Critical Temperature of Ferromagnetic Layered Ising Models

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Ising Model

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- A constant $J_e \geq 0$ (interaction) is associated to each edge $e \in E$; A spin variable σ_v is associated to each vertex v, σ_v can only be $+1$ or -1.

Ferromagnetism

- The probability of a configuration $\mathcal{C}$ is defined to be

$$Pr(\mathcal{C}) = \frac{1}{Z} exp(\beta H(\mathcal{C}))$$

where $\beta \geq 0$ is the reciprocal temperature, Z is a normalizing constant, and H is the Hamiltonian given by

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- Under the above definition, $J_e \geq 0$ implies that each pair of adjacent spins are more likely to have the same sign, this is called ferromagnetism.

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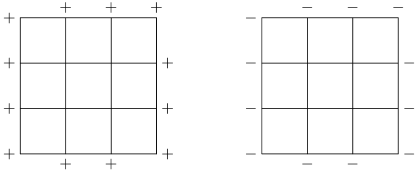
- A boundary condition b_Λ for a finite box Λ is given by specifying the configuration for all spins outside Λ
- The Hamiltonian with given boundary conditions b_Λ is

$$H_{\Lambda,b_\Lambda}(\mathcal{C}) = \sum_{u\in\Lambda, v\in\Lambda, u\sim v} J_e \sigma_u \sigma_v + \sum_{u\in\Lambda, v\in\Lambda^c, u\sim v} J_e \sigma_u \sigma_v$$

where the configuration of any spin in Λ^c is specified by b_Λ.

Gibbs Measure and Boundary Conditions



The weak limits of probability measures under the "+" boundary condition and the "-" boundary condition are known to exist.

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- The free energy, $\mathcal{F}$, is defined by

$$\mathcal{F} = \lim_{n \to \infty} \frac{1}{n^2} \log Z_{\Lambda_n, b_\Lambda}$$

  The limit exists and is independent of the boundary conditions.

Spontaneous Magnetization

The spontaneous magnetization $\mathcal{M}^*$ for an Ising model with period $1 \times s$, is defined by

$$\mathcal{M}^* = \frac{1}{s} \sum_{p=1}^{s} \langle \sigma_p \rangle_+$$

where $\langle \cdot \rangle_+$ is the expectation under the weak limit of probability measures given the "+" boundary condition.

Past Work

- Peierls (1936) noticed the phase transition exhibited by the 2-dimensional Ising model by comparing the behavior of the spins in different temperatures. In high temperature, spins far from each other are almost independent, while in low temperature, spins have nonzero correlations no matter how far away they are.

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Past Work

- Peierls (1936) noticed the phase transition exhibited by the 2-dimensional Ising model by comparing the behavior of the spins in different temperatures. In high temperature, spins far from each other are almost independent, while in low temperature, spins have nonzero correlations no matter how far away they are.
- Onsager (1944) solved the Ising model explicitly when the interactions on all edges are the same. He computed the free energy $\mathcal{F}$, as a function of the reciprocal temperature β, and found a unique β_0, where $\mathcal{F}$ fails to be an analytic function of β. He defined this β_0 to be the reciprocal critical temperature.

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- Lebowitz (1972) defined a critical temperature $T_c = \frac{1}{\beta_c}$ as the lowest temperature such that the spontaneous magnetization $\mathcal{M}^*$ is zero. He also identified β_c and β_0.

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Past Work

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- Aizenman (1987) proved the uniqueness of the critical temperature, T_c, for a large class of Ising-type model. T_c is characterized by

$$\mathcal{M}^* = \begin{cases} 0 & \text{if } \mathrm{T} > \mathrm{T_c} \\ > 0 & \text{if } \mathrm{T} < \mathrm{T_c} \end{cases}$$

$$\langle \sigma_i \sigma_j \rangle_+ \sim \begin{cases} e^{-\alpha|i-j|} & \text{if } \mathrm{T} > \mathrm{T_c} \\ C & \text{if } \mathrm{T} < \mathrm{T_c} \end{cases} \quad \alpha > 0, \mathrm{C} > 0 \text{ are constants}$$

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Dimer Model

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- A dimer configuration (perfect matching) of a graph is a collection of edge weights with the property that each vertex is incident to exactly one of these edges.
- The probability of a dimer configuration $\mathcal{D}$ on a finite graph is proportional to the product of weights of present edges.

$$Pr(\mathcal{D}) \propto \prod_{e \in \mathcal{D}} w_e$$

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Fisher Correspondence

The Fisher graph we consider is one obtained from the honeycomb lattice by replacing each vertex with a triangle.

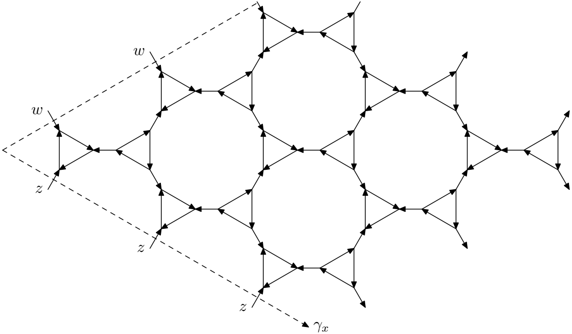
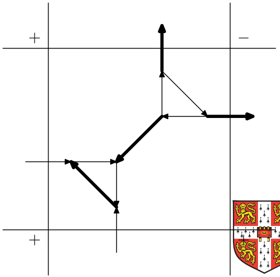


fig1.01

Fisher Correspondence

- There is a two-to-one correspondence between the Ising spin configurations on the square grid and the dimer configurations on a Fisher graph.



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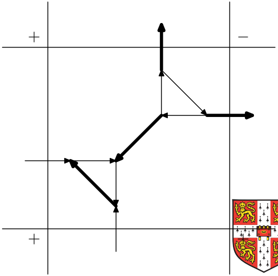
- If two adjacent spins have the same sign, the dual edge is not present in the dimer configuration; otherwise the dual edge is present.



Fisher Correspondence

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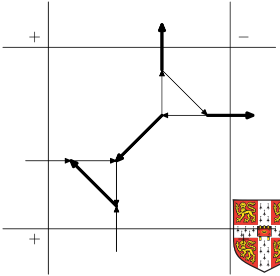
- If two adjacent spins have the same sign, the dual edge is not present in the dimer configuration; otherwise the dual edge is present.
- The correspondence is two-to-one because if we negate the spins at all vertices, we will end up with the same dimer configuration.



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- Multiply the entries of the weighted adjacency matrix of G_s, corresponding to edges crossed by γ_x, γ_y, by $z(w)$, or $\frac{1}{z}(\frac{1}{w})$, according to their orientation. Let $K(z, w)$ be the modified weighted adjacency matrix.

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- The characteristic polynomial is defined to be $\det K(z, w)$. The spectral curve is defined to be the zero locus $\det K(z, w) = 0$.

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- The characteristic polynomial is defined to be $\det K(z, w)$. The spectral curve is defined to be the zero locus $\det K(z, w) = 0$.
- Using a large torus to approximate the infinite periodic graph, Boutillier and de Tiliere computed the probability of any fixed finite set of edges present in the dimer configurations on a Fisher graph with edge weights satisfying $\det K(1, 1) = 0$.

Toeplitz Matrix

If ψ is a matrix-valued function defined on the unit circle with Fourier coefficient ψ_k, then $T[\psi]$, the semi-infinite block Toeplitz matrix is defined as

$$T[\psi] = \begin{pmatrix} \psi_0 & \psi_1 & \psi_2 & \cdots \\ \psi_{-1} & \psi_0 & \psi_1 & \cdots \\ \cdots & & & \end{pmatrix}$$

ψ is the symbol and let T_n be the first $n \times n$ block of T.

Widom's formula

Theorem

(Widom)If

$$\sum_{k=-\infty}^{\infty} \|\psi_k\| + (\sum_{k=-\infty}^{\infty} |k| \|\psi_k\|^2)^{\frac{1}{2}} < \infty$$

$$\det \psi(e^{i\theta}) \neq 0, \qquad \frac{1}{2\pi}\Delta_{0\leq\theta\leq 2\pi} \arg\det\psi(e^{i\theta}) = 0,$$

here $\|\psi_k\|$ *denotes the Hilbert-Schmidt norm of the matrix* ψ_k*, and* $\frac{1}{2\pi}\Delta_{0\leq\theta\leq 2\pi} \arg\det\psi(e^{i\theta})$ *is the index of the point* 0 *with respect to the curve* $\{\det\psi(e^{i\theta}) : 0 \leq \theta \leq 2\pi\}$*, namely,*

$$\frac{1}{2\pi}\Delta_{0\leq\theta\leq 2\pi} \arg\det\psi(e^{i\theta}) = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{1}{\det\psi(\zeta)} \frac{\partial \det\psi(\zeta)}{\partial\zeta} d\zeta,$$

Widom's formula

Theorem

(Widom) then

$$\lim_{n\to\infty} \frac{\det T_n(\psi)}{G[\psi]^{n+1}} = E[\psi],$$

with

$$\begin{aligned} G[\psi] &= \exp\{\frac{1}{2\pi}\int_0^{2\pi} \log\det\psi(e^{i\theta})d\theta\}, \\ E[\psi] &= \det T[\psi]\,T[\psi^{-1}]. \end{aligned}$$

where the last det *refers to the determinant defined for operators on Hilbert space differing from the identity by an operator of trace class.*

Main Theorems

Theorem

(z.Li) The intersection $\det K(z,w) = 0$ *with the unit torus* $\mathbb{T}^2 = \{(z,w) \| |z| = 1, |w| = 1\}$ *is either empty or a single real point. That is, if the intersection is nonempty,then it is a unique point* $(e^{i\theta_0}, e^{i\phi_0}) \in \{(\pm 1, \pm 1)\}$*, and in a neighborhood of* (θ_0, ϕ_0)*,*

$$\begin{aligned}\det K(e^{i\theta}, e^{i\phi}) &= \alpha(\theta-\theta_0)^2 + \beta(\theta-\theta_0)(\phi-\phi_0) + \gamma(\phi-\phi_0)^2 \\ &\quad O(|\theta-\theta_0|^3 + |\phi-\phi_0|^3\end{aligned}$$

where α, β, γ *are real constants satisfying*

$$\beta^2 - 4\alpha\gamma \leq 0$$

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Main Theorems

Theorem

(z.Li) The critical temperature of the periodic Ising model is defined by the lowest temperature where the spontaneous magnetization is zero. The layered Ising model is at critical temperature if and only if the spectral curve of the dimer model on the corresponding Fisher graph has a real zero on the unit torus $\mathbb{T}^2$.

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Proofs of Main Theorems:1

- Theorem 1 follows from combinatorics and estimates of coefficients of the Taylor expansion.

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- We express $\langle\sigma_{00}\sigma_{0n}\rangle$ as the determinant of a large truncated block Toeplitz matrix, following the computation of McCoy and Wu.
- Widom's formula is applied to compute the asymptotics of the large truncated block Toeplitz matrix, and we prove that the G in the formula for square lattice Ising model is identically 1.

Proofs of Main Theorems: 2

- With the help of spectral analysis, we derive that $\lim_{n\to\infty}\langle\sigma_{00}\sigma_{0n}\rangle$ is not analytic only when the spectral curve has a real zero on $\mathbb{T}^2$. From Lebowitz and Aizenmen's work, it is obvious that at the critical temperature β_c, $\lim_{n\to\infty}\langle\sigma_{00}\sigma_{0n}\rangle$ is not analytic. Hence, at the critical temperature, the spectral curve has a real zero on $\mathbb{T}^2$.

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- We prove that as β increases from 0 to ∞, there is a unique $0<\beta^*<\infty$, such that the spectral curve has a real node on $\mathbb{T}^2$, the existence and uniqueness of β_c then $\beta^*=\beta_c$.

Thank you!