The energy density correlations in the 2D Ising model

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The Ising model

Introduced by Lenz, 1920

Model for ferromagnetism

Quantum justification

Applications:

- Study of gases
- Image processing
- Ecology

(• Neural networks, economics)

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The Ising model on a graph G

Probability space $\{\pm 1\}^{V(G)}$

$$\mathbb{P}[\vec{\sigma}] = \frac{1}{Z_\beta} e^{-\beta H(\vec{\sigma})}$$

$\beta > 0$ (inverse temperature)

$$H(\vec{\sigma}) := -\sum_{\langle x,y\rangle \in E(G)} \sigma_x \sigma_y \quad \text{(energy)}$$

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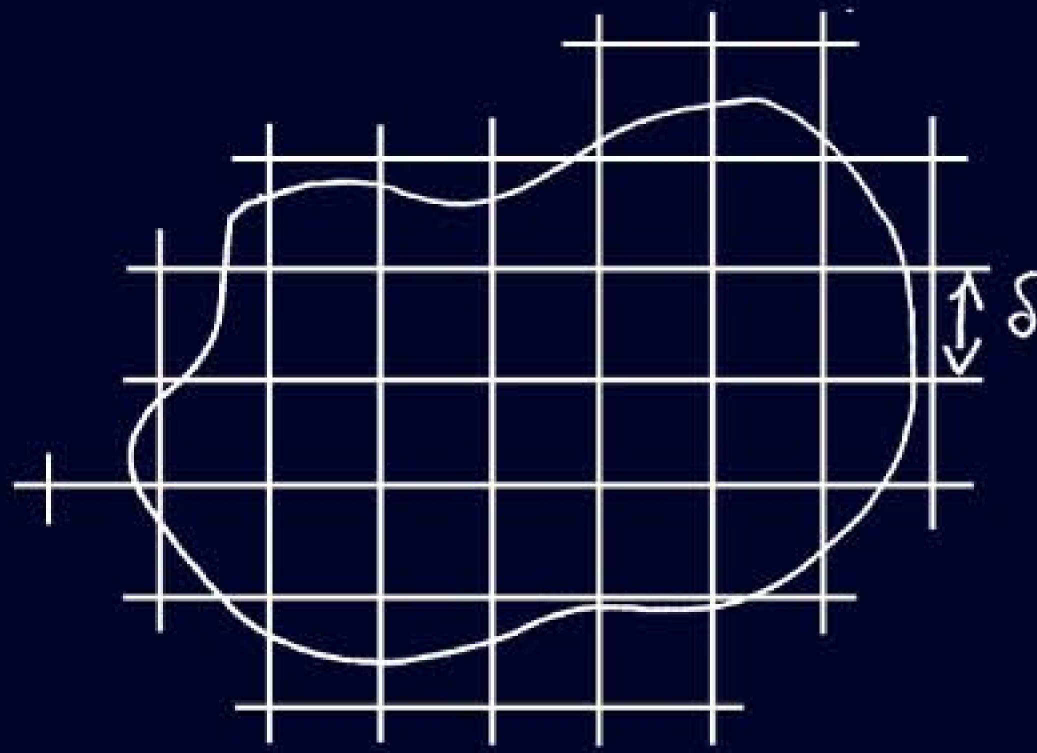
The Ising model on the square grid

Ω a Jordan domain, $\delta > 0$

Our graph $\Omega^\delta := \Omega \cap \delta \mathbb{Z}^2$

(largest connected component)



Impose + boundary condition

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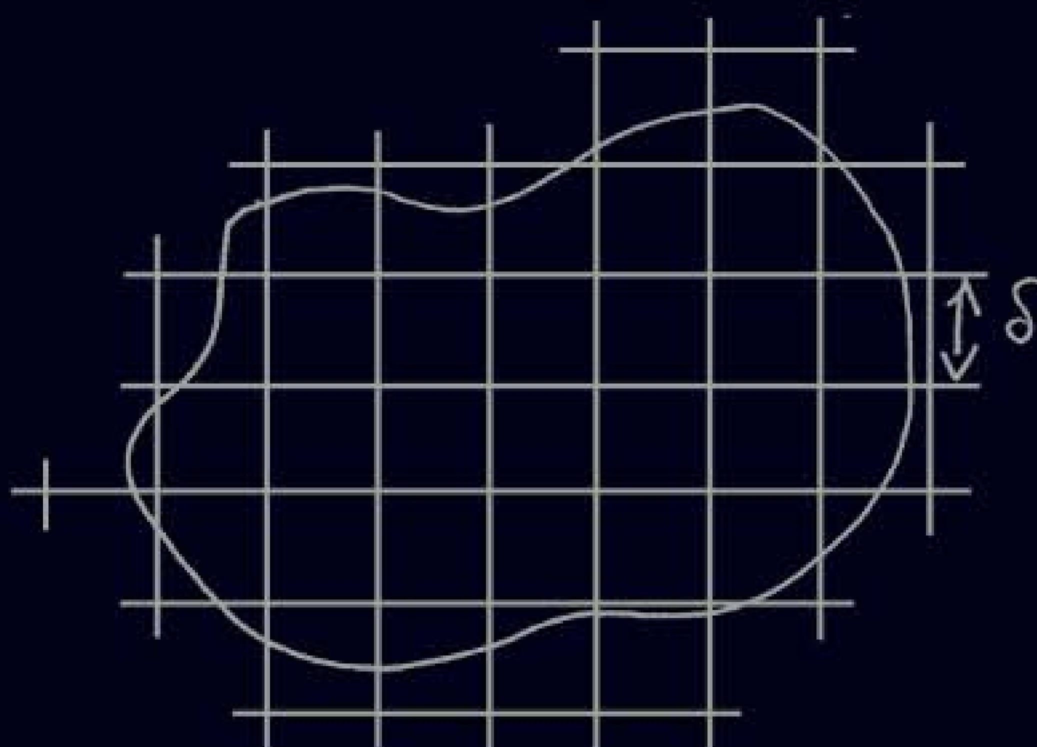
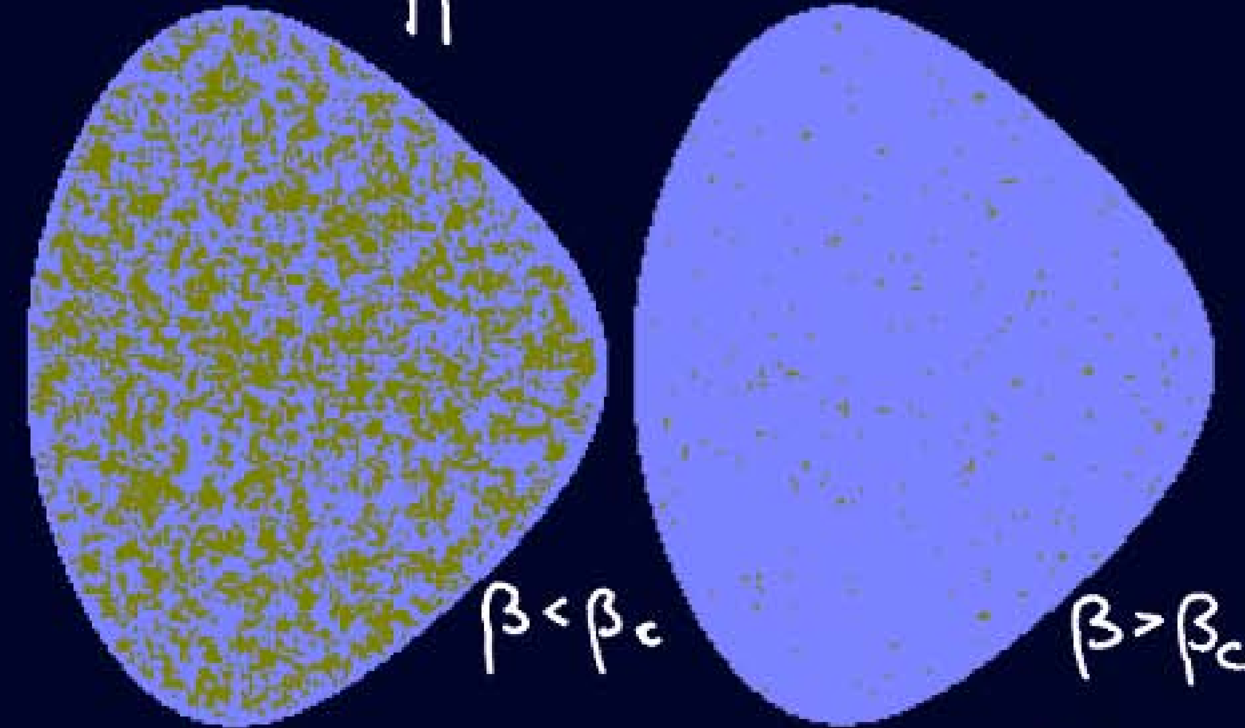


Impose + boundary condition

What happens as $\delta \to 0$?



$(\mathbb{P}[\vec{\sigma}] \propto e^{-\beta H(\vec{\sigma})})$

Thm (Onsager, 1944)

$$\mathbb{E}[\sigma_x^\delta] \xrightarrow{\delta \to 0} \begin{cases} 0 \text{ if } \beta \leq \beta_c \\ C(\beta) > 0 \text{ if } \beta > \beta_c \end{cases}$$

$$\beta_c = \frac{1}{2} \ln\left(\sqrt{2} + 1\right)$$

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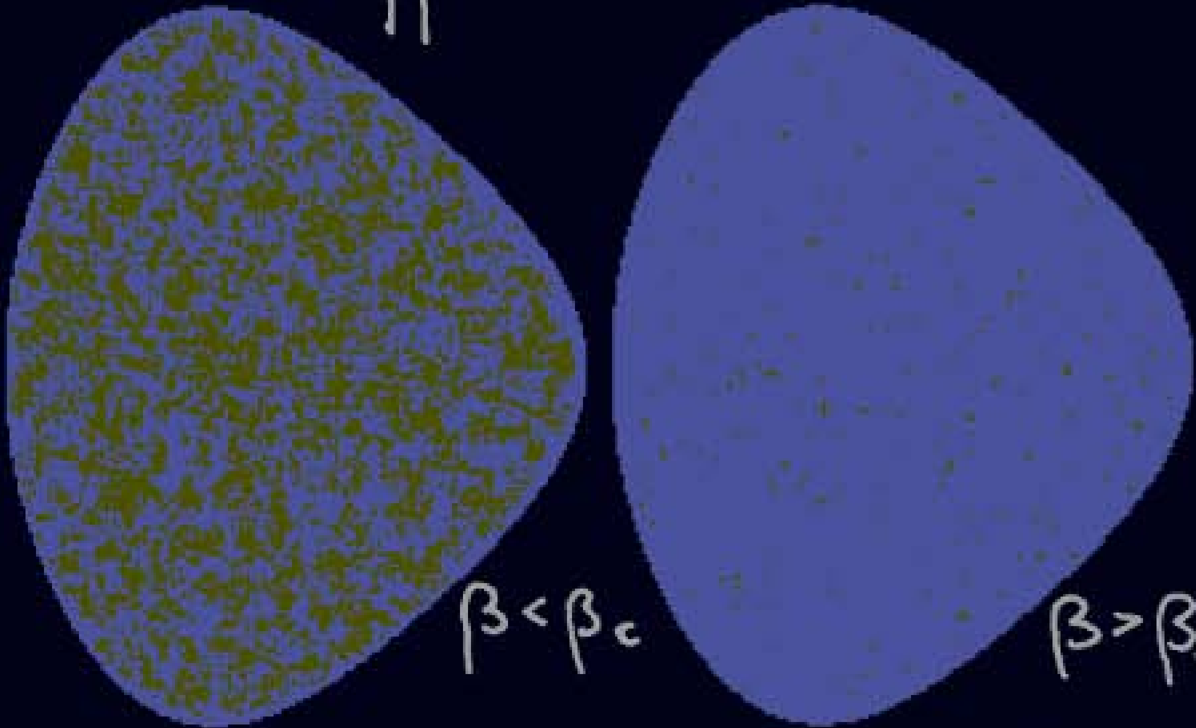

$\beta < \beta_c$ $\qquad$ $\beta > \beta_c$

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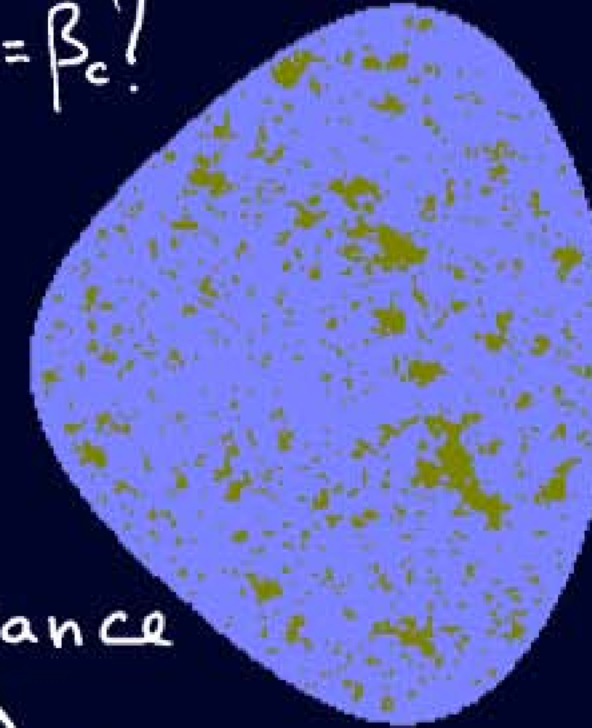

Universality

Conformal invariance

$\swarrow$ $\qquad$ $\searrow$

Conformal Field Theory $\qquad$ Schramm-Löwner Evolution

$\downarrow$ $\qquad$ $\downarrow$

Local observables (spin, energy) $\qquad$ Global geometry (interfaces)

$\searrow$ $\qquad$ $\swarrow$

Understand the effect of the shape of Ω

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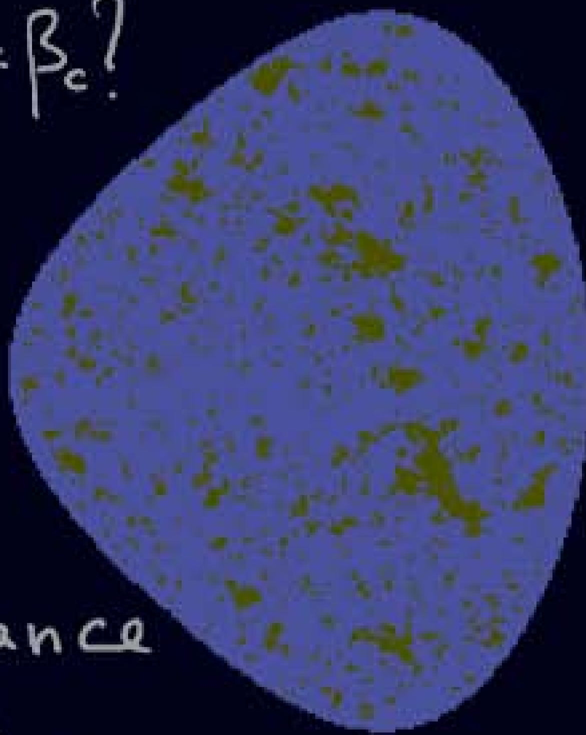

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→ Understand the effect of the shape of Ω

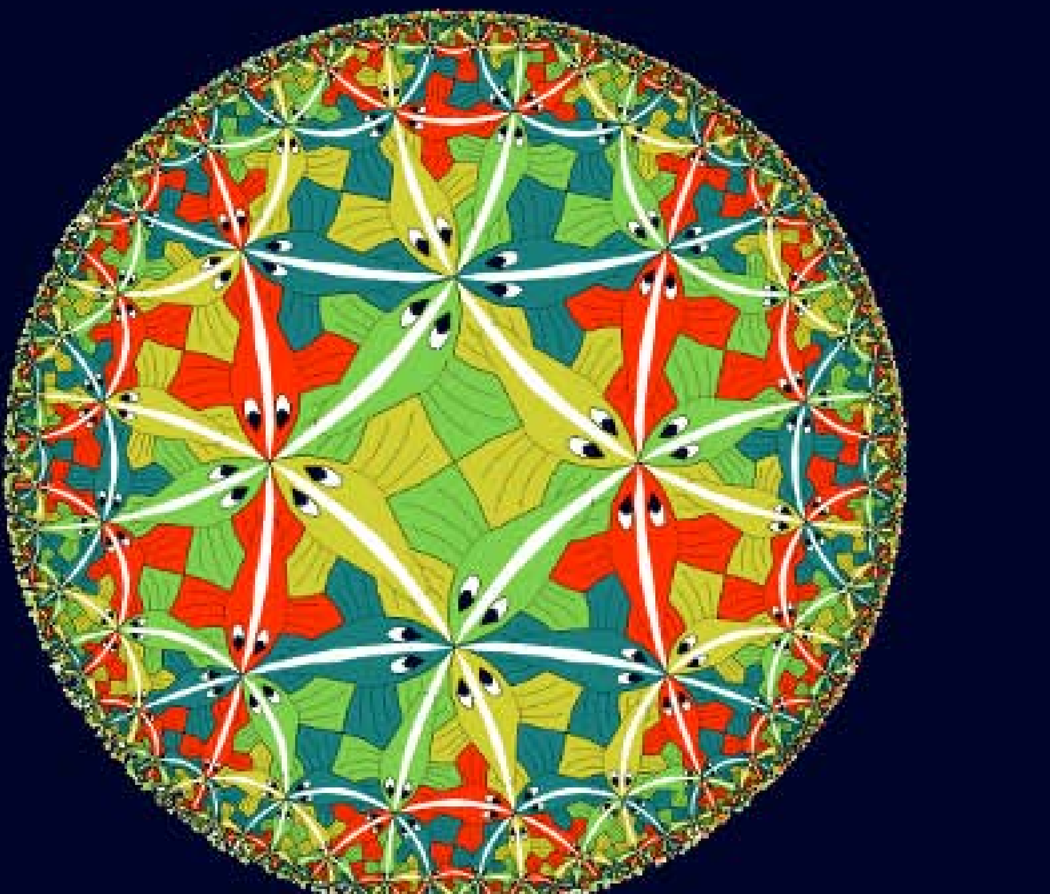
The energy density

The energy terms in

$$H(\vec{\sigma}) = -\sum_{\langle i,j\rangle \in E(\Omega)} \sigma_i \sigma_j$$

are given by the edges

How does the energy distribute across the domain?



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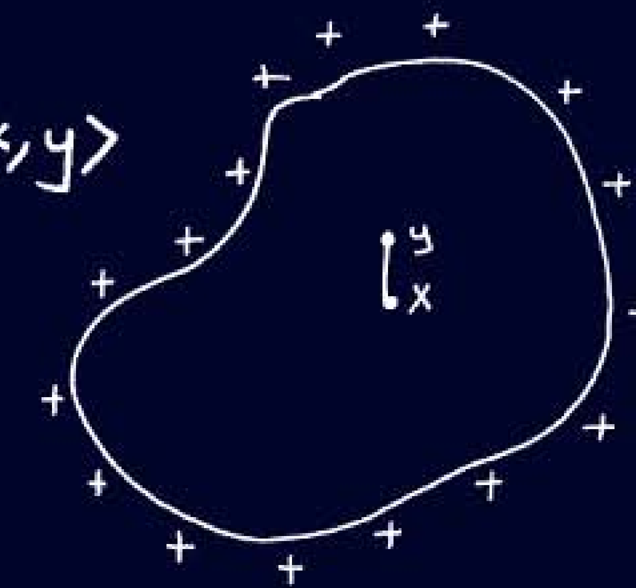

Thm (H., S.)

$$\frac{1}{\delta}\left(\mathbb{P}[\sigma_x = \sigma_y] - \frac{2+\sqrt{2}}{4}\right) \xrightarrow{\delta \to 0} \frac{1}{2\pi}\,\ell_\Omega(a)$$

ℓ_Ω is the hyperbolic metric element

If $\psi_\Omega : \Omega \longrightarrow D(0,1)$ is conformal

$$\ell_\Omega(a) := \frac{|\psi_\Omega'(a)|}{1 - |\psi_\Omega(a)|^2}$$

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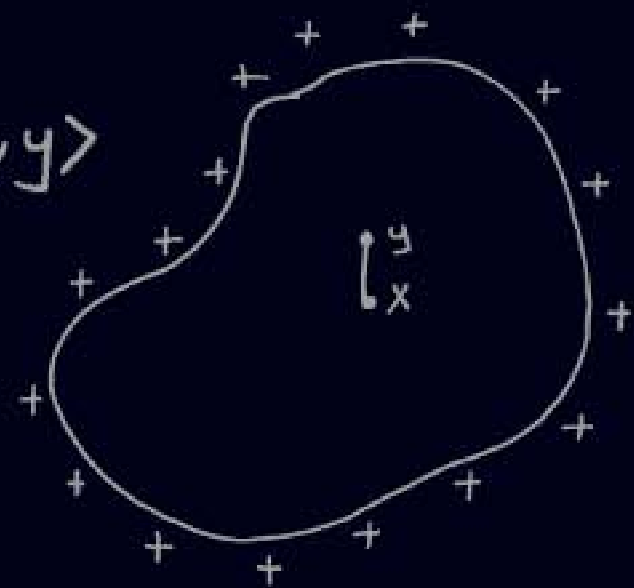

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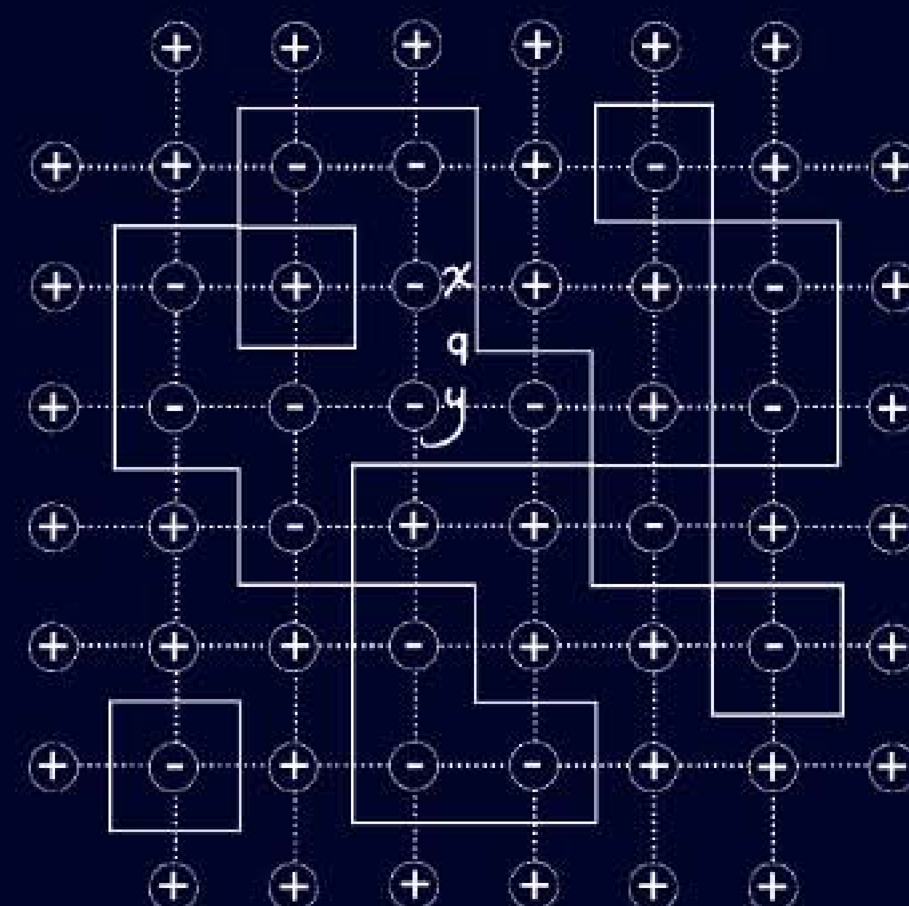
Sketch of the proof

(A) Contour representation

(B) Holomorphic deformation
(fermionic observable)

(C) Limit as $\delta \longrightarrow 0$



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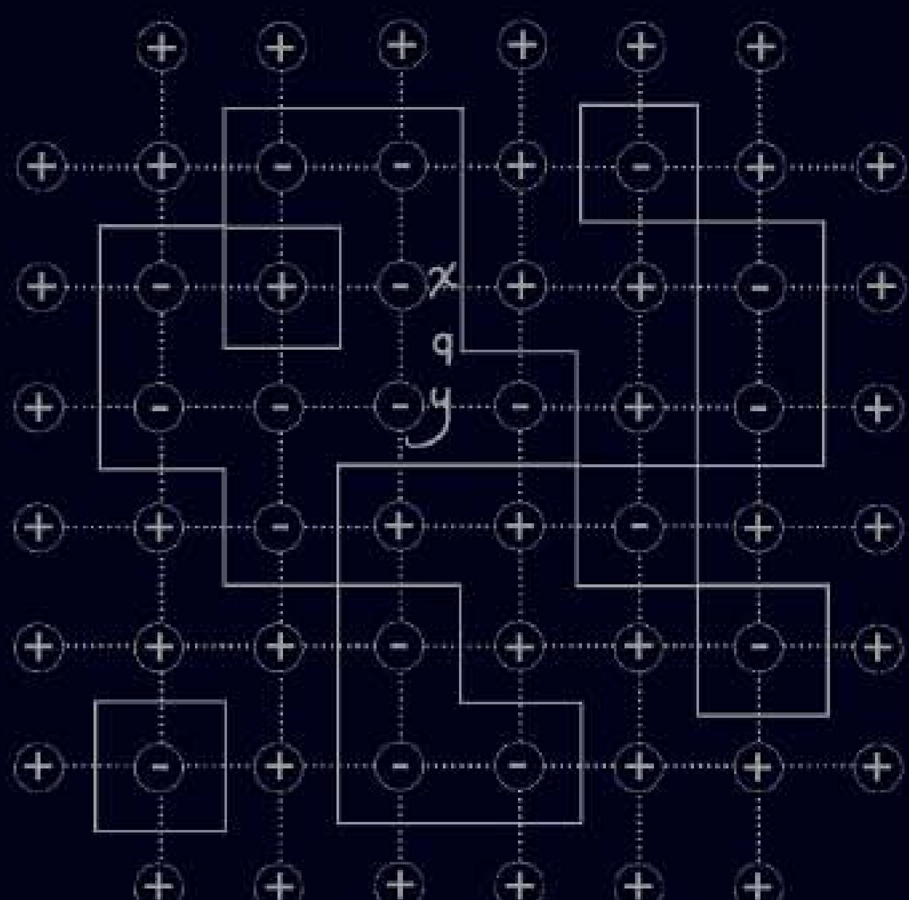

(A) Bijection $\{\pm 1\}^{V(\Omega^\delta)} \longleftrightarrow \mathcal{C}^\delta$

$\mathcal{C}^\delta := \{\gamma \subseteq E(\Omega^{\delta *}) : \text{coll. of closed loops}\}$

$$\mathbb{P}[\gamma] = \alpha^{\#\text{edges}(\gamma)} / Z^\delta$$

$\alpha = \alpha_c = \sqrt{2} - 1$

$$Z^\delta := \sum_{\gamma \in \mathcal{C}^\delta} \alpha^{\#\text{edges}(\gamma)}$$

$$\mathcal{C}_a^\delta := \{\gamma \in \mathcal{C}^\delta : a \notin \gamma\}$$

(identify a to its middlepoint)

$$Z_a^\delta := \sum_{\gamma \in \mathcal{C}_a^\delta} \alpha^{\#\text{edges}(\gamma)}$$

$$\mathbb{P}[\sigma_x = \sigma_y] = Z_a^\delta / Z^\delta$$

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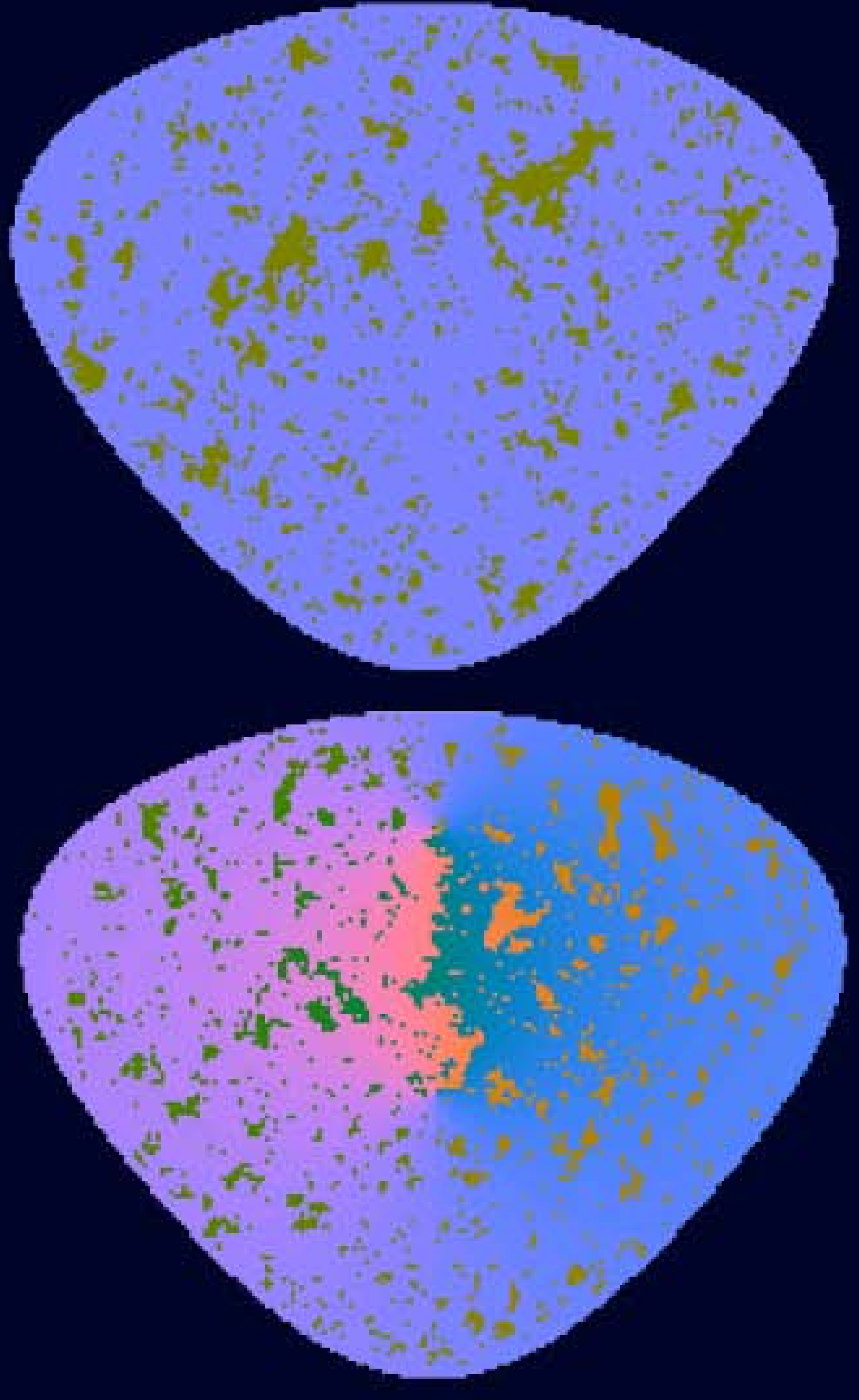

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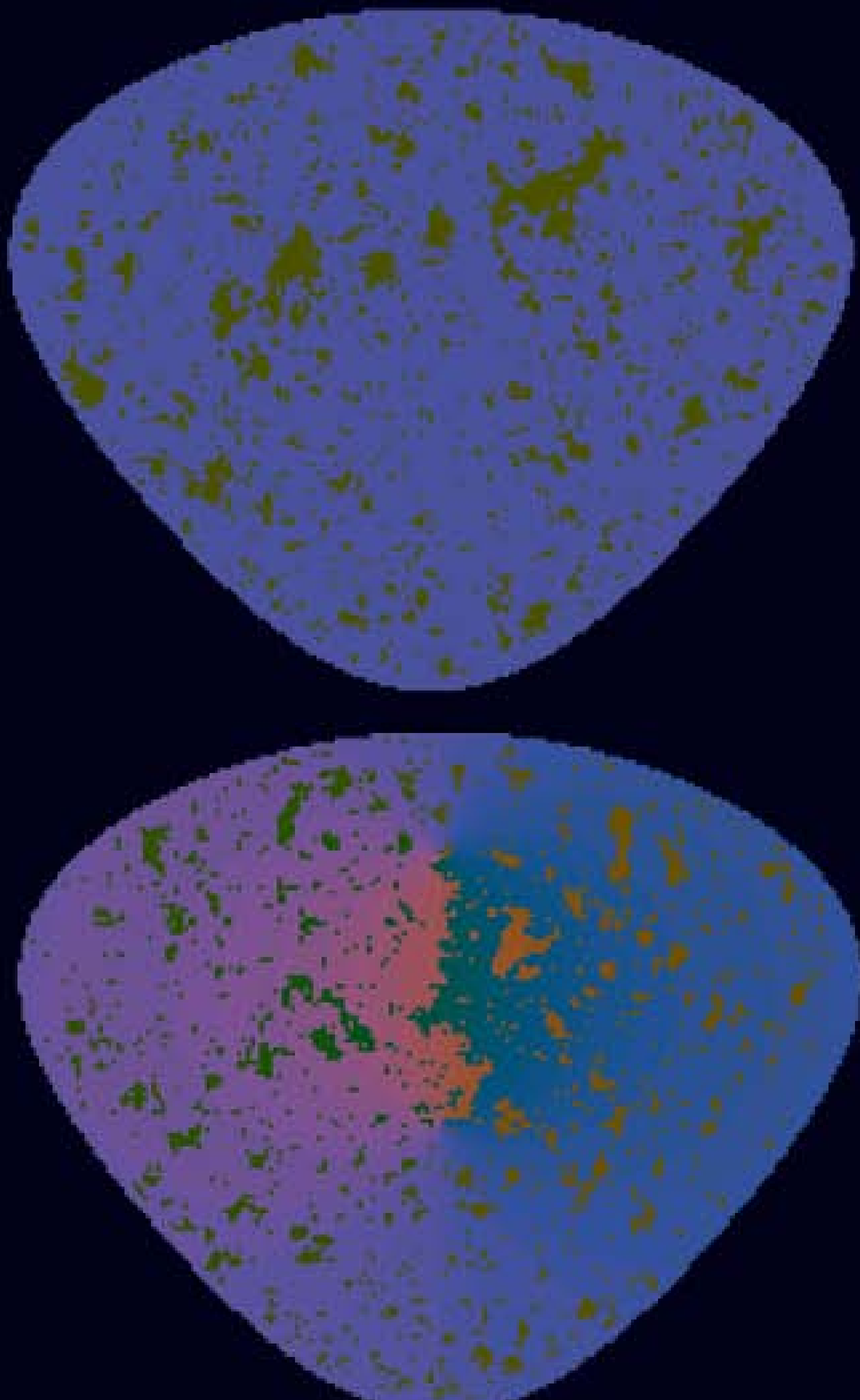
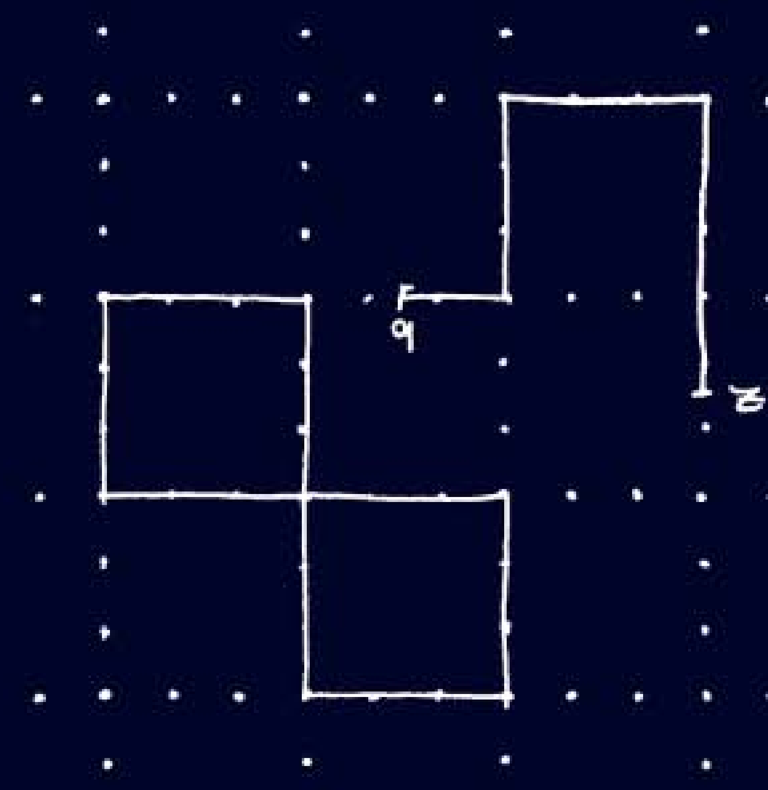


$\mathcal{C}^\delta_{a,z} :=$

$\{ \gamma \subseteq E(\Omega^\delta) : \gamma$ contains closed loops and a path from a to z starting from left to right at a $\}$

z: middlepoint of edge



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$\#$edges: 12

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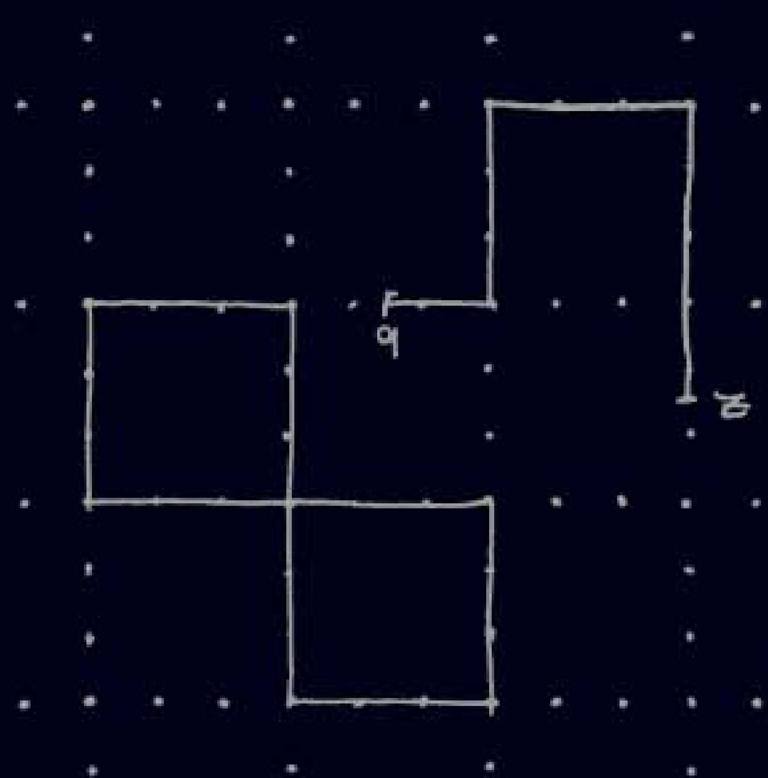

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(B) Holomorphic observable

W: winding number (i.e. total rotation) of the path from a to z (well-defined mod 4π)

$$f^{\delta}_{a}(z) := \frac{\sum_{\gamma \in \mathcal{C}^{\delta}_{a,z}} \alpha^{\#\mathrm{edges}(\gamma)} e^{-i\frac{W(\gamma)}{2}}}{Z^{\delta}}$$

$(\alpha = \sqrt{2} - 1)$

$$f^{\delta}_{a}(a) := \frac{Z^{\delta}_{a}}{Z^{\delta}}$$

(B) Holomorphic observable

W: winding number (i.e. total rotation) of the path from a to z (well-defined mod 4π)

$$f_a^\delta(z) := \frac{\sum_{\gamma \in C_{a,z}^\delta} \alpha^{\#\mathrm{edges}(\gamma)} e^{-i\frac{W(\gamma)}{2}}}{Z^\delta}$$

$(\alpha = \sqrt{2} - 1)$

$$f_a^\delta(a) := \frac{Z_a^\delta}{Z^\delta}$$

(B) Properties of $z \mapsto f_a^\delta(z)$

(1) Discrete holomorphic on $\Omega^\delta \setminus \{a\}$

Discrete Cauchy-Riemann equations

$$(\bar\partial^\delta f_a^\delta = 0)$$

(2) Discrete singularity at a

$$(\bar\partial^\delta f_a^\delta)(a) = \frac{1}{2}$$

(3) Boundary condition

$$f_a^\delta |_{\partial\Omega^\delta} \parallel \frac{1}{\sqrt{n}}$$

n: ext. normal

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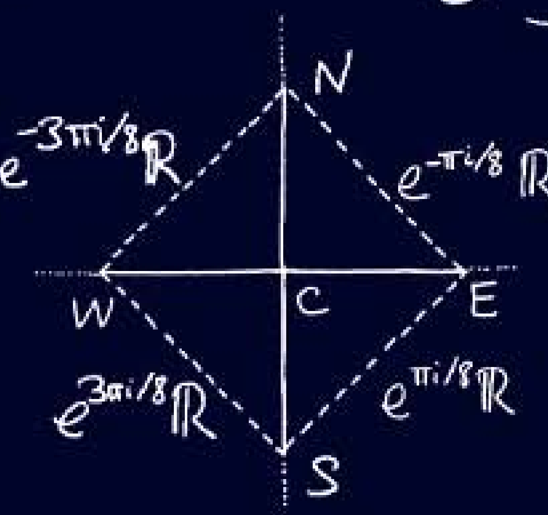
(B) Discrete analyticity



$$\mathrm{Proj}\left(f_a^\delta(E) - f_a^\delta(N); e^{-\pi i/8}\mathbb{R}\right) = 0$$

$$\mathrm{Proj}\left(f_a^\delta(N) - f_a^\delta(W); e^{-3\pi i/8}\mathbb{R}\right) = 0$$

$$\mathrm{Proj}\left(f_a^\delta(W) - f_a^\delta(S); e^{3\pi i/8}\mathbb{R}\right) = 0$$

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$$\rightsquigarrow \underbrace{f_a^\delta(E) - f_a^\delta(W) + i\left(f_a^\delta(N) - f_a^\delta(S)\right)}_{2\bar\partial^\delta f_a^\delta(C)} = 0$$

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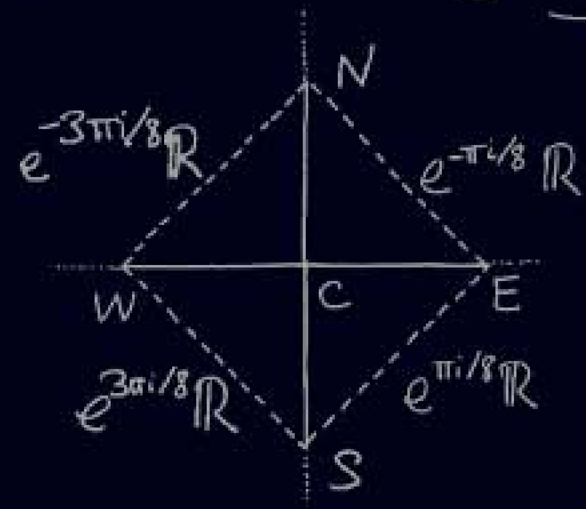

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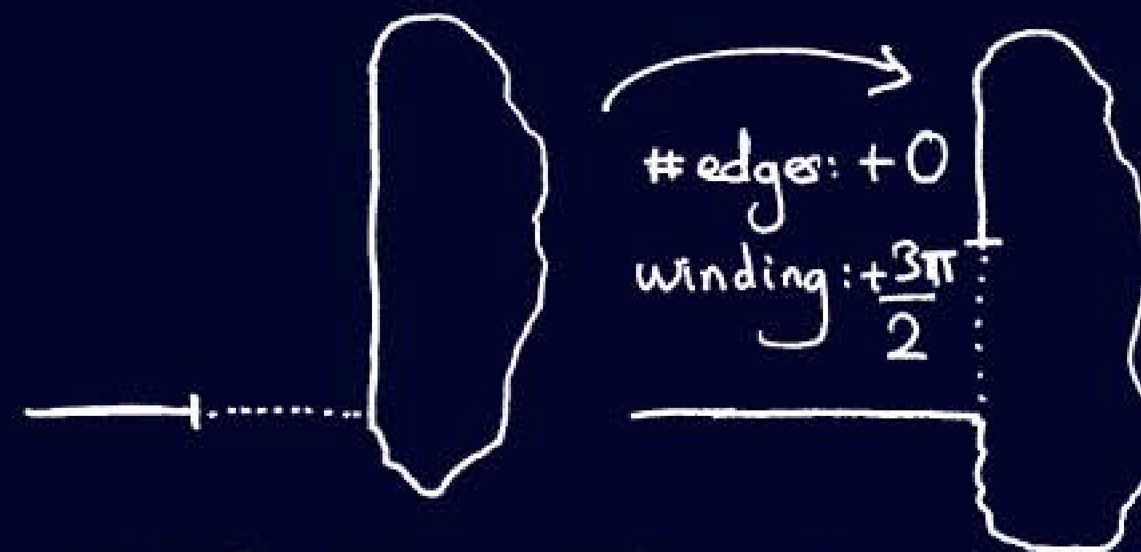
f_a^δ is discrete analytic on $\Omega\setminus\{a\}$

$$X\in\mathbb{R}:\ \mathrm{Proj}\left(X-\alpha e^{-\pi i/4}X;\, e^{-\frac{3\pi i}{8}}\mathbb{R}\right)=0$$



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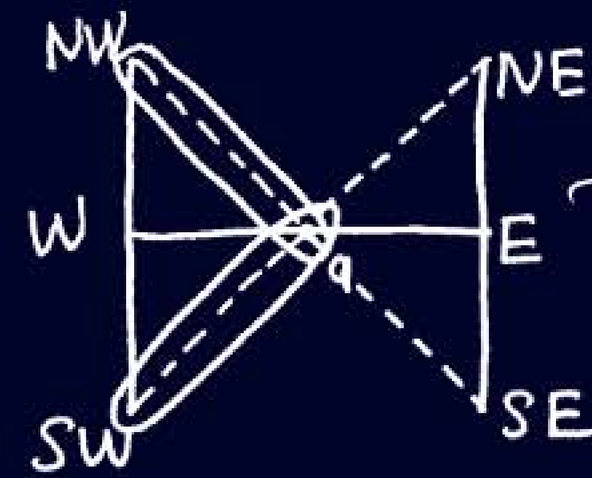
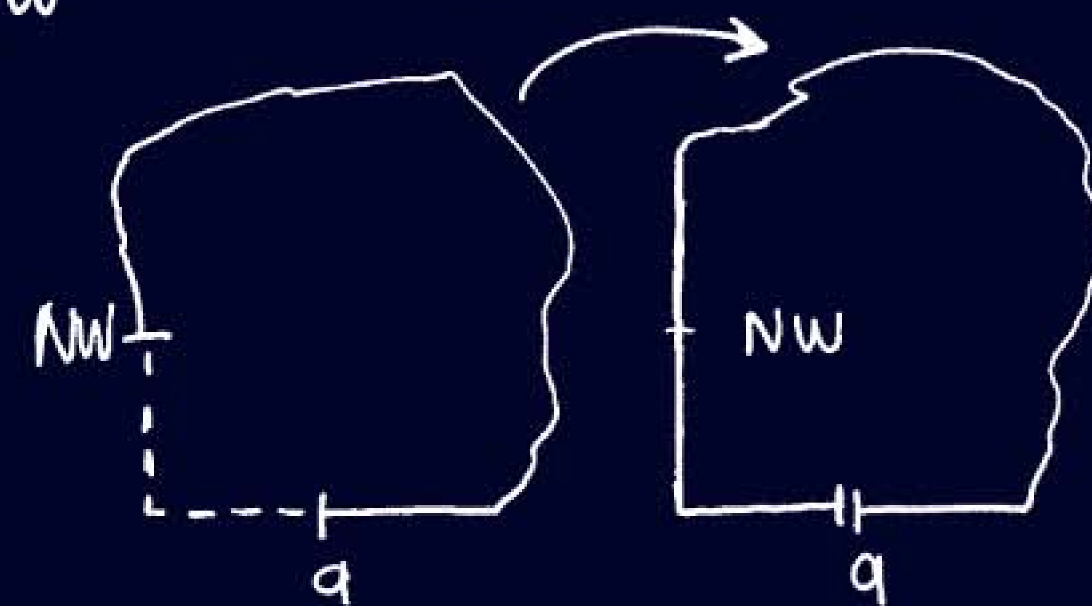


Bijections don't work at



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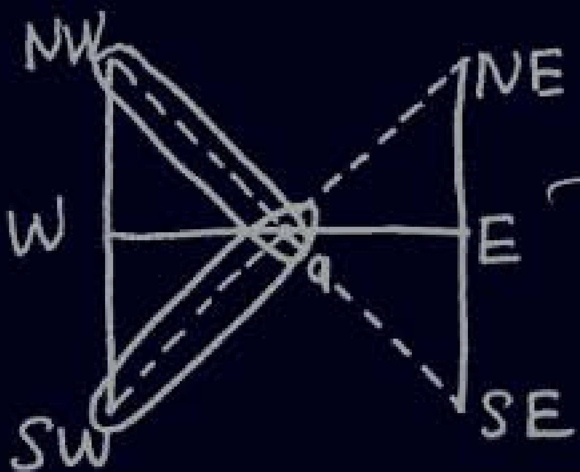

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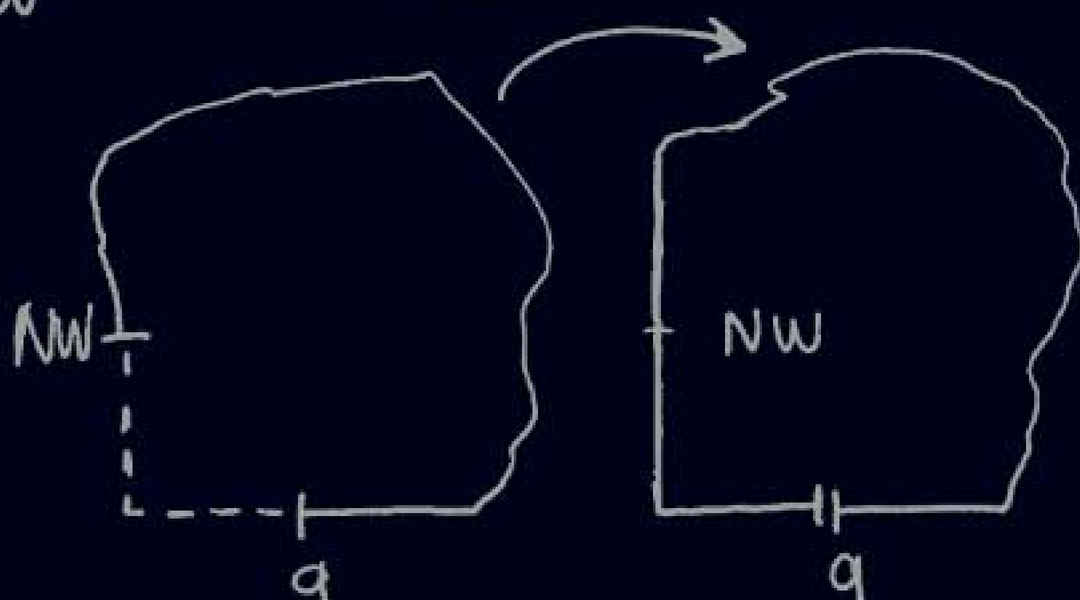

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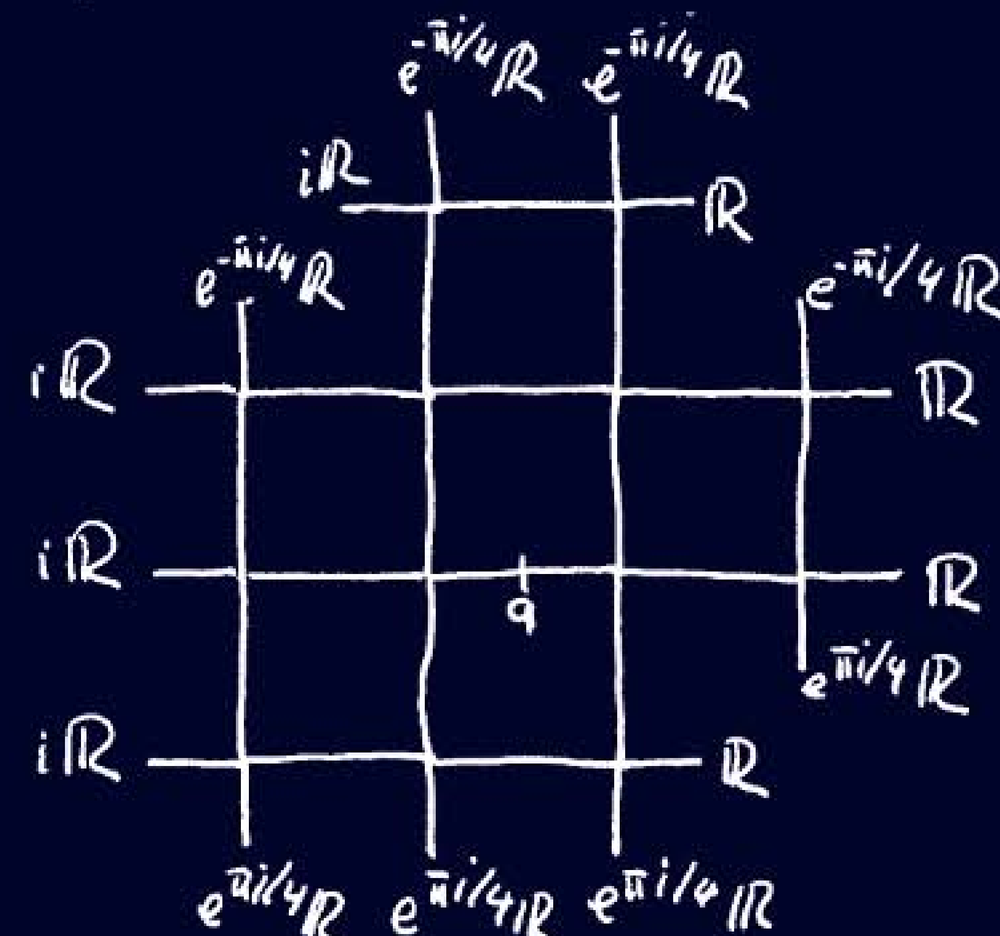
(B) Boundary condition

Winding determined up to $2k\pi$ on $\partial\Omega$

$\rightsquigarrow$ argument of weights of configurations determined up to $k\pi$



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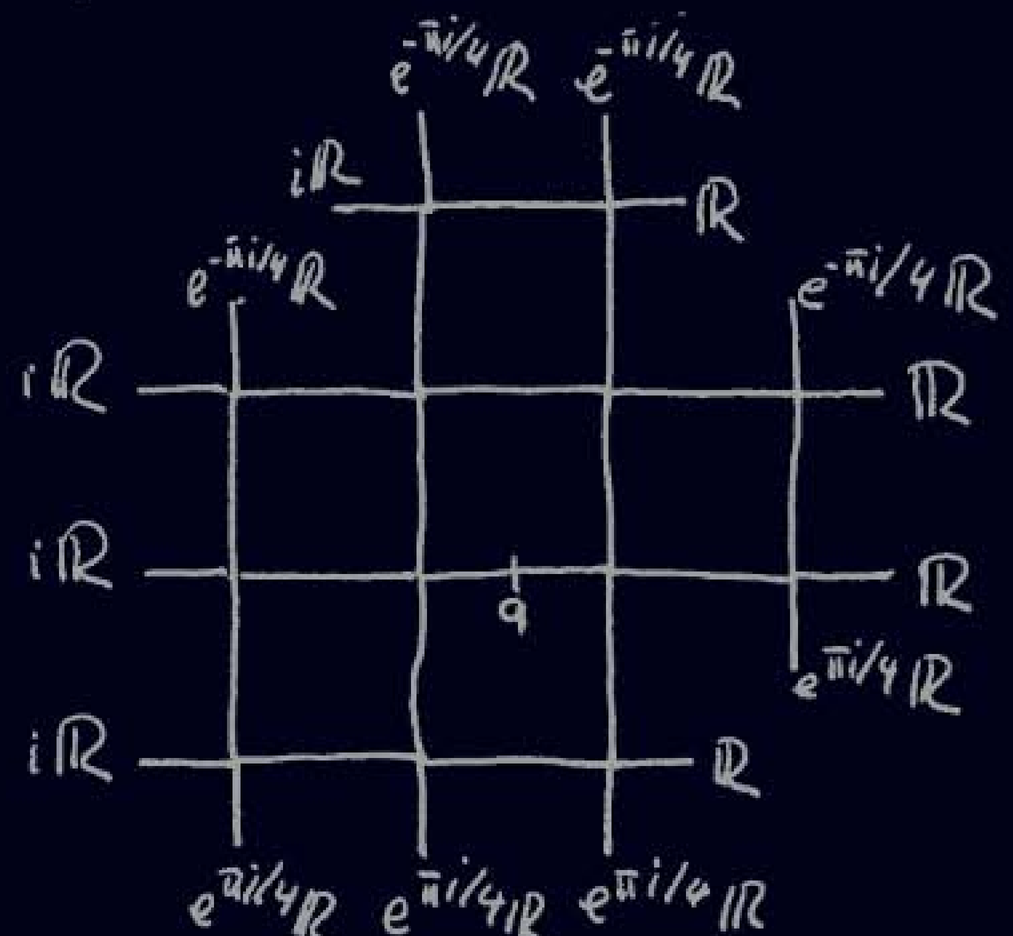

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(B) Derivation of the energy density

Conditions (1) and (2)

$$\Longleftrightarrow$$

$h_a^\delta := f_a^\delta - g_a^\delta$ discrete holomorphic on Ω^δ

- g_a^δ discrete $\bar\partial$ – Green function
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(C) Convergence results

$$\frac{1}{\delta} f_a^\delta(z) \longrightarrow \frac{\psi_a(z)+1}{2\pi\,\psi_a(z)} \sqrt{\psi_a'(a)\,\psi_a'(z)}$$

$$\left(\psi_a : \begin{array}{l}\Omega \longrightarrow D(0,1) \text{ conformal} \\ a \longmapsto 0 \\ \psi_a'(a) > 0\end{array}\right)$$

$$\frac{1}{\delta} g_a^\delta(z) \xrightarrow{\delta \to 0} \frac{1}{2\pi(z-a)} \quad \text{(Kenyon)}$$

$$\rightsquigarrow \frac{1}{\delta} h_a^\delta(a) \xrightarrow{\delta \to 0} \frac{1}{2\pi} \ell_\Omega(a)$$

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(1) Precompactness

Difficulty: control a function only knowing its argument on $\partial\Omega$

- control h_a^δ (we know g_a^δ converges)

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- integral trick: $f_a^\delta \parallel \sqrt{\frac{1}{n}}$

(Dirichlet) $\mathrm{Re}\left(\int (f_a^\delta)^2 \right)\Big|_{\partial\Omega} = cst$

Pole of order 1, residue $\frac{1}{2}$ with boundary condition: limit is uniquely determined

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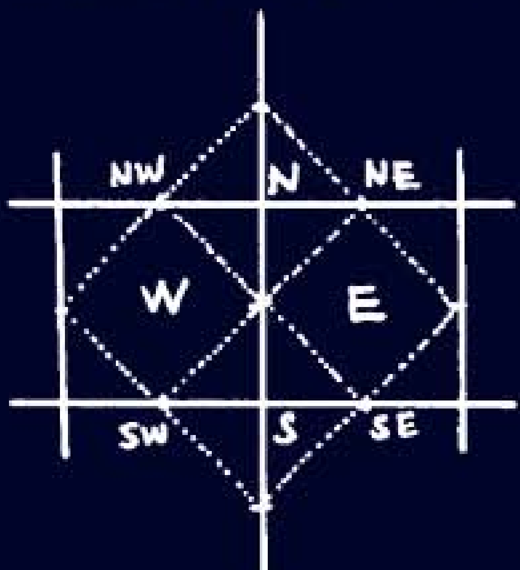
(C) Construction of $\mathcal{F}_a^\delta := \mathrm{Re}\left(\int (f_a^\delta)^2\right)$

- use projections on lines $e^{\pm \pi i/8}\mathbb{R}$, $e^{\pm 3\pi i/8}\mathbb{R}$
- $\mathcal{F}_a^\delta$ defined on the dual of the medial lattice.

$$\mathcal{F}_a^\delta(N) - \mathcal{F}_a^\delta(E) = \left|\mathrm{Proj}^2\left(f_a^\delta(NE); e^{\frac{\pi i}{8}}\mathbb{R}\right)\right|$$



$$\mathcal{F}_a^\delta(S) - \mathcal{F}_a^\delta(E) = \left|\mathrm{Proj}^2\left(f_a^\delta(SE); e^{-\frac{\pi i}{8}}\mathbb{R}\right)\right|$$

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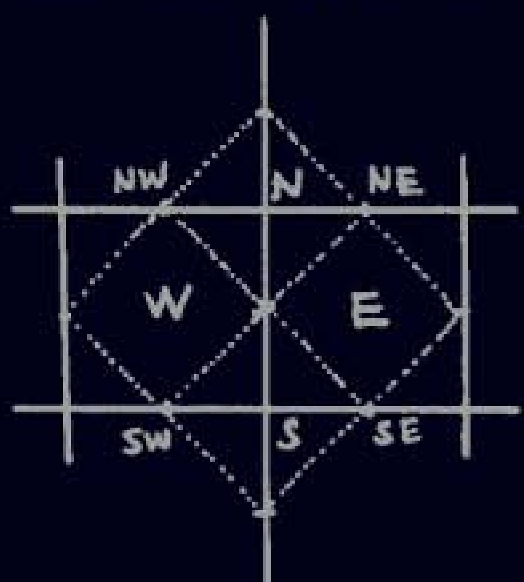
(C) Construction of $\mathcal{F}_a^\delta := \mathrm{Re}\left(\int (f_a^\delta)^2\right)$

- use projections on lines $e^{\pm\pi i/8}\mathbb{R}$, $e^{\pm 3\pi i/8}\mathbb{R}$
- $\mathcal{F}_a^\delta$ defined on the dual of the medial lattice.

$$\mathcal{F}_a^\delta(N)-\mathcal{F}_a^\delta(E) = \left|\mathrm{Proj}^2\left(f_a^\delta(NE); e^{\frac{\pi i}{8}}\mathbb{R}\right)\right|$$



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(C) Proof of precompactness

- Assume length$(\partial\Omega) < \infty$
- Control $\mathcal{H}_a^\delta := \mathrm{Re}\left(\int (h_a^\delta)^2\right)$

$\mathcal{H}_a^\delta$ is subharmonic

Boundary condition for f_a^δ

$$\Longrightarrow \quad \partial_n \mathcal{H}_a^\delta \le M\delta$$

$$0 \le \sum_\Omega \Delta \mathcal{H}_a^\delta = \sum_{\partial\Omega} (\partial_n \mathcal{H}_a^\delta)_+ - (\partial_n \mathcal{H}_a^\delta)_-$$

$$\sum (\partial_n \mathcal{H}_a^\delta)_- \le \sum (\partial_n \mathcal{H}_a^\delta)_+ \le M\,\mathrm{length}(\partial\Omega)$$

$$\sum |\partial_n \mathcal{H}_a^\delta| \le 2M\,\mathrm{length}(\partial\Omega)$$

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Write $\mathcal{H}_a^\delta =: H^\delta + S^\delta$

H^δ harmonic, $H^\delta|_{\partial\Omega^\delta} = \mathcal{H}_a^\delta|_{\partial\Omega^\delta}$

S^δ subharmonic, $S^\delta|_{\partial\Omega^\delta} = 0$

For S^δ: $\sum\sum \Delta^\delta S^\delta \leq M \,\mathrm{length}(\partial\Omega)$

$\rightsquigarrow S^\delta = \Delta S^\delta * G_\Delta^\delta$, G_Δ^δ: Green fct

For H^δ: $\sum |\partial_n H^\delta| \leq 3M \,\mathrm{length}(\partial\Omega)$

$\rightsquigarrow$ Use harmonic conjugate C

$\sum_{\partial\Omega} |\partial_t C| \leq 3M \,\mathrm{length}(\partial\Omega)$

(t: ccw tangent)

$\rightsquigarrow$ Bounded Dirichlet problem

Write $\mathcal{H}_\bullet^\delta =: \mathcal{H}^\delta + \mathcal{S}^\delta$

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For $\mathcal{S}^\delta$: $\sum\sum \Delta^\delta \mathcal{S}^\delta \le M \,\mathrm{length}(\partial\Omega)$

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⇝ Use harmonic conjugate $\mathcal{C}$

$\sum_{\partial\Omega} |\partial_t \mathcal{C}| \le 3M \,\mathrm{length}(\partial\Omega)$

(t: ccw tangent)

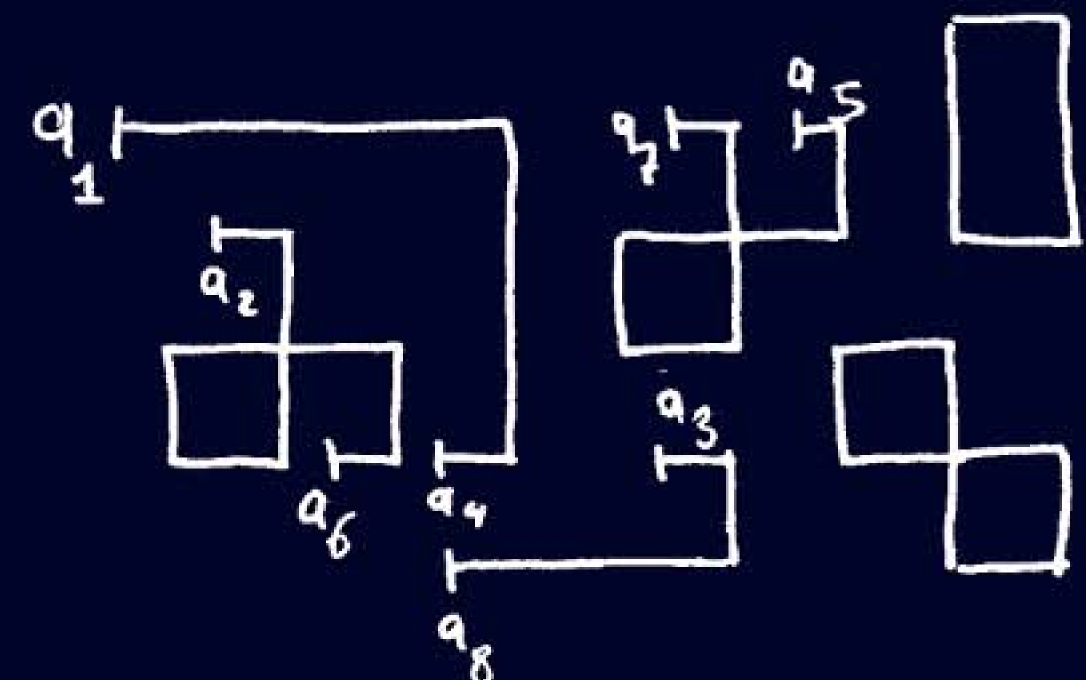
⇝ Bounded Dirichlet problem

N-point energy correlations

⇝ Define a $2N$-point holomorphic observable

$\mathcal{E}^\delta(a_1, \dots, a_{2N})$

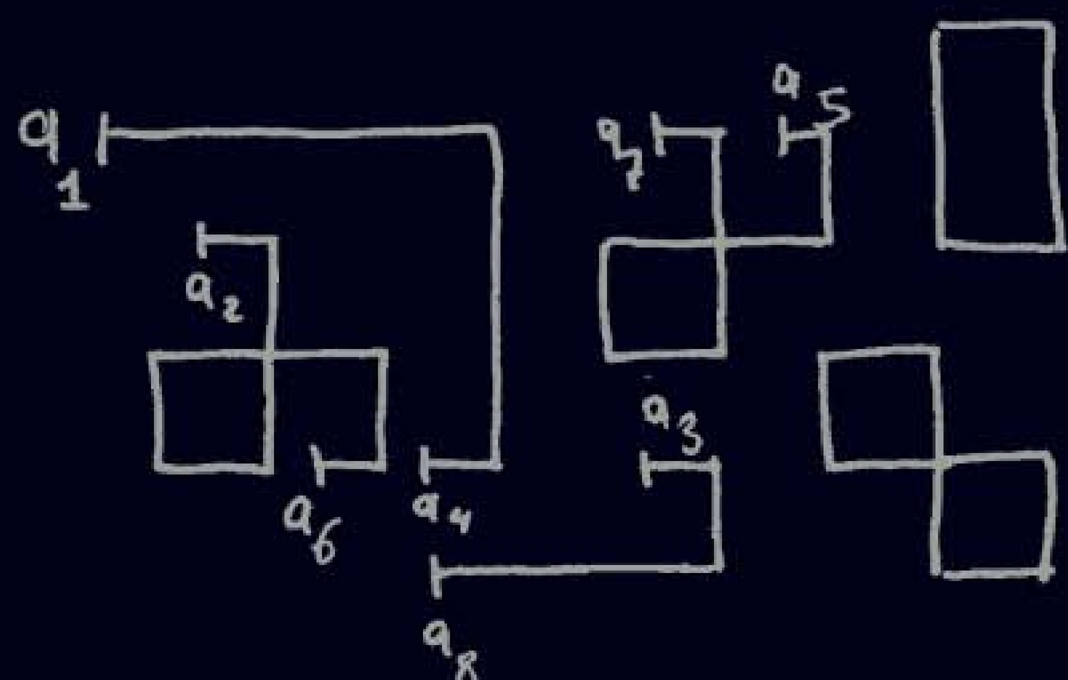
$:= \{\gamma \subseteq E(\Omega^\delta)$: γ contains closed loops and N paths linking $a_1, \dots, a_{2N}$ pairwise$\}$



$\mathcal{N}$-point energy correlations

$\rightsquigarrow$ Define a $2\mathcal{N}$-point holomorphic observable

$$\mathcal{C}^\delta(a_1,\dots,a_{2\mathcal{N}}) := \left\{ \begin{array}{l} \gamma \subseteq E(\Omega^\delta) :\ \gamma \text{ contains} \\ \text{closed loops and } \mathcal{N} \text{ paths} \\ \text{linking } a_1,\dots,a_{2\mathcal{N}} \text{ pairwise} \end{array} \right\}$$

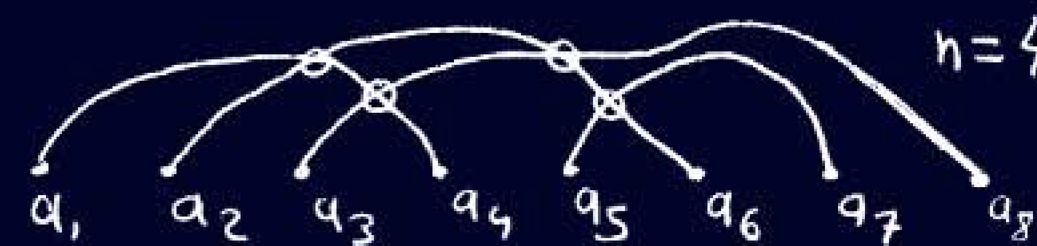


$2\mathcal{N}$-point observable

$$f^\delta(a_1,\dots,a_{2\mathcal{N}}) := \frac{\displaystyle\sum_{\gamma \in \mathcal{C}^\delta(a_1,\dots,a_{2\mathcal{N}})} \alpha^{\#\text{edges}} e^{-iW(\gamma)/2}}{Z^\delta}$$

$$W(\gamma) := 2\pi n + \sum_{\left\{ \begin{array}{l} (a_i,a_j) : i<j \\ a_i \overset{}{\underset{\gamma}{\leftrightarrow}} a_j \end{array} \right\}} W(a_i \to a_j)$$

n: number of crossings in the connection diagram



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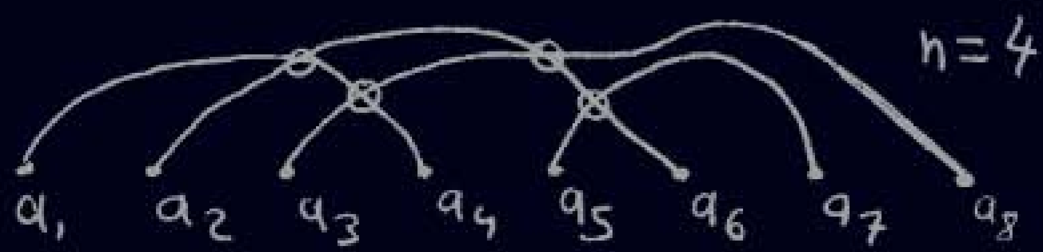

Thank you!