}_1$, with modified weights along edges crossing a dual horizontal and vertical cycle.



- $\det K_1(z,w)$ is the dimer characteristic polynomial.

Free energy and Gibbs measure

Theorem (Boutillier,dT)

- *The free energy of the dimer model on $\mathcal{G}$ is:*

$$f = -\frac{1}{2(2\pi i)^2}\iint_{\mathbb{T}^2} \log(\det K_1(z,w))\frac{dz}{z}\frac{dw}{w}$$

- *The weak limit of the Boltzmann measures P_n defines a Gibbs measure P_{dimer} on $\mathcal{G}$. The probability of the subset of edges $E=\{e_1=u_1v_1,\cdots,e_k=u_kv_k\}$ being in a dimer configuration of $\mathcal{G}$ is given by:*

$$\mathcal{P}(e_1,\cdots,e_k) = \left(\prod_{i=1}^{k} K_{u_i,v_i}\right)\mathrm{Pf}((K^{-1})_E), \quad \textit{where}$$

$$K^{-1}_{(v,x,y)(v',x',y')} = \frac{1}{(2\pi i)^2}\iint_{\mathbb{T}^2} \frac{\mathrm{Cof}(K_1(z,w))^t_{v,v'}}{\det K_1(z,w)} z^{x'-x} w^{y'-y}\frac{dz}{z}\frac{dw}{w}.$$

[Probab. Theory Related Fields, 2010]

Idea of the proof (Gibbs measure)

[Cohn, Kenyon, Propp; Kenyon, Okounkov, Sheffield]

Theorem (Kasteleyn,Kenyon,...)

The Boltzmann measure $P_n(e_1, \cdots, e_k)$ is equal to:

$$\frac{\left(\prod_{i=1}^{k} K_{u_i,v_i}\right)}{2Z_n}\left(-\mathrm{Pf}(K_n^{00})_{E^C} + \mathrm{Pf}(K_n^{10})_{E^C} + \mathrm{Pf}(K_n^{01})_{E^C} + \mathrm{Pf}(K_n^{11})_{E^C}\right)$$

- Use Jacobi's formula: $\mathrm{Pf}((K_n^{\theta\tau})_{E^C}) = \mathrm{Pf}(K_n^{\theta\tau})\mathrm{Pf}((K_n^{\theta\tau})_E^{-1})$.
- Use Fourier to block diagonalize $K_n^{\theta\tau}$.
- Obtain Riemann sums. Show that they converge on a subsequence to the corresponding integral.
- Use Sheffield's theorem, which proves a priori existence of the limit.

Proof, continued

- Sheffield's theorem does not hold for non bipartite graphs.
- K_n^{00} is never invertible $\Rightarrow$ delicate estimate, $\frac{\mathrm{Pf}(K_n^{00})_{EC}}{\mathcal{Z}^n} = O\left(\frac{1}{n}\right)$.
- To show convergence of the Riemann sums, need to know what are the zeros of $\det K_1(z,w)$ on the torus $\mathbb{T}^2$.

Characteristic polynomial and the Laplacian

- G : infinite $\mathbb{Z}^2$-periodic isoradial graph.
- Laplacian on G, with weights $\tan(\theta_e)$ is represented by the matrix Δ:

$$\Delta_{u,v} = \begin{cases} \tan(\theta_{uv}) & \text{if } u \sim v \\ -\sum_{w \sim u} \tan(\theta_{uw}) & \text{if } u = v. \end{cases}$$

- $\Delta_1(z,w)$: Laplacian matrix on G_1 with additional weights z and w.
- Laplacian characteristic polynomial: $\det(\Delta_1(z,w))$.

Theorem (Boutillier,dT)

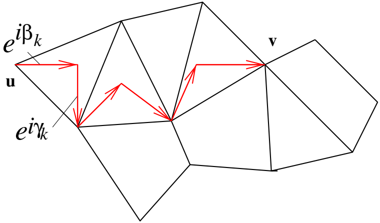
- *There exists a constant c such that:*

$$\det K_1(z,w) = c \det \Delta_1(z,w).$$

- *The curve $\{(z,w) \in \mathbb{C}^2 : \det K_1(z,w) = 0\}$ is a Harnack curve and $\det K_1(z,w)$ admits a unique double zero $(1,1)$ on $\mathbb{T}^2$.*

Local expression for K^{-1}, general case

- Discrete exponential function [Mercat,Kenyon]
  $\mathrm{Exp}: V(G) \times V(G) \times \mathbb{C} \to \mathbb{C}$,



$$\mathrm{Exp}_{\mathbf{u},\mathbf{u}}(\lambda) = 1$$

$$\mathrm{Exp}_{\mathbf{u},\mathbf{u}_{k+1}}(\lambda) = \mathrm{Exp}_{\mathbf{u},\mathbf{u}_k}(\lambda) \frac{(\lambda + e^{i\beta_k})(\lambda + e^{i\gamma_k})}{(\lambda - e^{i\beta_k})(\lambda - e^{i\gamma_k})}.$$

Local expression for K^{-1}

Theorem (Boutillier,dT)

The inverse of the Kasteleyn matrix K on $\mathcal{G}$ has the following local expression:

$$K^{-1}_{u,v} = \frac{1}{(2\pi i)^2} \oint_{C_{uv}} f_u(\lambda) f_v(\lambda) \mathrm{Exp}_{\mathbf{u},\mathbf{v}} \log(\lambda) d\lambda + c_{u,v},$$

where

- C_{uv} *is a closed contour containing all poles of the integrand, and avoiding a half-line* d_{uv}.
- $c_{u,v} = \begin{cases} 0 & \textit{if } \mathbf{u} \neq \mathbf{v} \\ \pm\frac{1}{4} & \textit{else.} \end{cases}$

[Comm. Math. Phys. 2010]

Proof (sketch)

- Idea [Kenyon]
- $f_v(\lambda)\mathrm{Exp}_{\mathbf{u},\mathbf{v}}(\lambda)$ is in the kernel of K.
- Use singularities of the log: define contours of integrations in such a way that:

$$(KK^{-1})(u,v) = \begin{cases} 0 & \text{if } u \neq v \\ 1 & \text{if } u = v. \end{cases}$$

Gibbs measure, local expression

Theorem (Boutillier,dT)

$$\mathcal{P}(e_1, \cdots, e_k) = \prod_{i=1}^{k} K(u_i, v_i) \mathrm{Pf}((K^{-1})_E),$$

defines a Gibbs measure in dimer configurations of $\mathcal{G}$*.*

Proof.

- [dT]: every finite, simply connected subgraph of a rhombus tiling can be completed by rhombi in order to become a periodic rhombus tiling of the plane.
- Convergence of the Boltzmann measures in the periodic case.
- Locality of the inverse Kasteleyn matrix.
- Uniqueness of the inverse Kasteleyn matrix in the periodic case.
- Kolmogorov's extension theorem.

□

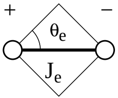
Consequences

- **Theorem (Baxter's formula)**

  Let G be a periodic isoradial graph. Then, the free energy of the critical Ising model on G is:

$$f_{\text{Ising}} = -\frac{\log 2}{2} - \frac{1}{|V(G_1)|} \sum_{e \in E(G_1)} \frac{\theta_e}{\pi} \log(\theta_e) + \frac{1}{\pi}\left(L(\theta_e) + L\left(\frac{\pi}{2} - \theta_e\right)\right),$$

  where $L(\theta) = -\int_0^\theta \log(2\sin(t))dt$ *is Lobachevski's function.*

  - $P_{\text{Ising}}\left(\right.$ 

$\left. \right) = \frac{1}{4} - \frac{\theta_e}{2\pi \sin\theta_e}$.
  - Spin/spin correlations are local.
  - Asymptotics computations for the dimer Gibbs measure.

Critical Ising model and CRSFs

- The dimer characteristic polynomial $\det K_1(z,w)$ is related to Ising configurations.
- The Laplacian characteristic polynomial $\det \Delta_1(z,w)$ counts CRSFs.

Theorem (dT)

There exists an explicit correspondence between weighted "double-dimer" configurations of $\mathcal{G}_1$ counted by $\det K_1(z,w)$, and CRSFs counted by $\det \Delta_1(z,w)$.

Proof.

- Matrix-tree theorem for the Kasteleyn matrix K:

$$\det K_1(z,w) = \sum_{F\in\mathcal{F}(\mathcal{G}_1)} \left(\prod_{e=(x,y)\in F} f_x \bar{f}_y K_{x,y} \right) \prod_{T\in F} (1 - z^{h(T)} w^{v(T)}).$$

- To every CRSF of G_1 corresponds a family of CRSFs of $\mathcal{G}_1$. Everything is explicit.