A renormalisation group analysis of the 4-dimensional weakly self-avoiding walk

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Abstract

We prove $|x|^{-2}$ decay of the critical two-point function for the continuous-time weakly self-avoiding walk on $\mathbb{Z}^4$. The walk two-point function is identified as the two-point function of a supersymmetric field theory with quartic self-interaction, and the field theory is then analysed using renormalisation group methods.

This is joint work with David Brydges.

Papers at `http://www.math.ubc.ca/~slade`, more in preparation.

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Self-avoiding walk

Discrete-time model: Let $\mathcal{S}_n(x)$ be the set of $\omega : \{0, 1, \ldots, n\} \to \mathbb{Z}^d$ with: $\omega(0) = 0$, $\omega(n) = x$, $|\omega(i+1) - \omega(i)| = 1$, and $\omega(i) \neq \omega(j)$ for all $i \neq j$. Let $\mathcal{S}_n = \cup_{x \in \mathbb{Z}^d} \mathcal{S}_n(x)$.

Let $c_n(x) = |\mathcal{S}_n(x)|$. Let $c_n = \sum_x c_n(x) = |\mathcal{S}_n|$. Easy: $c_n^{1/n} \to \mu$. Declare all walks in $\mathcal{S}_n$ to be equally likely: each has probability c_n^{-1}.

Two-point function: $G_z(x) = \sum_{n=0}^{\infty} c_n(x) z^n$, radius of convergence $z_c = \mu^{-1}$.

Predicted asymptotic behaviour:

$$c_n \sim A\mu^n n^{\gamma-1}, \quad \mathbb{E}_n|\omega(n)|^2 \sim Dn^{2\nu}, \quad G_{z_c}(x) \sim c|x|^{-(d-2+\eta)},$$

with universal critical exponents γ, ν, η obeying $\gamma = (2-\eta)\nu$.

Dimensions other than $d = 4$

Theorem. (Brydges–Spencer (1985); Hara–Slade (1992); Hara (2008)...)
For $d \geq 5$,

$$c_n \sim A\mu^n, \quad \mathbb{E}_n|\omega(n)|^2 \sim Dn, \quad G_{z_c}(x) \sim c|x|^{-(d-2)}, \quad \frac{1}{\sqrt{Dn}}\omega(\lfloor nt \rfloor) \Rightarrow B_t.$$

Proof uses lace expansion, requires $d > 4$.

$d = 2$. Prediction: $\gamma = \frac{43}{32}$, $\nu = \frac{3}{4}$, $\eta = \frac{5}{24}$,
Nienhuis (1982); Lawler–Schramm-Werner (2004) — connection with $\mathrm{SLE}_{8/3}$.

$d = 3$. Numerical: $\gamma \approx 1.16$, $\nu \approx 0.588$, $\eta \approx 0.031$.
E.g., Clisby (2011): $\nu = 0.587597(7)$.

Theorem (Madras 2012+): $\mathbb{E}_n|\omega(n)|^2 \geq \frac{1}{6}n^{4/3d}$, so $\nu \geq 2/(3d)$.

Not proved for $d = 2, 3, 4$: $\mathbb{E}_n|\omega(n)|^2 \leq O(n^{2-\epsilon})$, i.e., that $\nu < 1$.

Predictions for $d = 4$

Prediction is that upper critical dimension is 4, and asymptotic behaviour for $\mathbb{Z}^4$ has log corrections (e.g., Brézin, Le Guillou, Zinn-Justin 1973):

$$c_n \sim A\mu^n (\log n)^{1/4}, \quad \mathbb{E}_n|\omega(n)|^2 \sim Dn(\log n)^{1/4}, \quad G_{z_c}(x) \sim c|x|^{-2}.$$

The susceptibility and correlation length are defined by:

$$\chi(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad \frac{1}{\xi(z)} = -\lim_{n\to\infty} \frac{1}{n} \log G_z(ne_1).$$

For these the prediction is:

$$\chi(z) \sim \frac{A'|\log(1 - z/z_c)|^{1/4}}{1 - z/z_c}, \qquad \xi(z) \sim \frac{D'|\log(1 - z/z_c)|^{1/8}}{(1 - z/z_c)^{1/2}} \qquad \text{as } z \uparrow z_c.$$

Continuous-time weakly self-avoiding walk

A.k.a. discrete Edwards model.

Let E_0 denote the expectation for continuous-time nearest-neighbour simple random walk $X(t)$ on $\mathbb{Z}^d$ started from 0 (steps at events of rate-1 Poisson process).

Let
$$I(T) = \int_0^T \int_0^T \delta_{X(s),X(t)} ds\, dt.$$
Let $g \in (0,\infty)$, $\nu \in (-\infty,\infty)$. The *two-point function* is

$$G_{g,\nu}(x) = \int_0^\infty E_0\left(e^{-gI(T)}\, \mathbb{1}_{X(T)=x}\right) e^{-\nu T} dT$$

(compare $\sum_n c_n(x) z^n$).

Subadditivity argument $\Rightarrow$ $\exists \nu_c(g)$ s.t. *susceptibility* $\chi_g(\nu) = \sum_{x\in\mathbb{Z}^d} G_{g,\nu}(x)$ obeys

$$\chi_g(\nu) \begin{cases} < \infty & \nu > \nu_c(g) \\ = \infty & \nu < \nu_c(g). \end{cases}$$

Main result

Theorem (Brydges–Slade). Let $d \geq 4$. There exists $g_0 > 0$ such that for $0 < g \leq g_0$,

$$G_{g,\nu_c(g)}(x) = \frac{c_{d,g}}{|x|^{d-2}} + o\left(\frac{1}{|x|^{d-2}}\right).$$

Related results:

- weakly SAW on 4-dimensional hierarchical lattice (replacement of $\mathbb{Z}^4$ by a recursive structure well-suited to RG): Brydges–Evans–Imbrie (1992); Brydges–Imbrie (2003); and with different RG method Ohno–Hara (2012+).
- Iagolnitzer–Magnen (1994): related continuum model, different RG method
- weakly self-avoiding Lévy walk on $\mathbb{Z}^3$ ($\alpha = \frac{3+\epsilon}{2}$, $d_c = 3 + \epsilon$): Mitter–Scoppola (2008).

Outlook: Method of proof has potential to (but has not yet achieved—work in progress with Bauerschmidt and Brydges):

- prove same result also with small nearest-neighbour attraction
- prove same result for a particular spread-out model of *discrete-time* strictly self-avoiding walk with exponentially decaying step weights
- prove logarithmic corrections for susceptibility and correlation length for $d = 4$

Finite-volume approximation

Fix $g > 0$.

Standard methods (Simon–Lieb inequality) show that

$$G_{\nu_c}(x) = \lim_{\nu \downarrow \nu_c} \lim_{\Lambda \uparrow \mathbb{Z}^d} G_{\Lambda,\nu}(x),$$

where $\Lambda = \mathbb{Z}^d / R\mathbb{Z}$ is a torus approximating $\mathbb{Z}^d$ and

$$G_{\Lambda,\nu}(x) = \int_0^\infty E_0^\Lambda \left(e^{-gI_\Lambda(T)} \mathbb{1}_{X(T)=x} \right) e^{-\nu T} dT,$$

with E_0^Λ the expectation for the continuous-time simple random walk on Λ.

Thus we can work in finite volume, and slightly subcritical, but must maintain sufficient control to take the limits.

Functional integral representation

Let $\phi : \Lambda \to \mathbb{C}$, with complex conjugate $\bar{\phi}$.
Let Δ denote the discrete Laplacian on Λ, i.e., $\Delta\phi_x = \sum_{y:|y-x|=1}(\phi_y - \phi_x)$.
Let

$$\tau_x = \phi_x\bar{\phi}_x + \tfrac{1}{2\pi i} d\phi_x \wedge d\bar{\phi}_x,$$

$$\tau_{\Delta,x} = \frac{1}{2}\Big(\phi_x(-\Delta\bar{\phi})_x + \tfrac{1}{2\pi i} d\phi_x \wedge (-\Delta d\bar{\phi})_x + \text{c.c.}\Big),$$

where $\wedge$ is the standard anti-commutative wedge product.

Theorem.

$$G_{\Lambda,\nu}(x) = \int_{\mathbb{C}^\Lambda} e^{-\sum_{u\in\Lambda}(\tau_{\Delta,u}+g\tau_u^2+\nu\tau_u)}\bar{\phi}_0\phi_x.$$

RHS is the two-point function of a supersymmetric field theory with boson field $(\phi, \bar{\phi})$ and fermion field $(d\phi, d\bar{\phi})$.

(Parisi–Sourlas 1980; McKane 1980; Luttinger 1983; Dynkin 1983; Le Jan 1987; Brydges–Evans–Imbrie 1992; Brydges–Imbrie 2003; Brydges–Imbrie–Slade 2009).

Meaning of the integral

The definition of an integral such as

$$G_{\Lambda,\nu}(x) = \int_{\mathbb{C}^\Lambda} e^{-\sum_{u\in\Lambda}(\tau_{\Delta,u}+g\tau_u^2+\nu\tau_u)}\bar{\phi}_0\phi_x$$

is as follows (very closely related to Grassmann integration):

• expand entire integrand in power series about degree-zero part (*finite* sum), e.g.,

$$e^{\tau_u} = e^{\phi_u\bar{\phi}_u+\frac{1}{2\pi i}d\phi_u d\bar{\phi}_u} = e^{\phi_u\bar{\phi}_u}\left(1+\tfrac{1}{2\pi i}d\phi_u d\bar{\phi}_u\right),$$

• keep only terms with one factor $d\phi_x$ and one $d\bar{\phi}_x$ for each $x \in \Lambda$,
• write $\phi_x = u_x + iv_x$, $\bar{\phi}_x = u_x - iv_x$ and similarly for differentials,
• then use anti-commutativity to rearrange the differentials to $\prod_{x\in\Lambda} du_x dv_x$,
• and finally perform Lebesgue integral over $\mathbb{R}^{2|\Lambda|}$.

Such integrals have nice properties. Let $S(\Lambda) = \sum_{u\in\Lambda}(\tau_{\Delta,u} + m^2\tau_u)$. Then:

$$\int e^{-S(\Lambda)}\bar{\phi}_0\phi_x = (-\Delta_\Lambda + m^2)^{-1}(0,x), \qquad \int e^{-S(\Lambda)}F(\tau) = F(0).$$

Now we study the integral and forget about the walks.

Auxiliary variables and Gaussian approximation

For $m^2 > 0$ and $z_0 > -1$, let

$$G_\Lambda(m^2, g_0, \nu_0, z_0) = \int e^{-S(\Lambda) - \widetilde{V}_0(\Lambda)} \bar{\phi}_0 \phi_x, \qquad \text{where}$$

$$S(\Lambda) = \sum_{u \in \Lambda} \left(\tau_{\Delta,u} + m^2 \tau_u\right), \qquad \widetilde{V}_0(\Lambda) = \sum_{u \in \Lambda} \left(g_0 \tau_u^2 + \nu_0 \tau_u + z_0 \tau_{\Delta,u}\right).$$

Change of variable $\phi_x \mapsto \sqrt{1 + z_0}\phi_x$ gives

$$G_\Lambda(m^2, g_0, \nu_0, z_0) = \tfrac{1}{1+z_0} G_\Lambda^{\text{old}} \left(\tfrac{g_0}{(1+z_0)^2}, \tfrac{\nu_0 + m^2}{1+z_0} \right).$$

Case of $\widetilde{V}_0 = 0$: $G_\Lambda(m^2, 0, 0, 0) = (-\Delta_\Lambda + m^2)^{-1}(a, b)$ so

$$\lim_{m^2 \downarrow 0} \lim_{\Lambda \uparrow \mathbb{Z}^d} G_\Lambda(m^2, 0, 0, 0) = (-\Delta_{\mathbb{Z}^d})^{-1}(0, x) \sim c_0 |x|^{-(d-2)}.$$

Objective: given $g_0 > 0$ show that there exist ν_0, z_0 (yield ν_c and c) such that

$$\lim_{m^2 \downarrow 0} \lim_{\Lambda \uparrow \mathbb{Z}^d} G_\Lambda(m^2, g_0, \nu_0, z_0) \sim c(g_0) |x|^{-(d-2)}.$$

External field

Introducing an *external field* $\sigma \in \mathbb{C}$, let

$$S(\Lambda) = \sum_{u \in \Lambda} \left(\tau_{\Delta,u} + m^2 \tau_u \right), \qquad V_0(\Lambda) = \widetilde{V}_0(\Lambda) - \sigma \bar{\phi}_0 - \bar{\sigma} \phi_x.$$

Then

$$G_{\Lambda,\nu}(m^2, g_0, \nu_0, z_0) = \int e^{-S(\Lambda) - \widetilde{V}_0(\Lambda)} \bar{\phi}_0 \phi_x = \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \bar{\sigma}} \bigg|_0 \int e^{-S(\Lambda) - V_0(\Lambda)}.$$

Thus we are led to study the integral

$$\frac{\partial}{\partial \sigma} \frac{\partial}{\partial \bar{\sigma}} \bigg|_0 \int e^{-S(\Lambda) - V_0(\Lambda)}.$$

Gaussian super-expectation

For a positive definite $\Lambda \times \Lambda$ matrix C, and $A = C^{-1}$, let

$$S_A(\Lambda) = \sum_{x,y\in\Lambda} \left(\phi_x A_{xy} \bar{\phi}_x + \tfrac{1}{2\pi i} d\phi_x A_{xy} d\bar{\phi}_y \right)$$

and, for a form F (lin. comb. of products of $d\phi$ and $d\bar{\phi}$ with coeffs depending on ϕ),

$$\mathbb{E}_C F = \int_{\mathbb{C}^\Lambda} e^{-S_A(\Lambda)} F.$$

Then $\mathbb{E}_C 1 = 1$. With $C = (-\Delta_\Lambda + m^2)^{-1}$, the integral of interest is

$$G_{\Lambda,\nu}(m^2, g_0, \nu_0, z_0) = \frac{\partial}{\partial\sigma}\frac{\partial}{\partial\bar{\sigma}}\bigg|_0 \int e^{-S_A(\Lambda)-V_0(\Lambda)} = \frac{\partial}{\partial\sigma}\frac{\partial}{\partial\bar{\sigma}}\bigg|_0 \mathbb{E}_C e^{-V_0(\Lambda)}.$$

Much of the standard theory of Gaussian integration carries over to this setting, even though $\mathbb{E}_C$ involves Grassmann integration and will take values in a space of differential forms.

Convolution integrals and progressive integration

Recall that a random variable $X \sim N(0, \sigma_1^2 + \sigma_2^2)$ has the same distribution as $X_1 + X_2$ where $X_1 \sim N(0, \sigma_1^2)$ and $X_2 \sim N(0, \sigma_2^2)$ are independent, and

$$Ef(X) = E_2\left(E_1(f(X_1 + X_2)|X_2)\right).$$

This finds expression for $\mathbb{E}_C$ via:

$$\mathbb{E}_{C_2+C_1} F = \mathbb{E}_{C_2} \circ \mathbb{E}_{C_1} \theta F,$$

where

$$(\theta F)(\phi, \xi, d\phi, d\xi) = F(\phi + \xi, d\phi + d\xi),$$

$\mathbb{E}_{C_1}$ integrates out ξ and $d\xi$, leaving ϕ and $d\phi$ fixed, and
$\mathbb{E}_{C_2}$ integrates out ϕ and $d\phi$.

More generally,

$$\mathbb{E}_{C_1+\cdots+C_N} = \mathbb{E}_{C_N} \circ \cdots \circ \mathbb{E}_{C_2}\theta \circ \mathbb{E}_{C_1}\theta.$$

Finite-range decomposition of covariance

Theorem (Brydges–Guadagni–Mitter 2004). Let $d > 2$. Fix a large L and suppose $|\Lambda| = L^{Nd}$. Let $C = (-\Delta_\Lambda + m^2)^{-1}$. Then there exist $C_1, \ldots, C_N$ such that:

- $C = \sum_{j=1}^{N} C_j$
- C_j positive definite,
- $C_j(x,y) = 0$ if $|x-y| \geq \frac{1}{2}L^j$
- for $j = 1, \ldots, N-1$ and with $[\phi] = \frac{1}{2}(d-2)$ (so $[\phi] = 1$ for $d = 4$),

$$|C_j(x,x)| \leq O(L^{-2[\phi]j}),$$

$$|\nabla_x^\alpha \nabla_y^\beta C_j(x,x)| \leq O(L^{-(2[\phi]+|\alpha|_1+|\beta|_1)j}).$$

The dynamical system

The covariance decomposition induces a field decomposition and allows the expectation to be done progressively:

$$\phi = \sum_{j=1}^{N} \xi_j, \qquad d\phi = \sum_{j=1}^{N} d\xi_j, \qquad \mathbb{E}_C = \mathbb{E}_{C_N} \circ \cdots \circ \mathbb{E}_{C_2} \circ \mathbb{E}_{C_1}.$$

Write $\phi_j = \sum_{i=j+1}^{N} \xi_i$, with $\phi_0 = \phi$, $\phi_N = 0$. Then $\phi_j = \phi_{j+1} + \xi_{j+1}$. Let

$$Z_0 = Z_0(\phi, d\phi) = e^{-V_0(\Lambda)},$$

and

$$Z_j(\phi_j, d\phi_j) = \mathbb{E}_{C_j} \circ \cdots \circ \mathbb{E}_{C_1} Z_0.$$

In particular, our goal is to compute

$$Z_N = \mathbb{E}_C Z_0 = \mathbb{E}_C e^{-V_0(\Lambda)} \qquad \text{in the limit } N \to \infty,$$

and we are led to study the dynamical system:

$$Z_{j+1} = \mathbb{E}_{C_{j+1}} Z_j.$$

Relevant, marginal, irrelevant directions

Recall that $[\phi] = \frac{d-2}{2}$. The covariance estimates suggest that $\phi_{j,x} \approx L^{-j[\phi]}$ and that this field is approximately constant over distance L^j. Thus, for a block B of side L^j,

$$\sum_{x \in B} |\phi_{j,x}|^p \approx |B| L^{-jp[\phi]} = L^{j(d-p[\phi])}.$$

For $d = 4$ the RHS is $L^{j(4-p)}$, which is
relevant for $p < 4$, *marginal* for $p = 4$, *irrelevant* for $p > 4$.

Taking symmetries and derivatives into account, the relevant and marginal monomials are:

$$\tau, \qquad \tau_\Delta, \qquad \tau^2.$$

The role of $d = 4$: τ^2 is *relevant* for $d < 4$ and *irrelevant* for $d > 4$:

$$\sum_{x \in B} |\phi_{j,x}|^4 = L^{j(4-d)}.$$

The RG map

Up to an error that must be controlled, seek approximation $Z_j \approx e^{-V_j(\Lambda)}$, with

$$V_j(\Lambda) = \sum_{u\in\Lambda} \Big((g_j \tau_u^2 + \nu_j \tau_u + z_j \tau_{\Delta,u}) + \lambda_j(\sigma\bar{\phi}_0 + \bar{\sigma}\phi_x) + q_j \sigma\bar{\sigma} \Big)$$

The error in the approximation is described by a family of forms, written $K_j = (K_j(X))$:

$$Z_j = \sum_{X\subset\Lambda} e^{-V_j(\Lambda\setminus X)} K_j(X).$$

Then

$$Z_j \text{ is characterised by } (g_j, \nu_j, z_j, K_j, \lambda_j, q_j).$$

The main effort: to devise an appropriate Banach space whose norm measures the size of K_j, and calculate how the coupling constants in V_j should evolve with j in such a way that K_j remains small.

The RG map is the description of the dynamical system $Z_j \mapsto Z_{j+1} = \mathbb{E}_{C_{j+1}} Z_j$ via

$$(g_j, \nu_j, z_j, K_j) \mapsto (g_{j+1}, \nu_{j+1}, z_{j+1}, K_{j+1}) \quad \text{which determines} \quad (\lambda_j, q_j) \mapsto (\lambda_{j+1}, q_{j+1}).$$

Flow of coupling constants

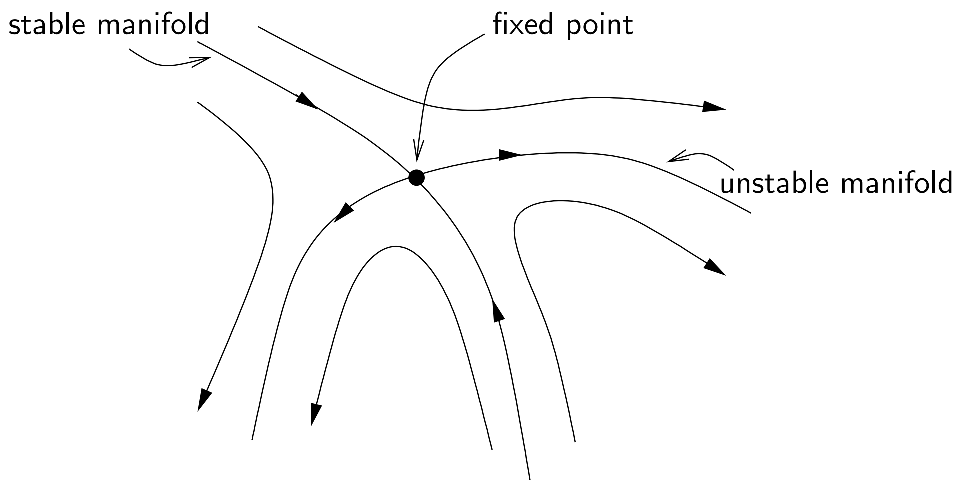
Theorem. Let $d \geq 4$. The flow of the coupling constants is given by

$$\begin{aligned} g_{j+1} &= g_j - c g_j^2 + r_{g,j} \\ \nu_{j+1} &= \nu_j + 2 g_j C_{j+1}(0,0) + r_{\mu,j} \\ z_{j+1} &= z_j + r_{z,j} \\ K_{j+1} &= r_{K,j}, \end{aligned}$$

where the r's are error terms within an appropriately defined Banach space, and Lipschitz in (g_j, ν_j, z_j, K_j). K_j enters only in the error terms and these are independent of λ_j, q_j. Also, with $w_j = \sum_{y \in \Lambda} \sum_{i=1}^{j} C_i(0,y)$,

$$\begin{aligned} \lambda_{j+1} &= (1 + \nu_{j+1} w_{j+1} - \nu_j w_j)\lambda_j + r_{\lambda,j} \\ q_{j+1} &= q_j + \lambda_j C_{j+1}(0,x) + r_{q,j}. \end{aligned}$$

Flow diagram



Schematic depiction of the stable manifold for the flow of the RG map.

Stable manifold theorem

Theorem. For small g_0 there is a choice of initial conditions z_0, ν_0 such that the solution $(g_j, \nu_j, z_j, K_j)_{0 \le j \le N}$, in the limits $N \to \infty$, $m^2 \to 0$, $j \to \infty$ obeys

$$(g_j, \nu_j, z_j, K_j) \to (0,0,0,0) \quad \text{“infrared asymptotic freedom.”}$$

From this and estimates on K_N,

$$\begin{aligned} G_{\mathbb{Z}^d}(0, g_0, \nu_0, z_0) &= \lim_{m^2 \downarrow 0} \lim_{N \to \infty} \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \bar\sigma}\bigg|_0 Z_N \\ &= \lim_{m^2 \downarrow 0} \lim_{N \to \infty} \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \bar\sigma}\bigg|_0 e^{-V_N(\Lambda)|_{\phi = d\phi = 0}} \\ &= \lim_{m^2 \downarrow 0} \lim_{N \to \infty} \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \bar\sigma}\bigg|_0 e^{\sigma \bar\sigma q_N} = \lim_{m^2 \downarrow 0} \lim_{N \to \infty} q_N. \end{aligned}$$

By solving the q recursion we obtain

$$\lim_{m^2 \downarrow 0} \lim_{N \to \infty} q_N = c(-\Delta_{\mathbb{Z}^4})^{-1}(0,x),$$

and this proves the main result for the critical two-point function.