Limit shapes in the Ising model

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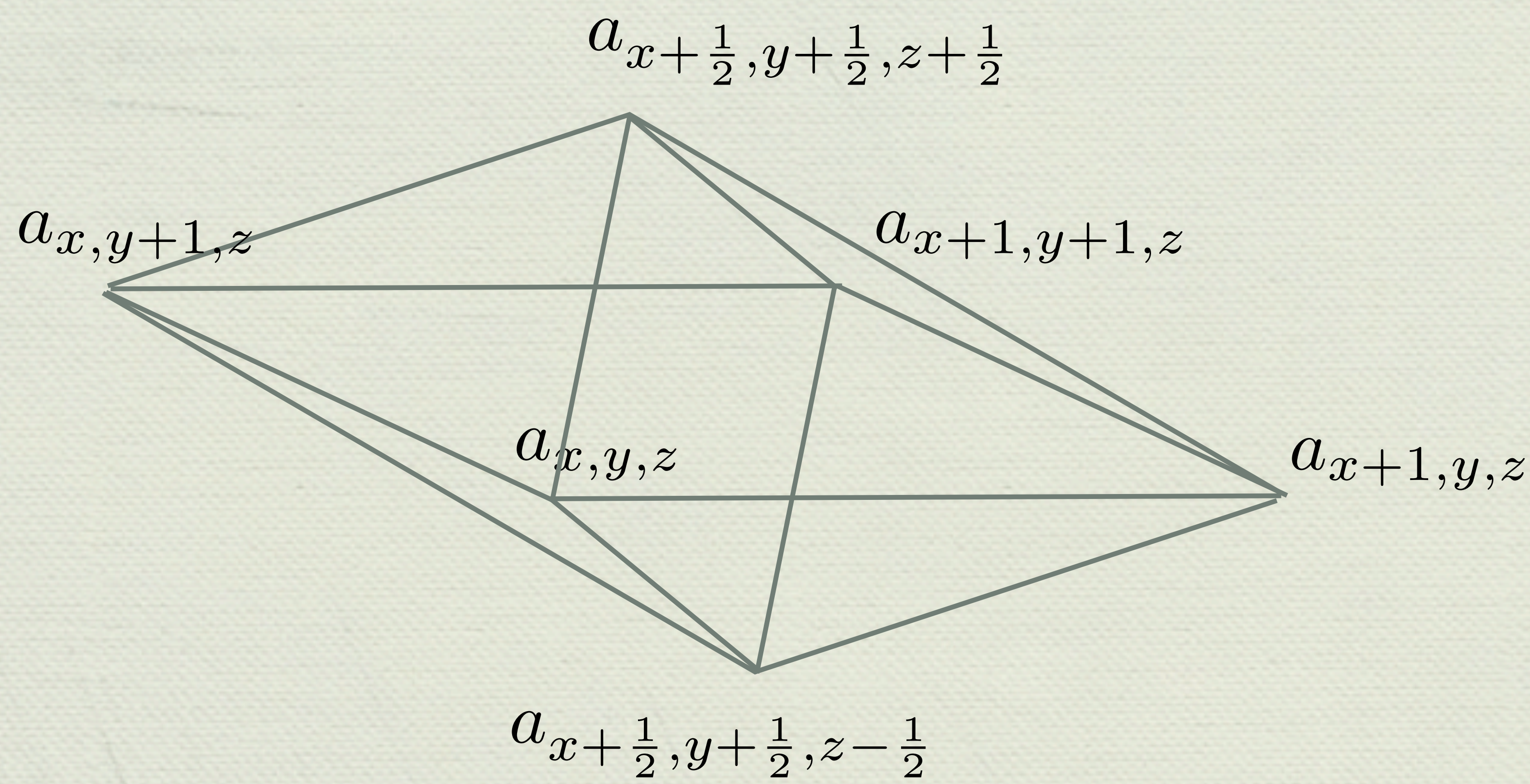
Octahedron recurrence = Hirota bilinear difference equation (HBDE)

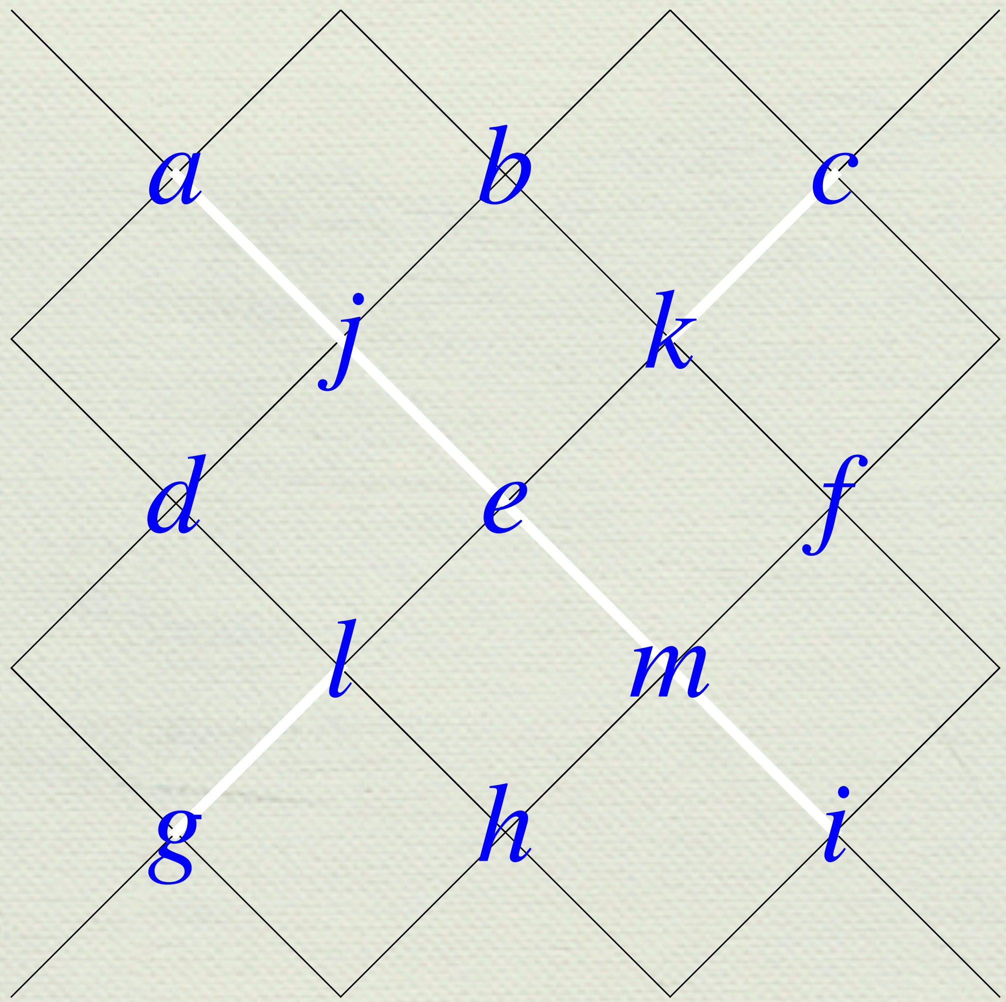
$$a_{x+\frac{1}{2},y+\frac{1}{2},z+\frac{1}{2}} = \frac{a_{x,y,z}a_{x+1,y+1,z} + a_{x,y+1,z}a_{x+1,y,z}}{a_{x+\frac{1}{2},y+\frac{1}{2},z-\frac{1}{2}}}$$



$$
\begin{array}{ccccc}
a & & b & & c \\
& j & & k & \\
d & & e & & f \\
& l & & m & \\
g & & h & & i
\end{array}
$$

$$a_{0,0,1} = \frac{\frac{ae+bd}{j}\,\frac{ei+fh}{m} + \frac{bf+ce}{k}\,\frac{dh+eg}{l}}{e}$$

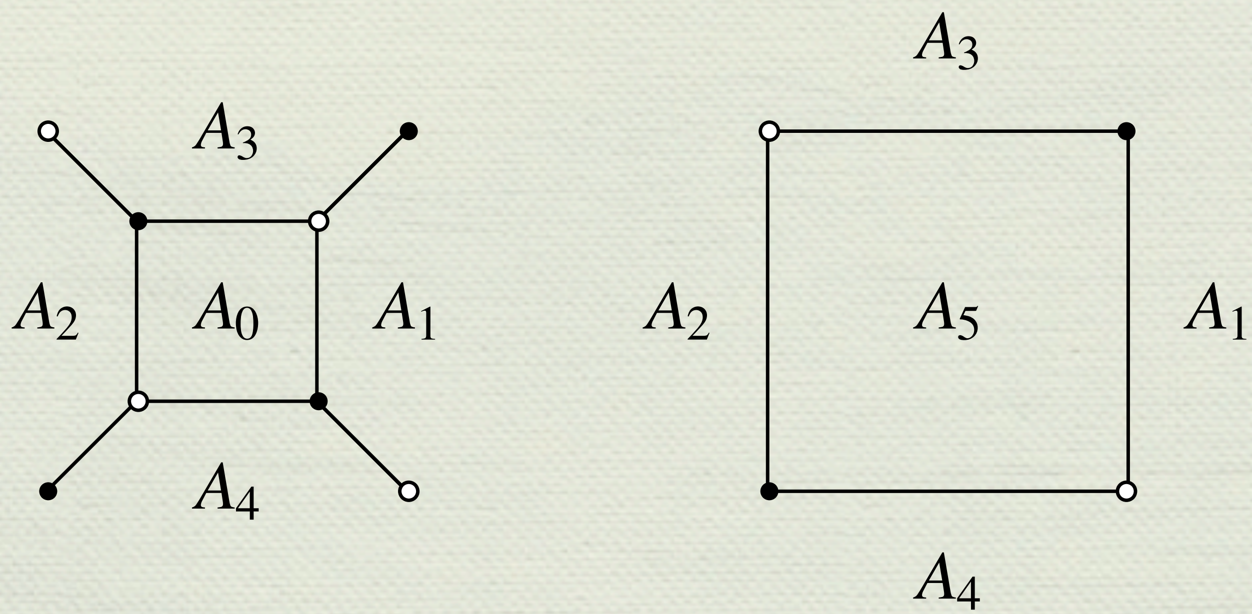
$$a_{0,0,1} = \frac{aei}{jm} + \frac{bdi}{jm} + \frac{afh}{jm} + \frac{bdfh}{\underbrace{ejm}} + \frac{bdfh}{ekl} + \frac{bfg}{kl} + \frac{cdh}{kl} + \frac{ceg}{kl}$$



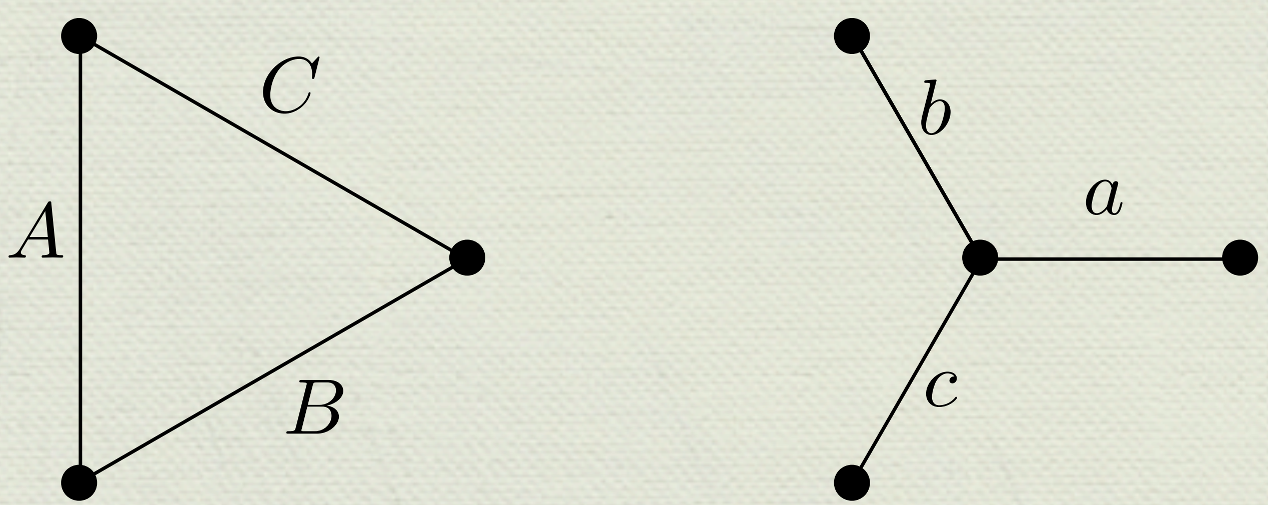
Tilings of the Aztec diamond.

(Speyer)

$$a_{0,0,1} = \frac{aei}{jm} + \frac{bdi}{jm} + \frac{afh}{jm} + \underbrace{\frac{bdfh}{ejm}} + \frac{bdfh}{ekl} + \frac{bfg}{kl} + \frac{cdh}{kl} + \frac{ceg}{kl}$$

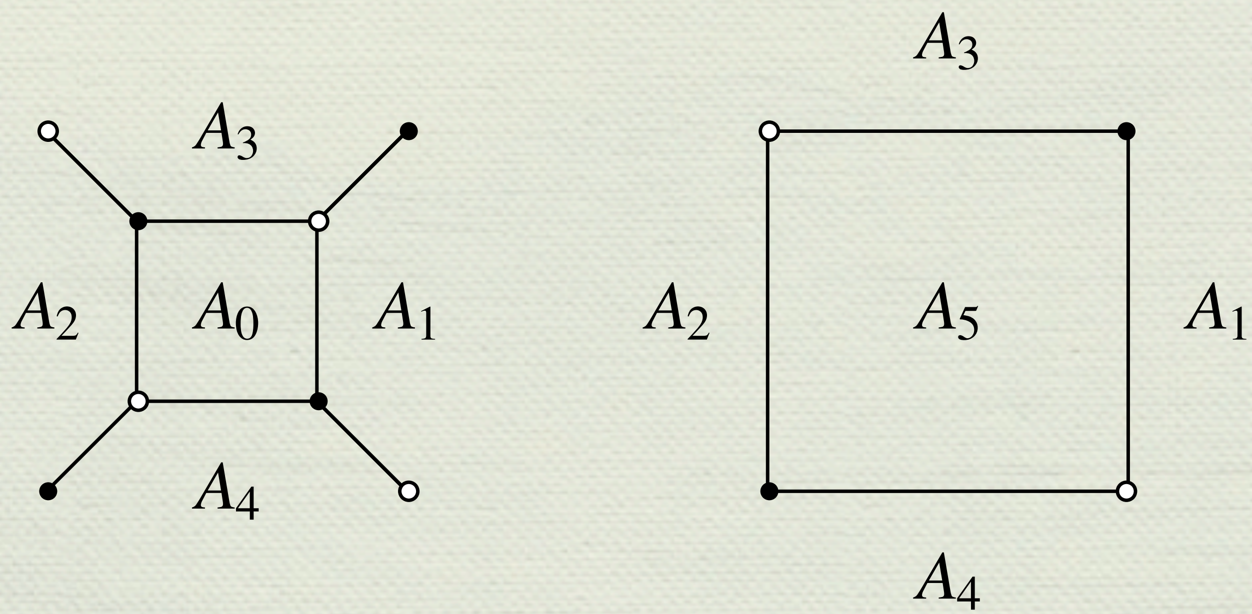


"Urban renewal" $A_5 A_0 = A_1 A_2 + A_3 A_4$

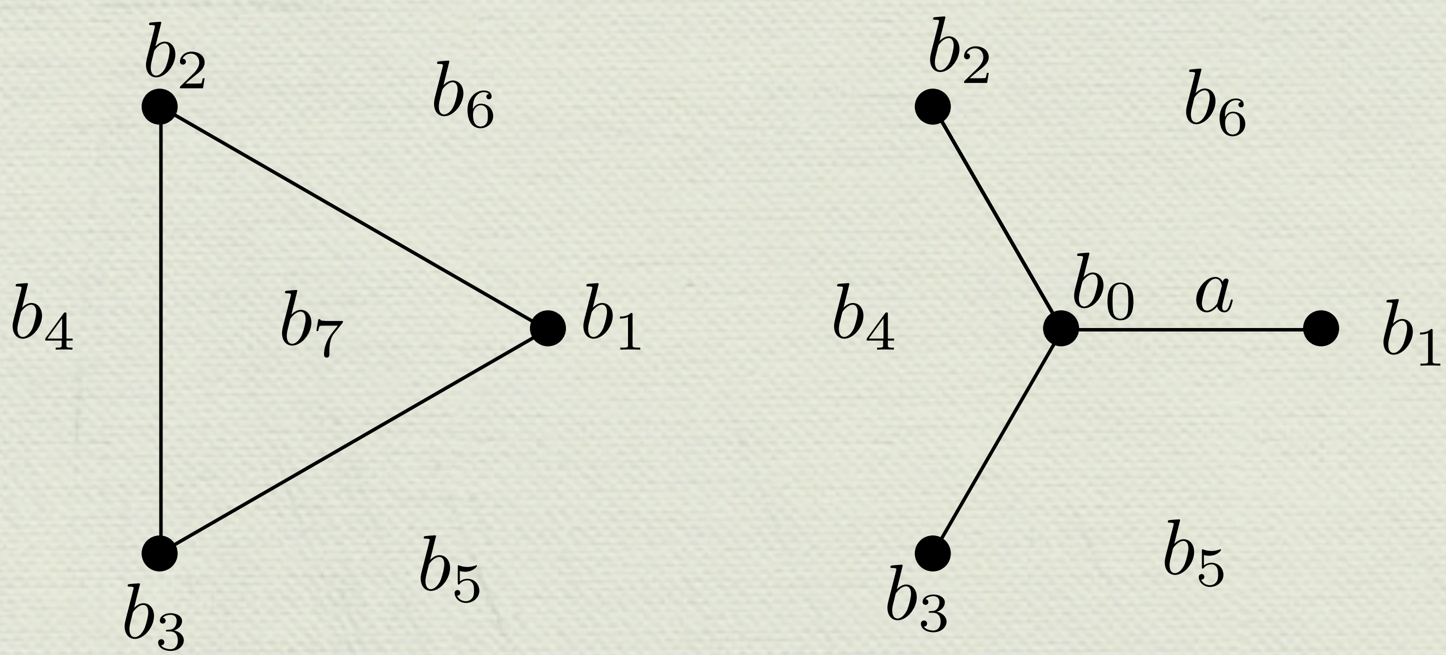


$$A = \frac{bc}{a+b+c}$$
$$B = \frac{ac}{a+b+c}$$
$$C = \frac{ab}{a+b+c}$$

Y-Delta transformation for resistor networks



"Urban renewal" $A_5 A_0 = A_1 A_2 + A_3 A_4$



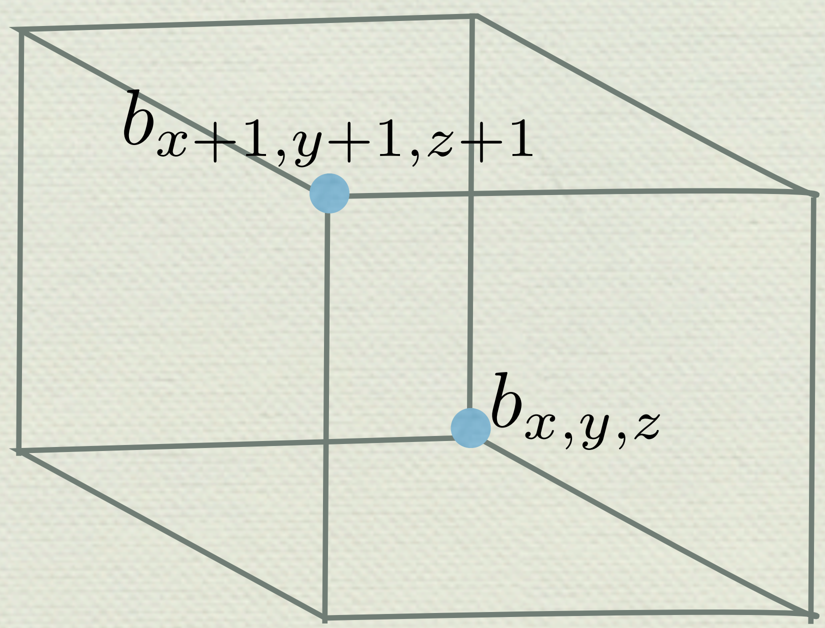
where $a = \frac{b_0 b_1}{b_5 b_6}$ etc.

$$b_7 = \frac{b_1 b_4 + b_2 b_5 + b_3 b_6}{b_0}$$

Y-Delta transformation for resistor networks

Cube recurrence = Miwa equation

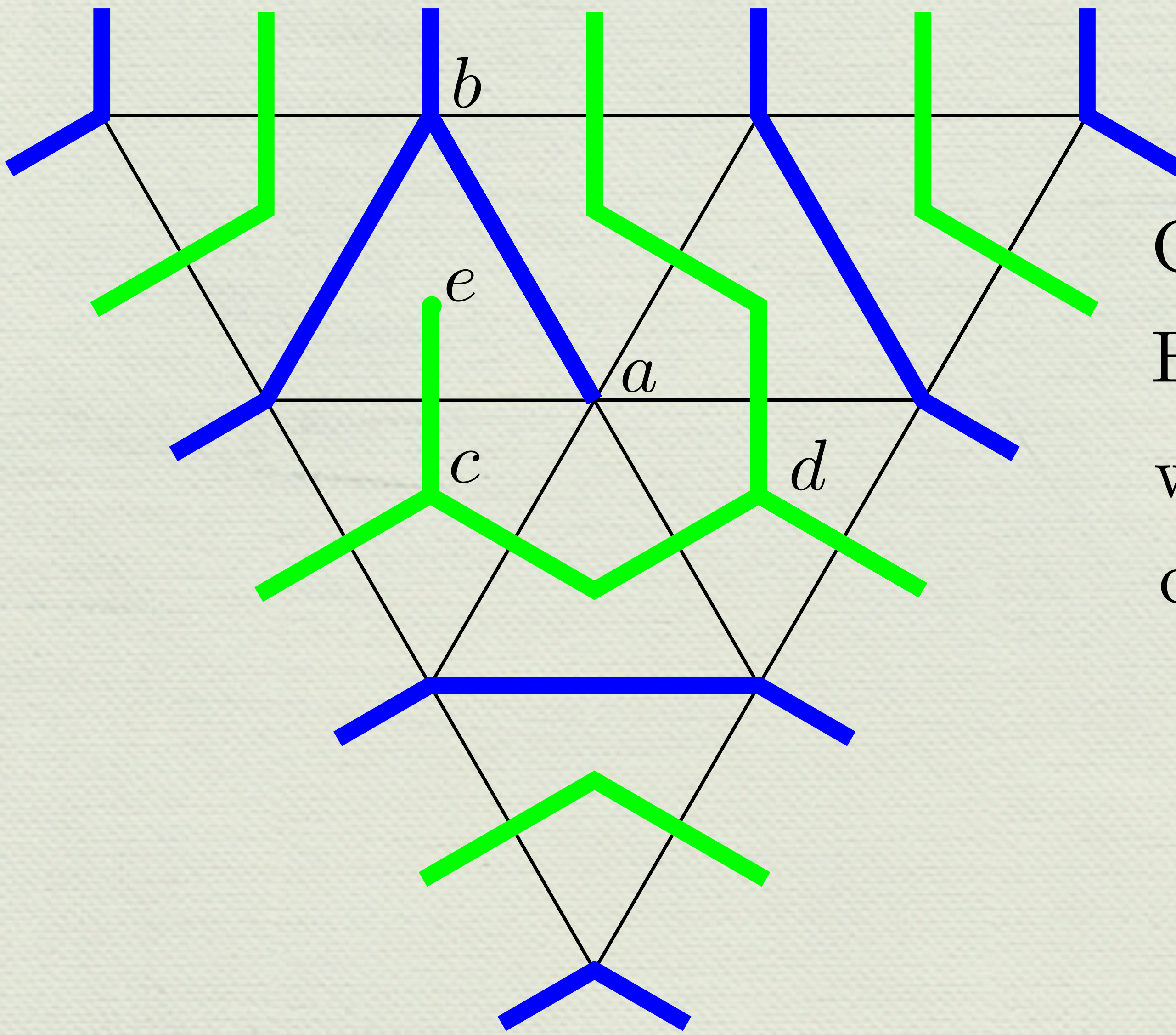
$$b_{x+1,y+1,z+1} = \frac{b_{x+1,y,z}b_{x,y+1,z+1} + b_{x,y+1,z}b_{x+1,y,z+1} + b_{x,y,z+1}b_{x+1,y+1,z}}{b_{x,y,z}}$$



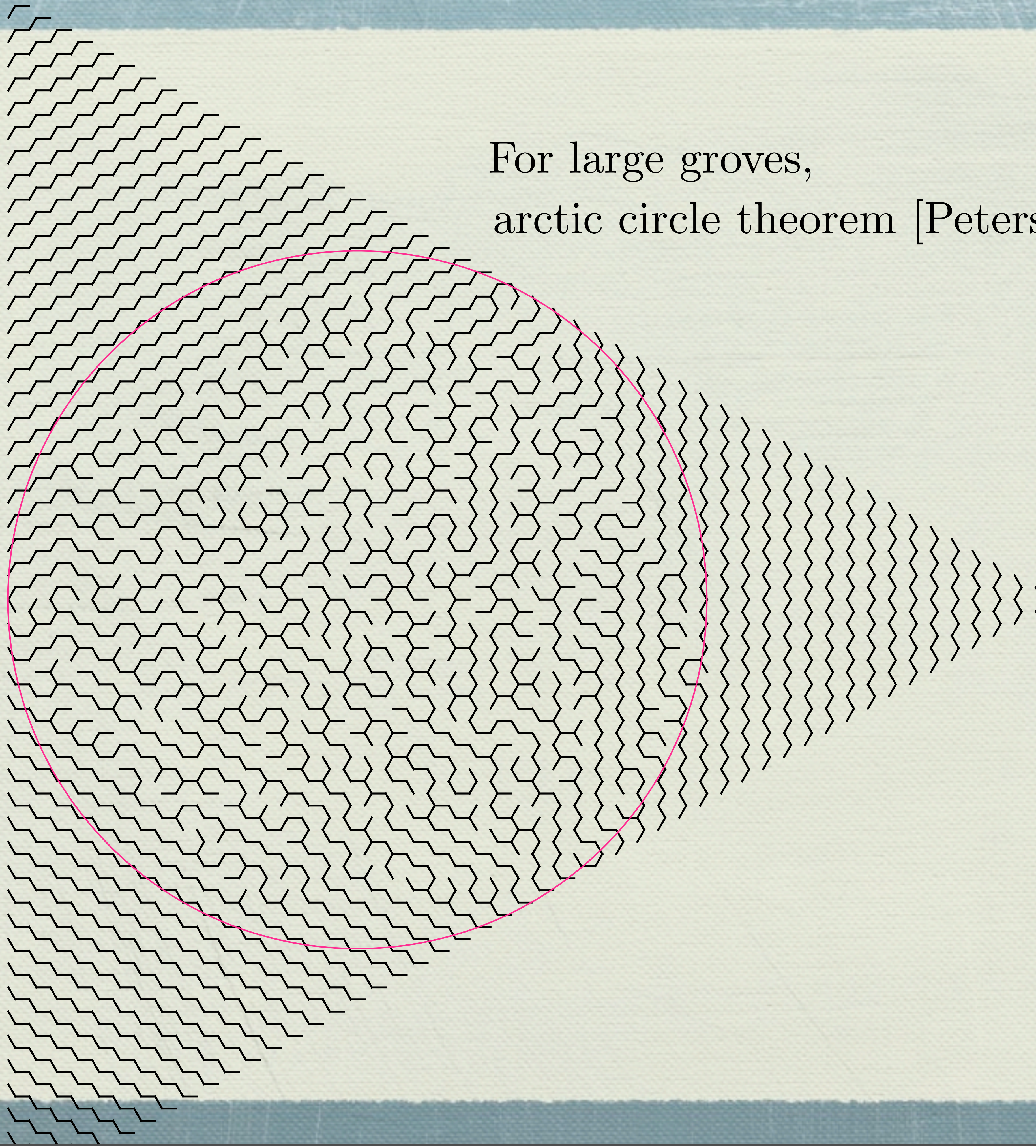
From the values on $0 \leq x+y+z \leq 2$ and the recurrence, get all $b_{x,y,z}$.

The Laurent property holds:

$$b_{2,1,1} = \cdots + \frac{bd}{ace} + \ldots$$



Carroll/Speyer "groves":
Essential spanning forests
with "tripartite" boundary
connections



For large groves,

arctic circle theorem [Peterson-Speyer]

G a graph, $c : E \to \mathbb{R}_{>0}$ edge weights.

Ising model: $\Omega = \{1, -1\}^G$,

$$Z = \sum_{\sigma \in \Omega} \prod_{i \sim j:\ \sigma_i = \sigma_j} c_{ij}$$

$$= \sum_{\sigma} \prod_{i \sim j} \Big(1 + (c_{ij} - 1)\delta_{\sigma_i = \sigma_j}\Big)$$

$$= \sum_{\sigma} \sum_{S \subset E} \prod_{ij \in S} (c_{ij} - 1)\delta_{\sigma_i = \sigma_j}$$

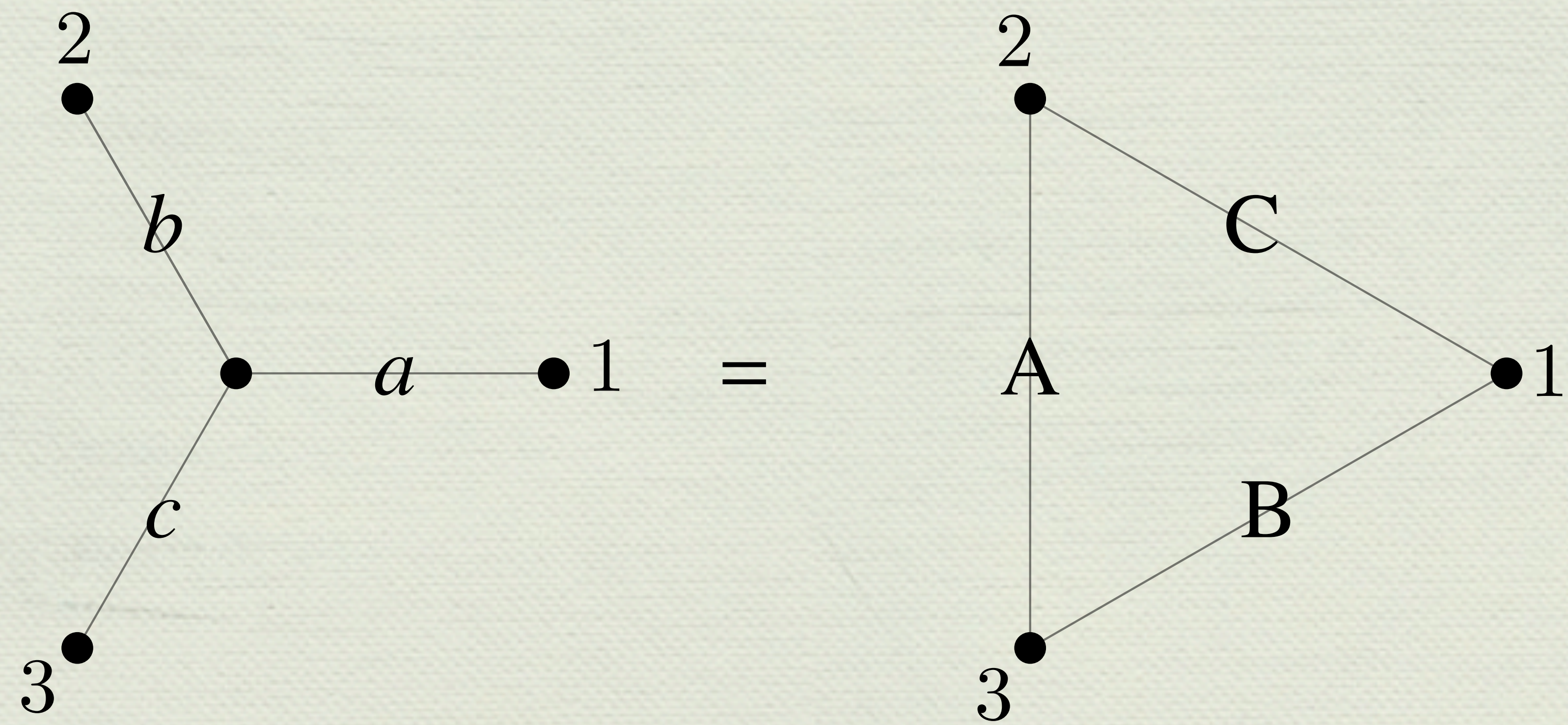
$$= \sum_{S \subset E} \prod_{ij \in S} (c_{ij} - 1) 2^k.$$

FK_2 model:

$$= \sum_{S \subset E} \prod_{ij \in S} d_{ij} 2^k$$

Here $k =$ number of components of S.

Ising model Y-Delta transformation



spins		
$+ + +$	$abc + 1$	ABC
$- + +$	$a + bc$	A
$+ - +$	$b + ac$	B
$+ + -$	$c + ab$	C

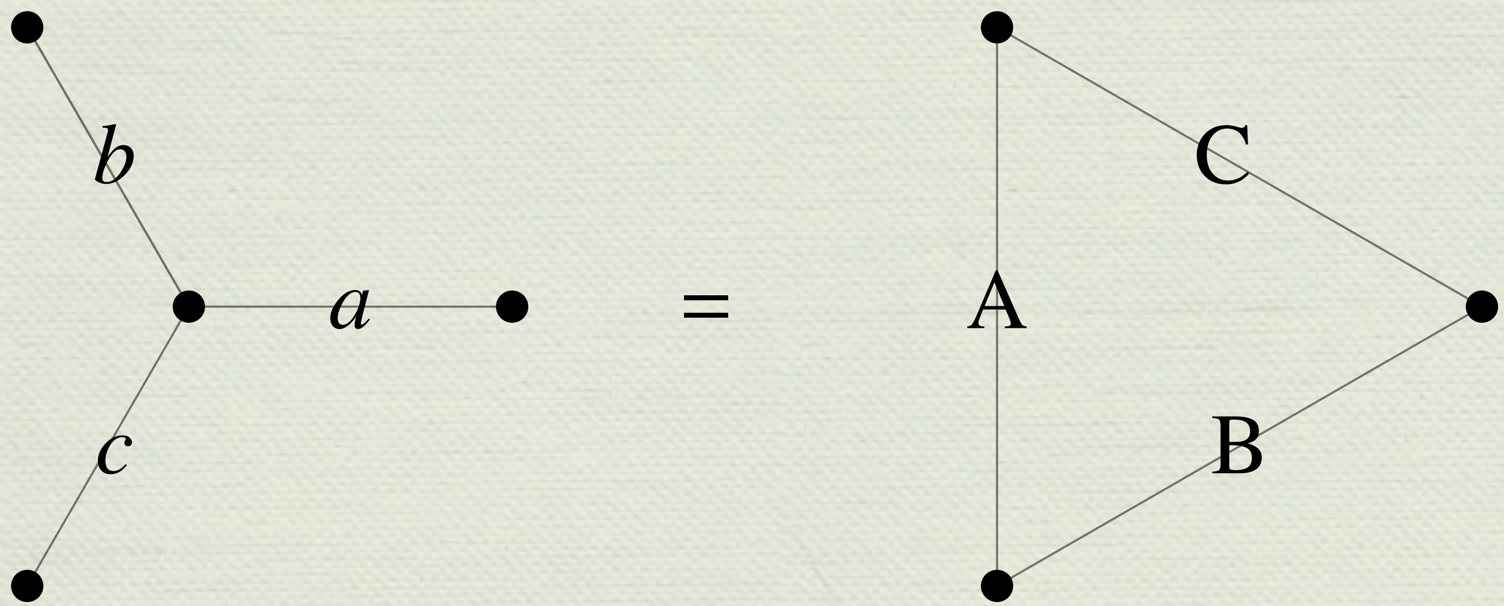
"Before" and "after" are proportional (Ising measure preserved) iff

$$A = \sqrt{\frac{(abc+1)(a+bc)}{(b+ac)(c+ab)}}$$

$$B = \sqrt{\frac{(abc+1)(b+ac)}{(a+bc)(c+ab)}}$$

$$C = \sqrt{\frac{(abc+1)(c+ab)}{(a+bc)(b+ac)}}$$

FK_q model Y-Delta transformation



Y weights	connected in exterior	Delta weights
$abc + ab + ac + bc + a + b + c + q$	123	$(A+1)(B+1)(C+1)$
$\frac{abc+ab+ac}{q} + bc + a + b + c + q$	23	$\frac{ABC+AB+AC+BC+B+C}{q} + A + 1$
$\frac{abc+ab+bc}{q} + ac + a + b + c + q$	13	$\frac{ABC+AB+AC+BC+A+C}{q} + B + 1$
$\frac{abc+bc+ac}{q} + ab + a + b + c + q$	12	$\frac{ABC+AB+AC+BC+A+B}{q} + C + 1$
$\frac{abc}{q^2} + \frac{ab+ac+bc}{q} + a + b + c + q$	$\emptyset$	$\frac{ABC+AB+AC+BC}{q^2} + \frac{A+B+C}{q} + 1$

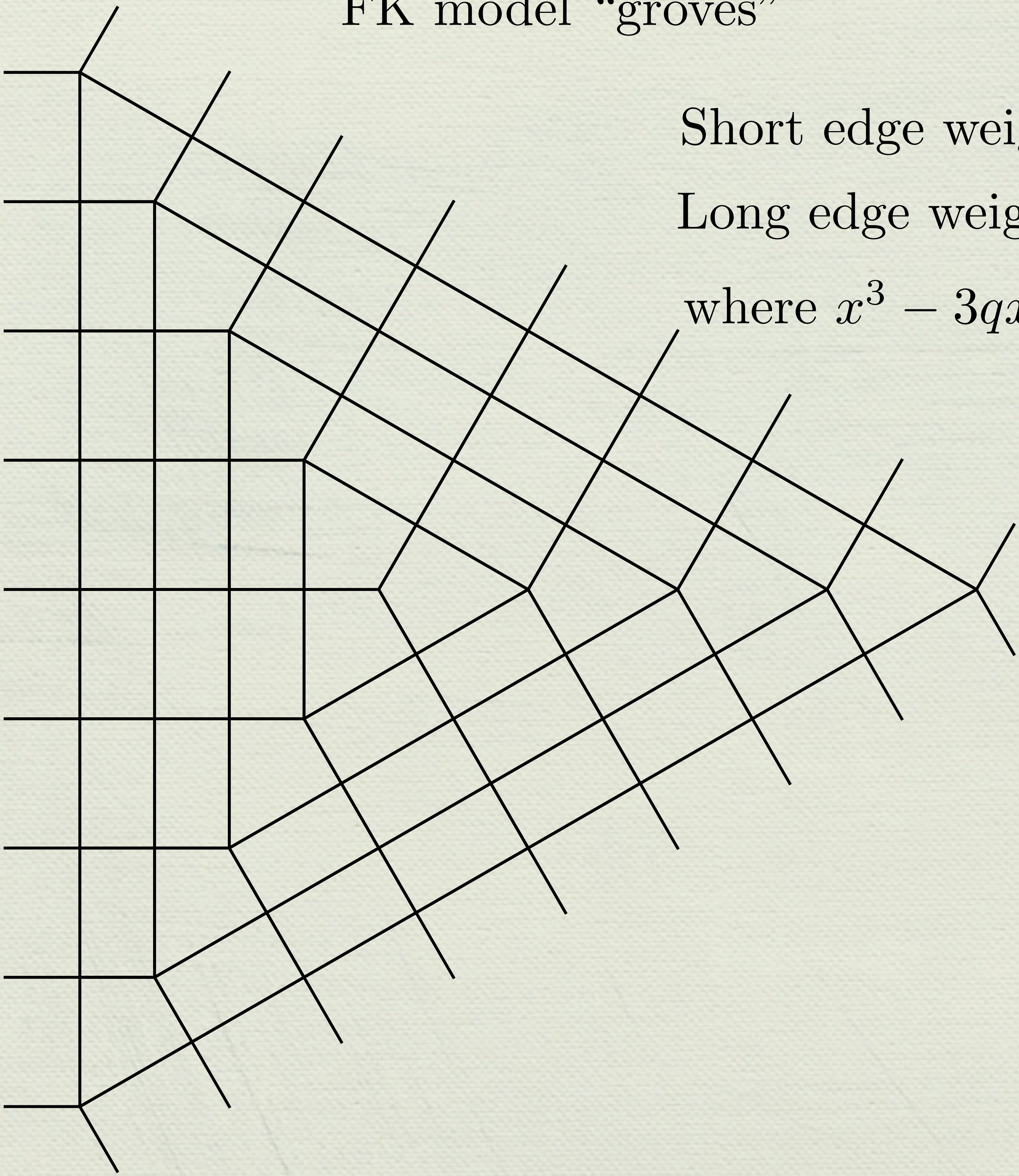
Lemma: There is a solution iff

$$abc - q(a+b+c) - q^2 = 0$$

$$(\text{or } ABC + AB + AC + BC - q = 0)$$

in which case $A = \frac{q}{a}, B = \frac{q}{b}, C = \frac{q}{c}$.

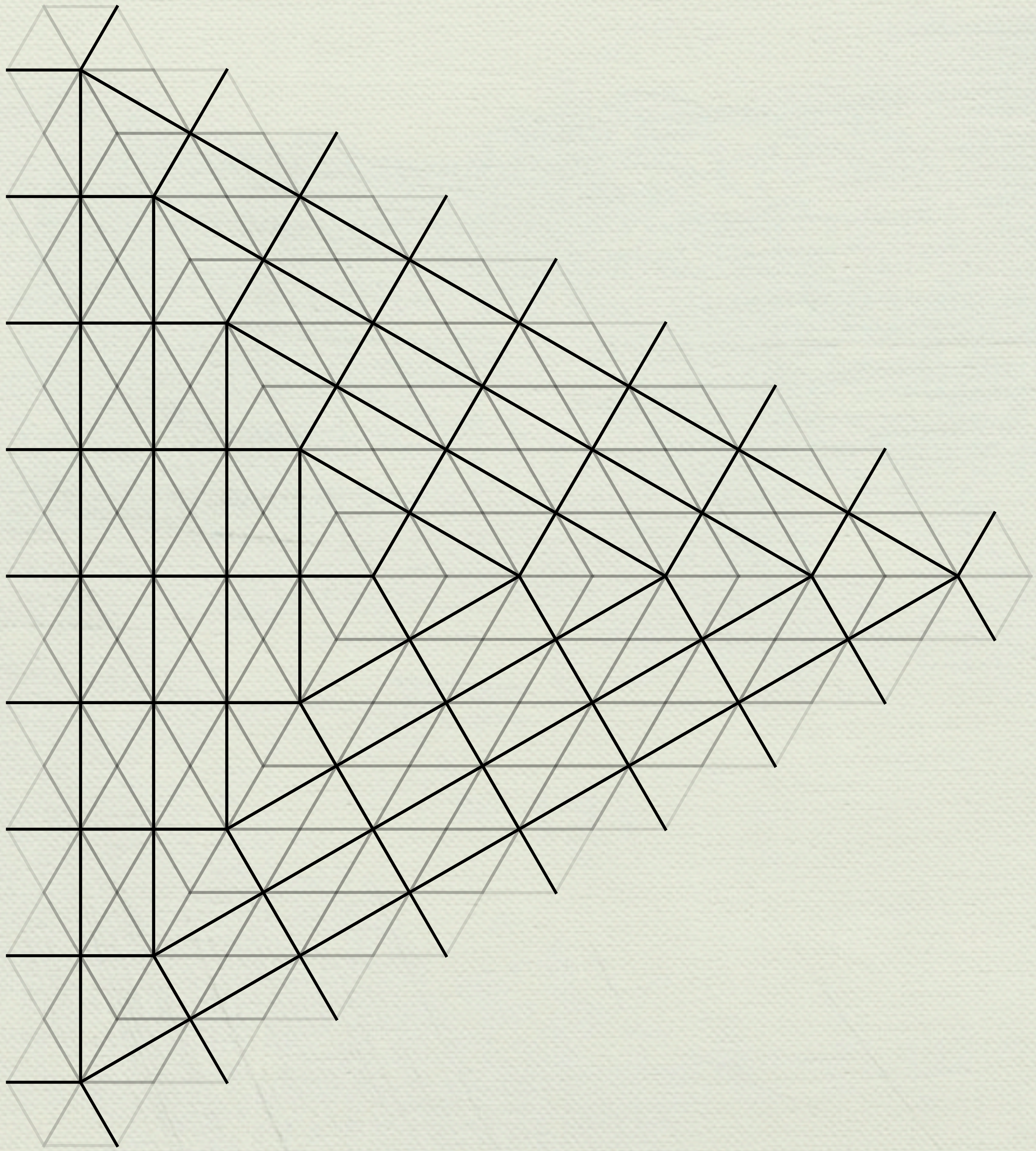
FK model "groves"

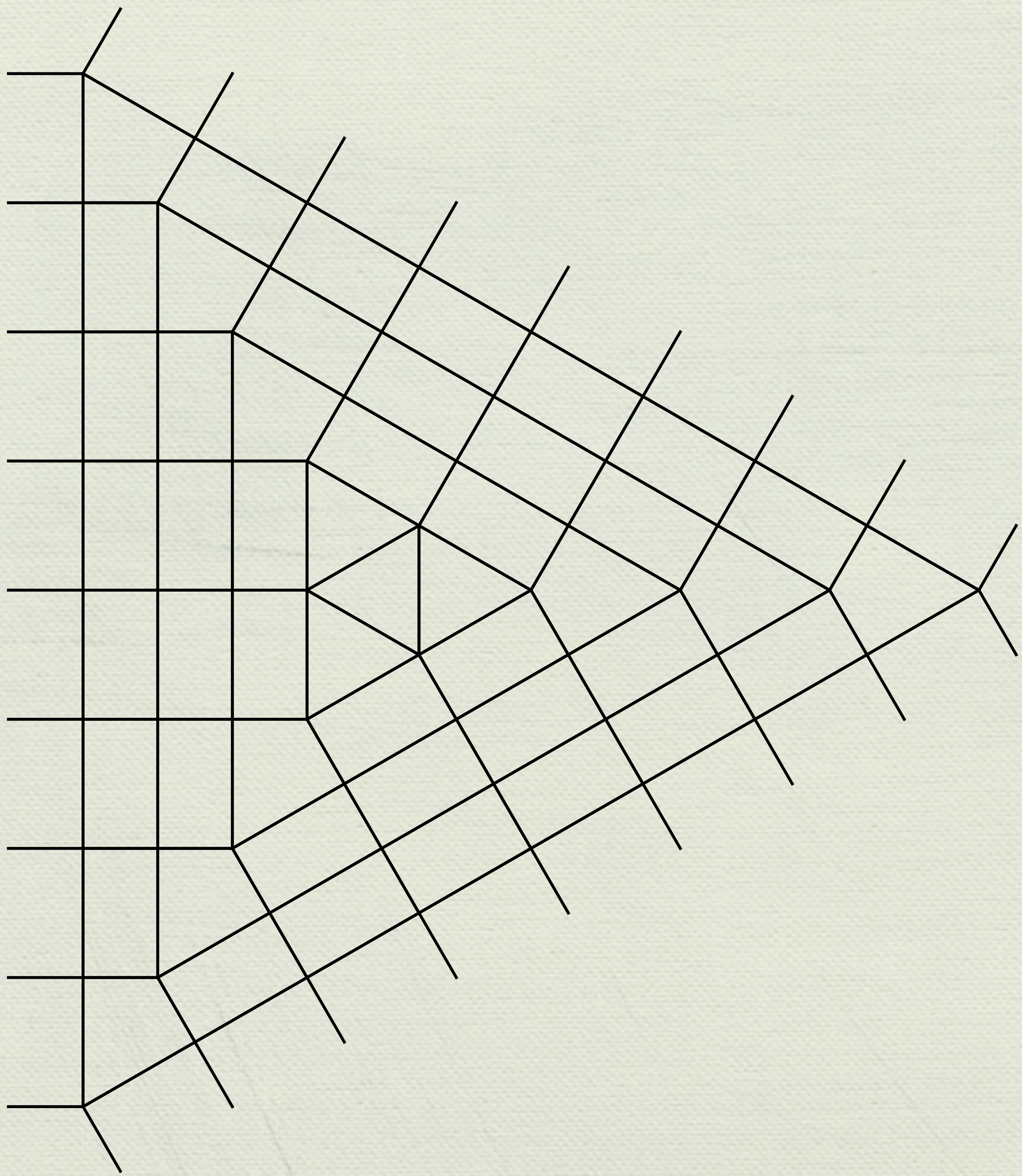


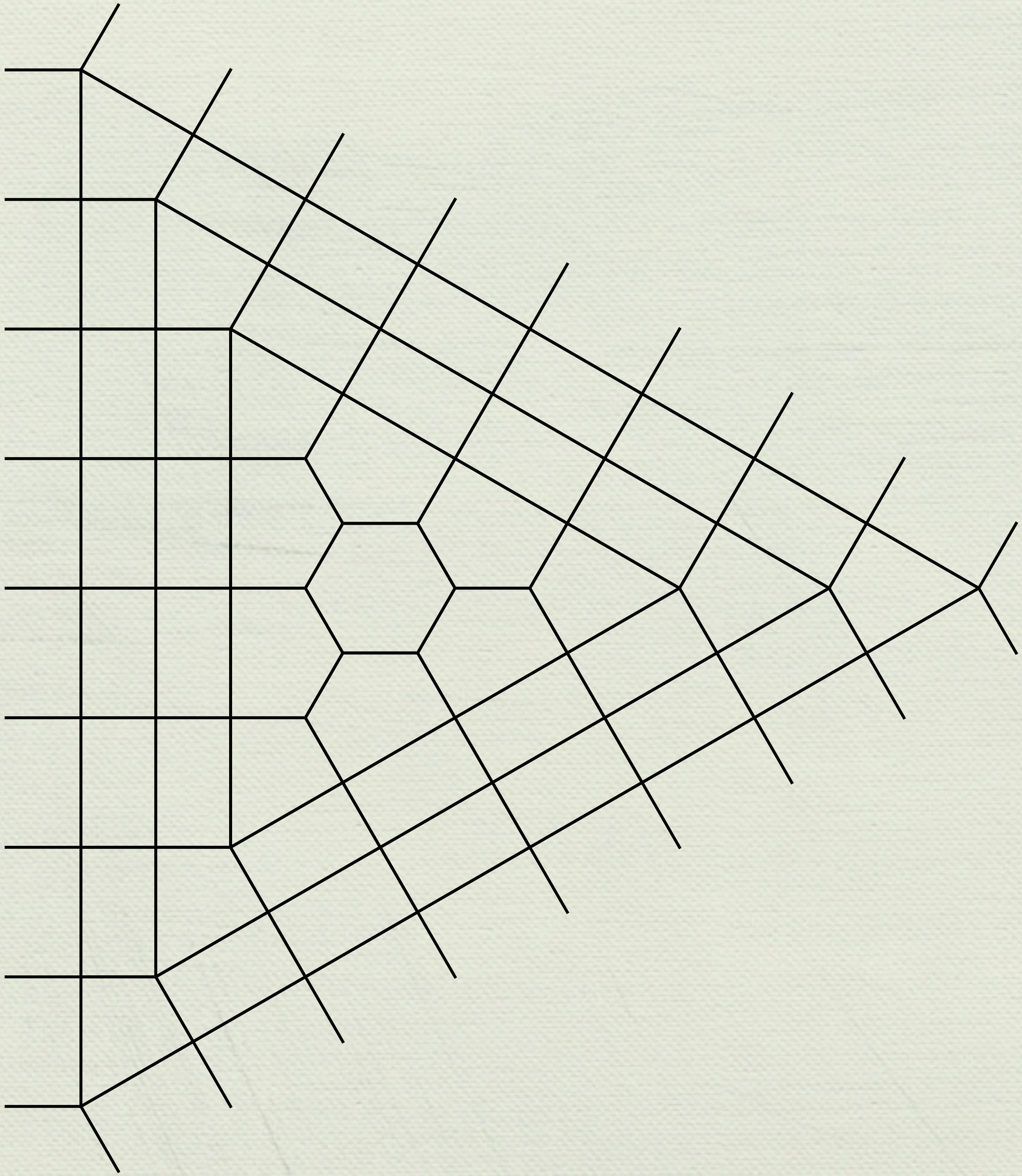
Short edge weights: x

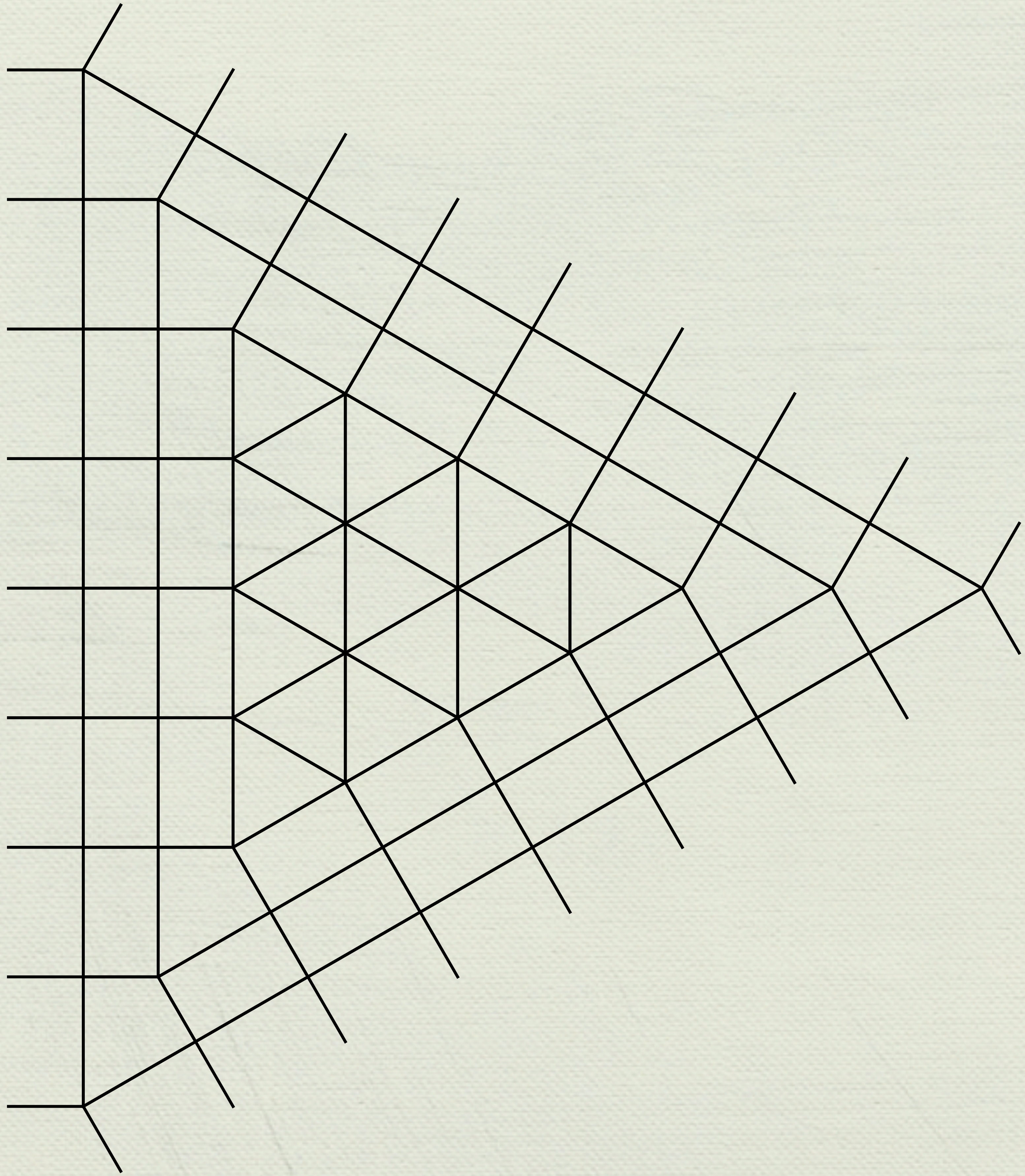
Long edge weights: q/x

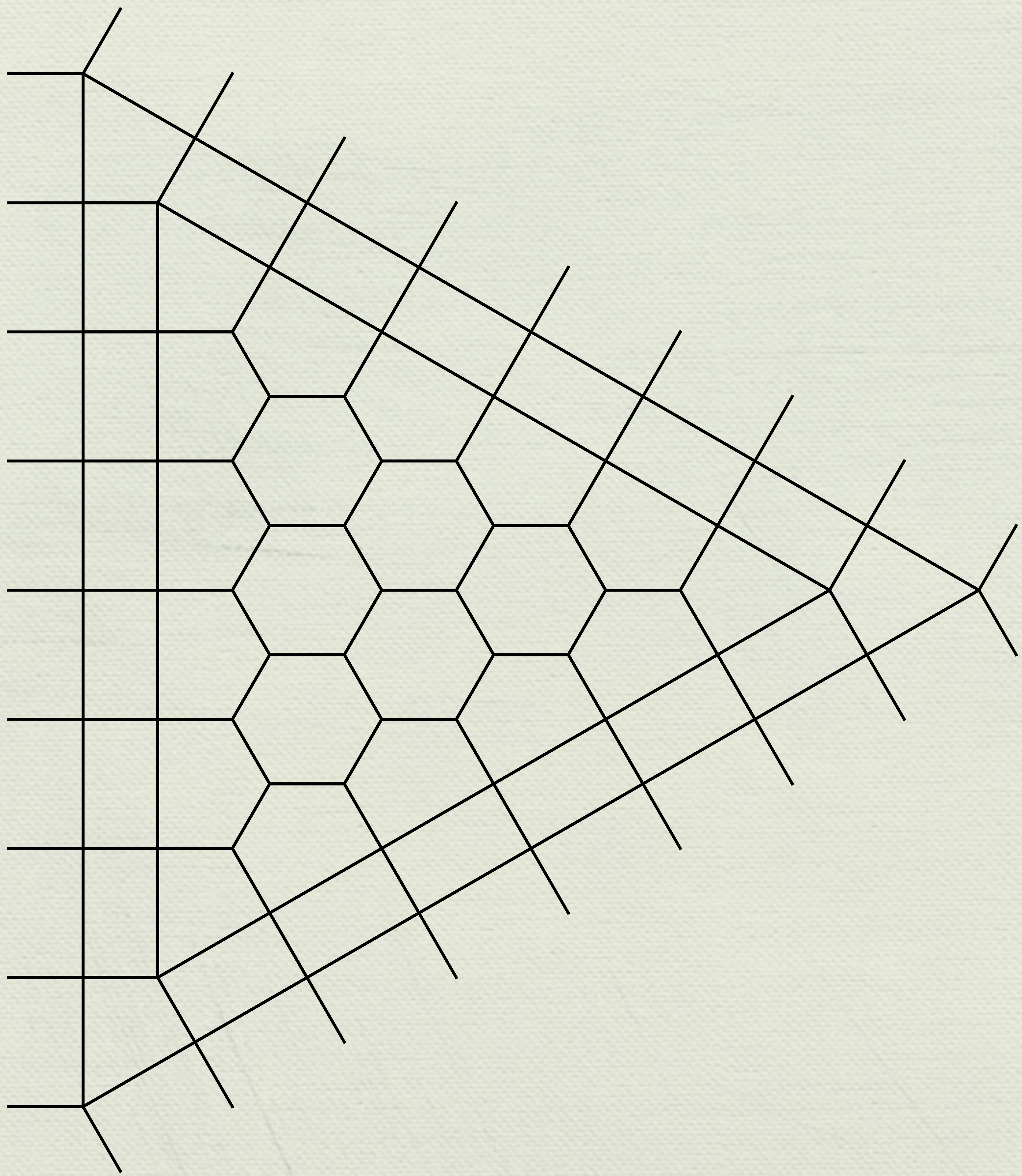
where $x^3 - 3qx - q^2 = 0.$

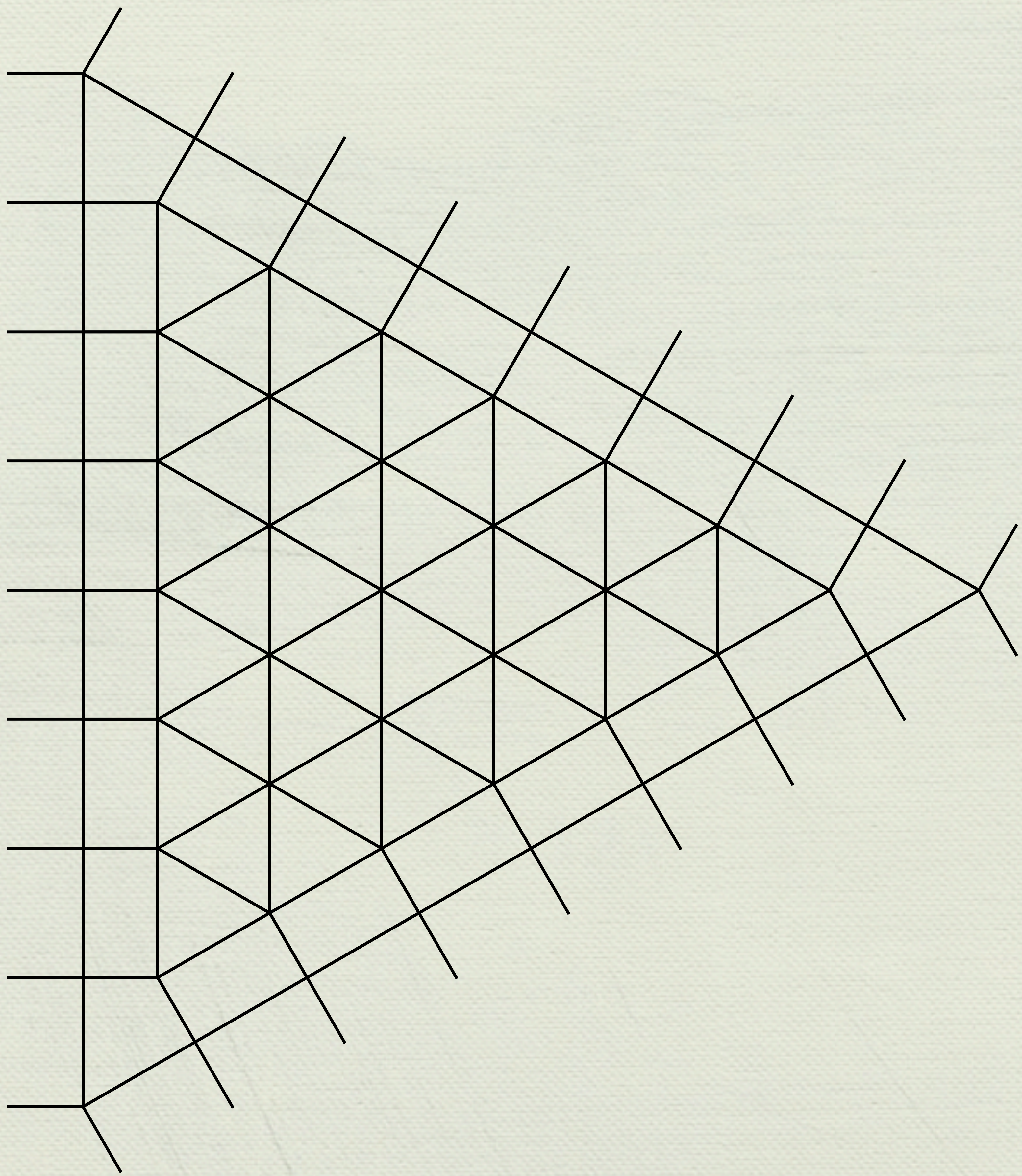


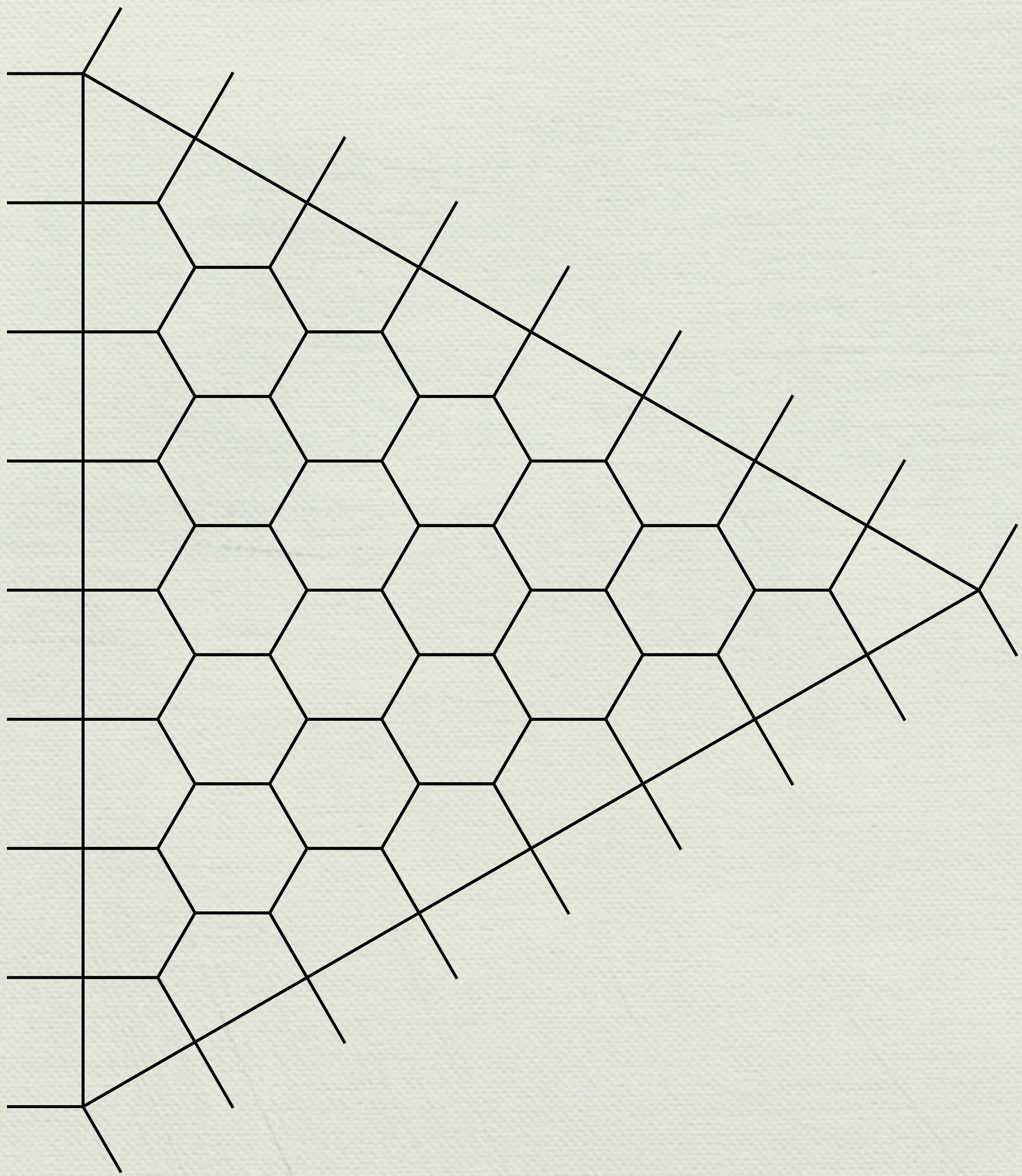


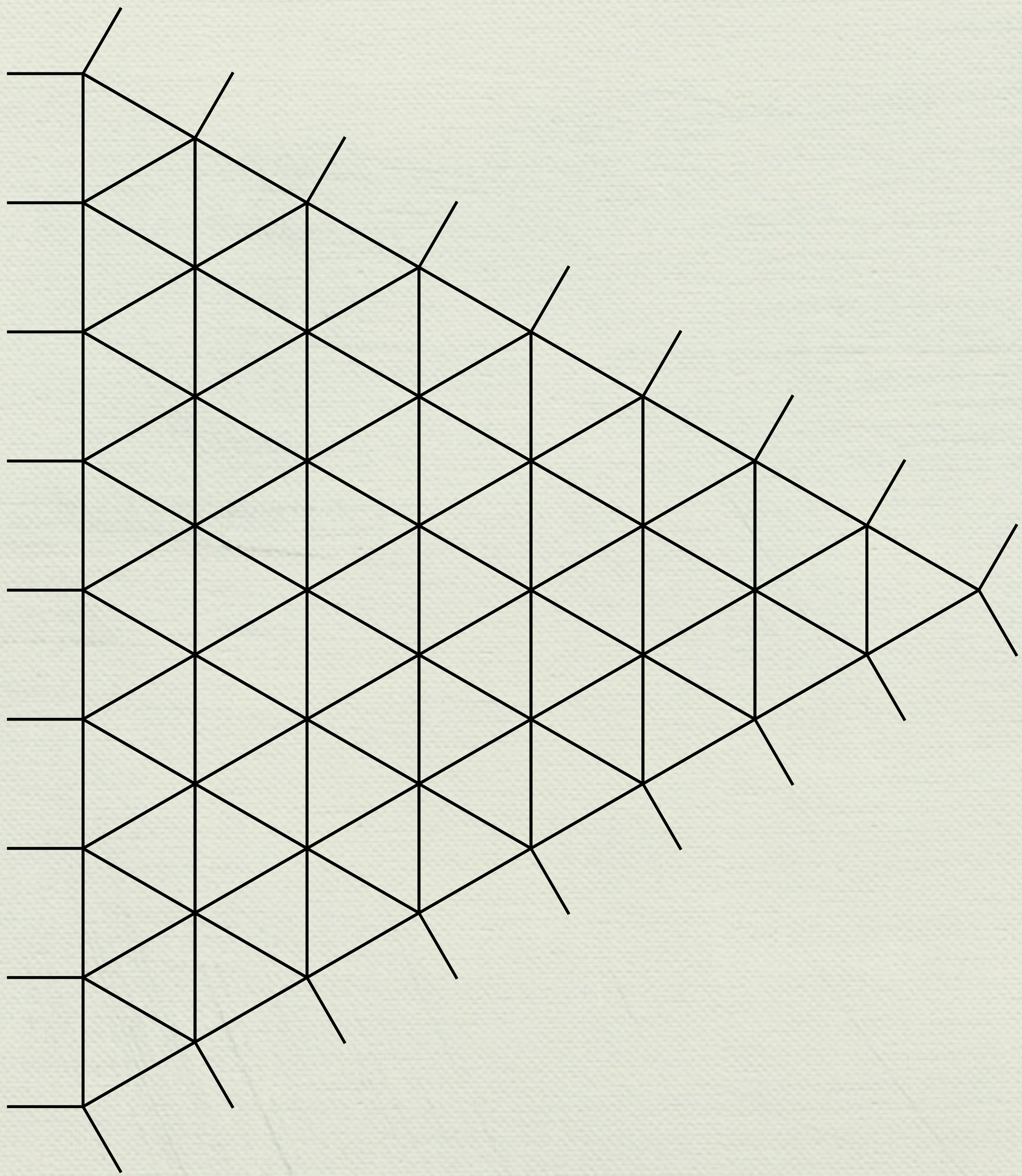


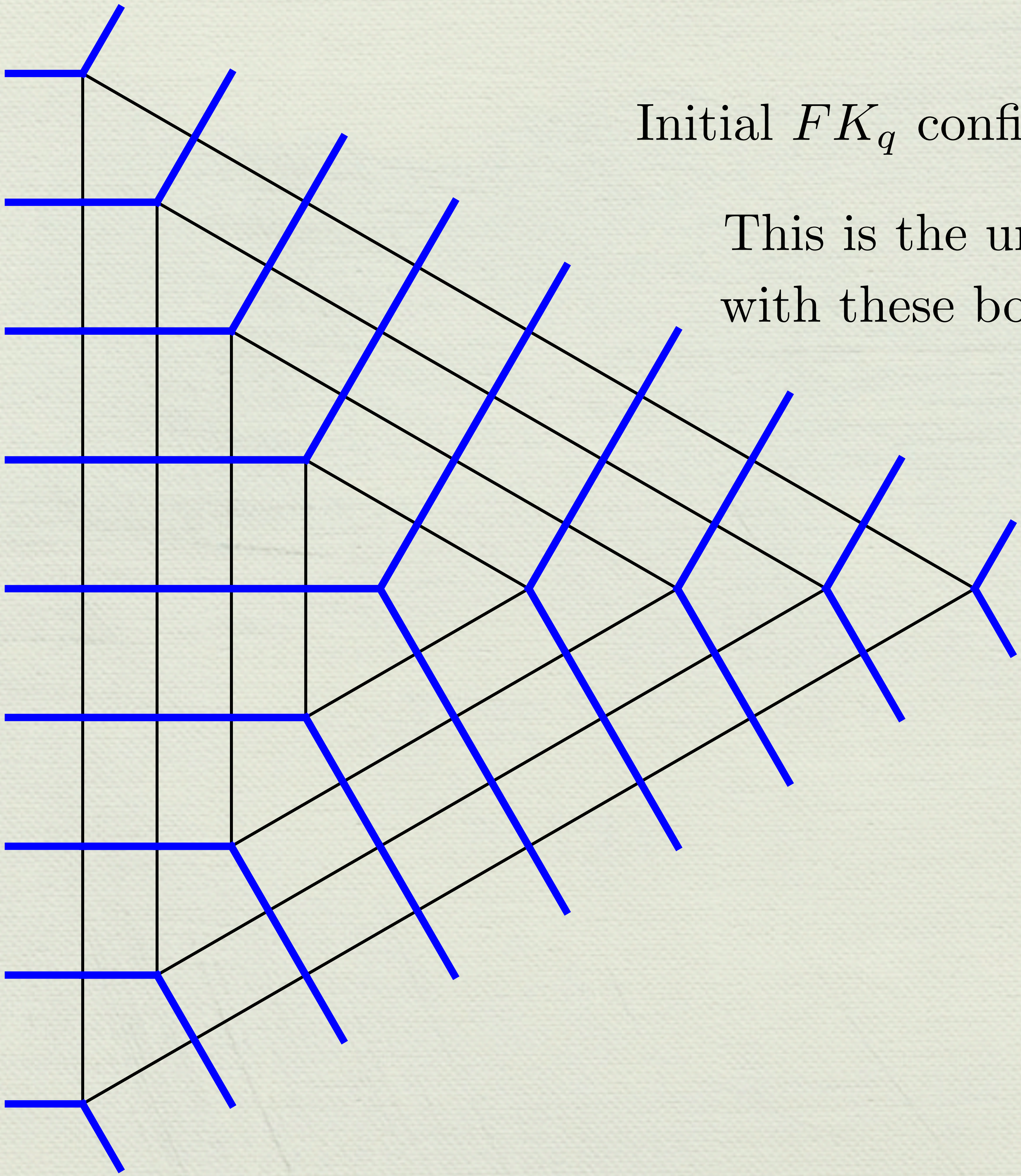






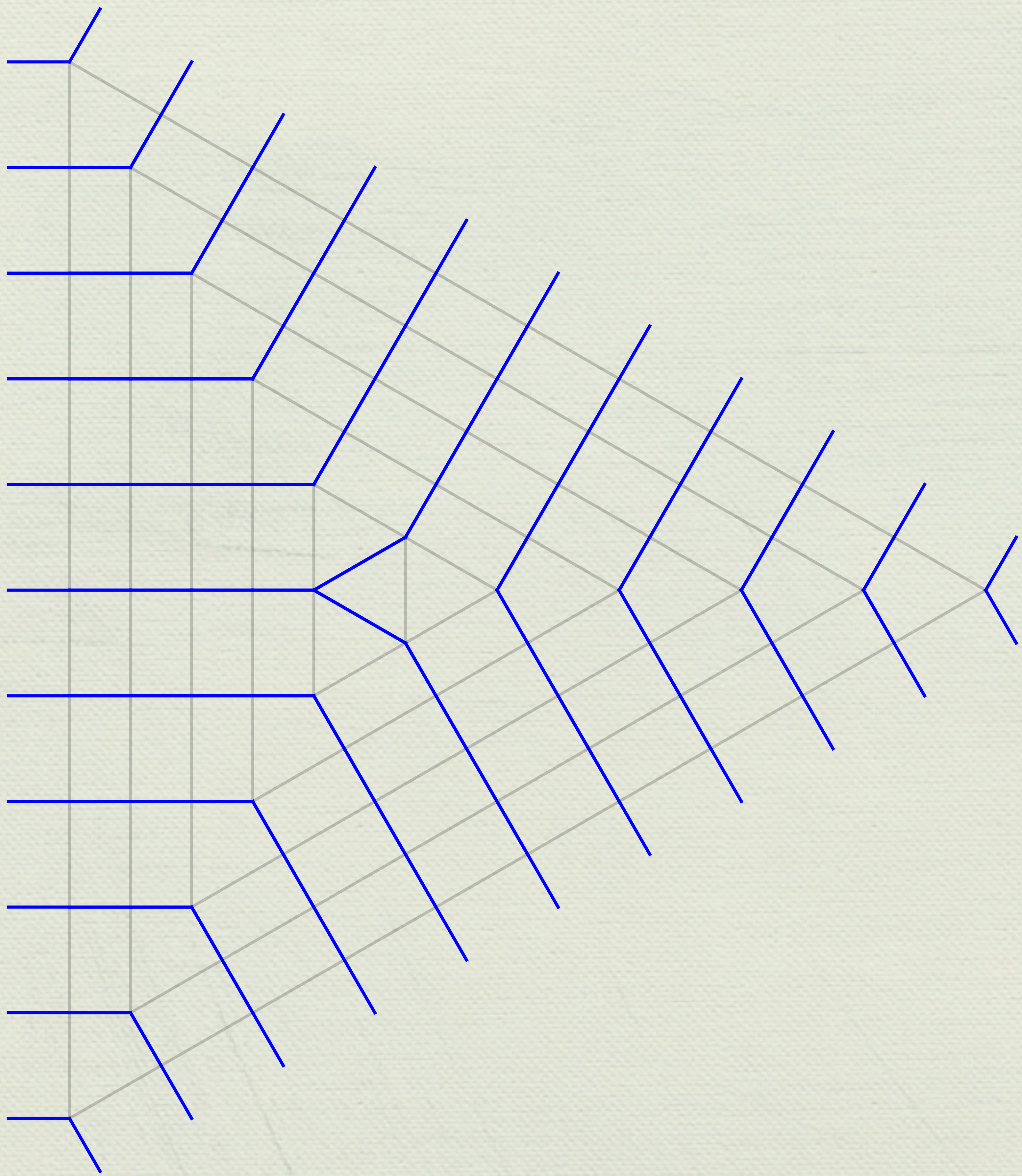


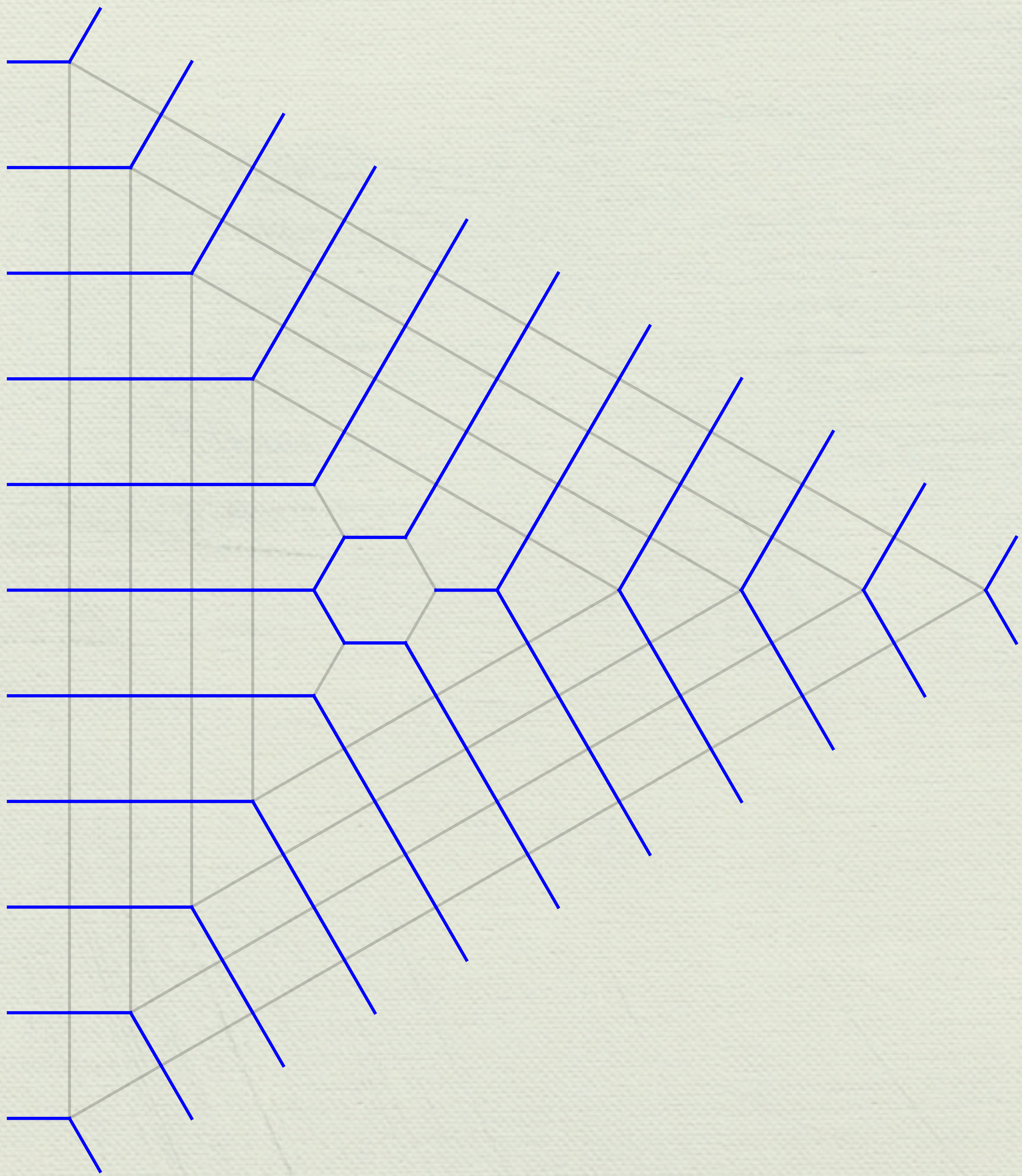


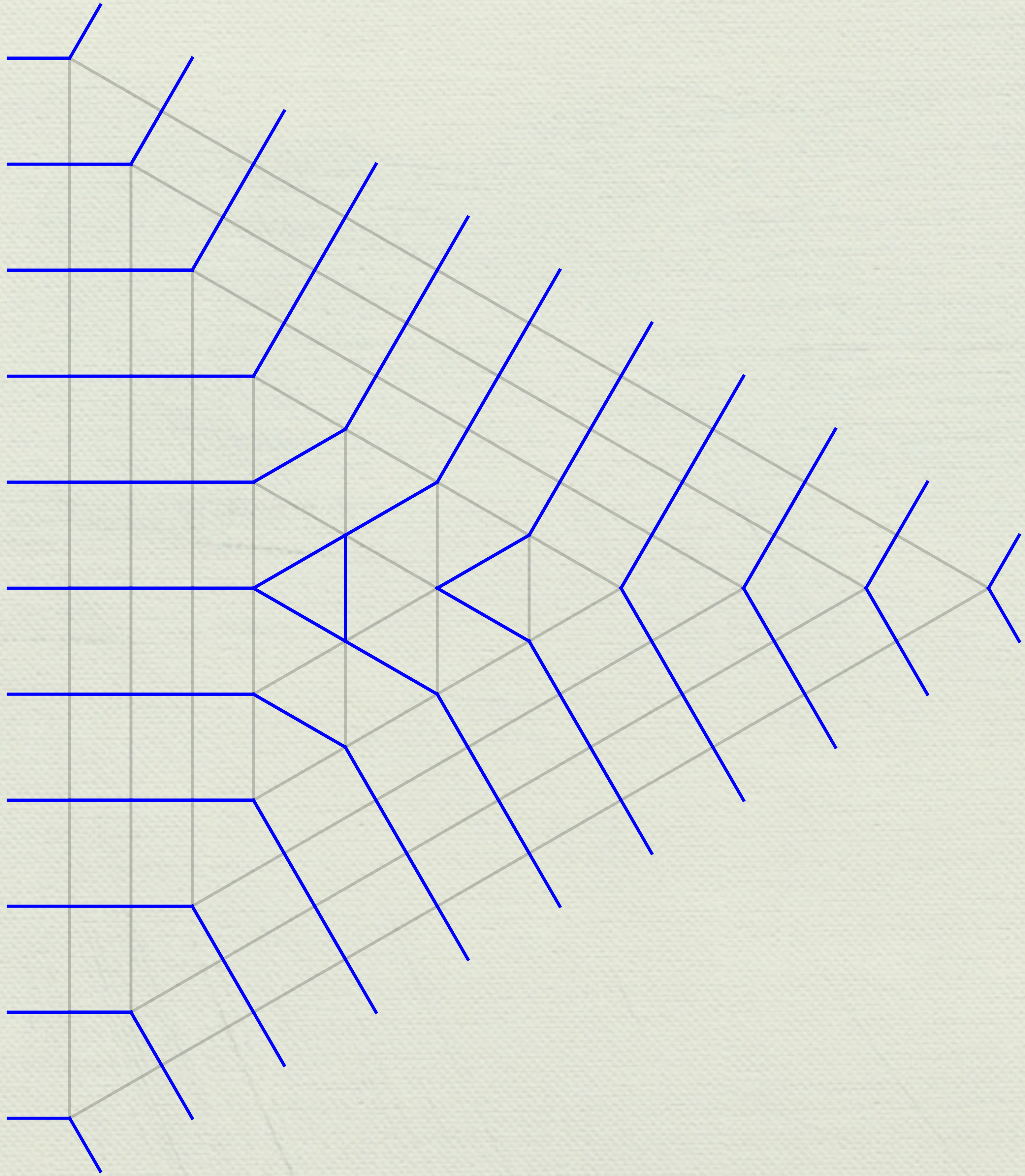


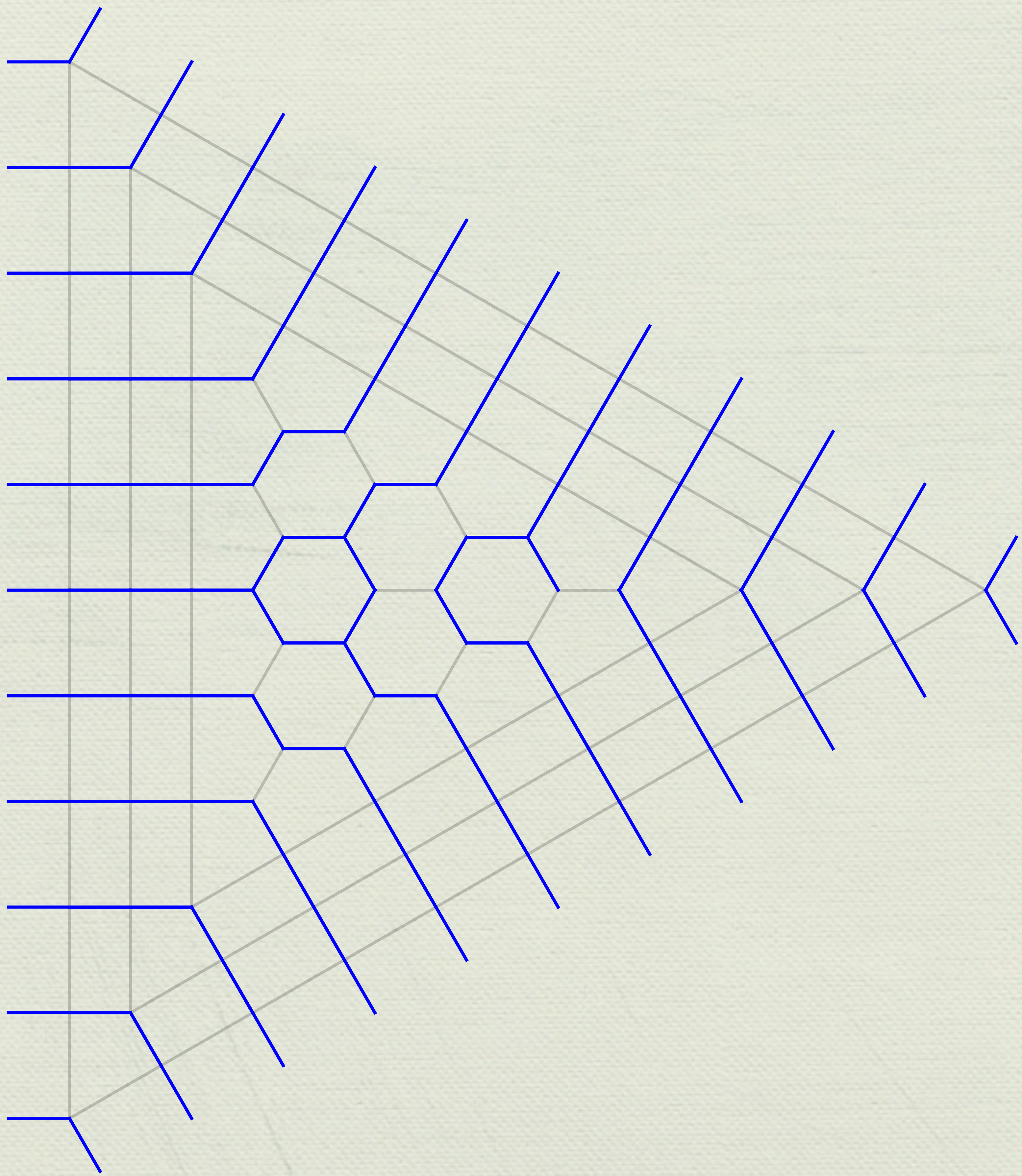
Initial FK_q configuration.

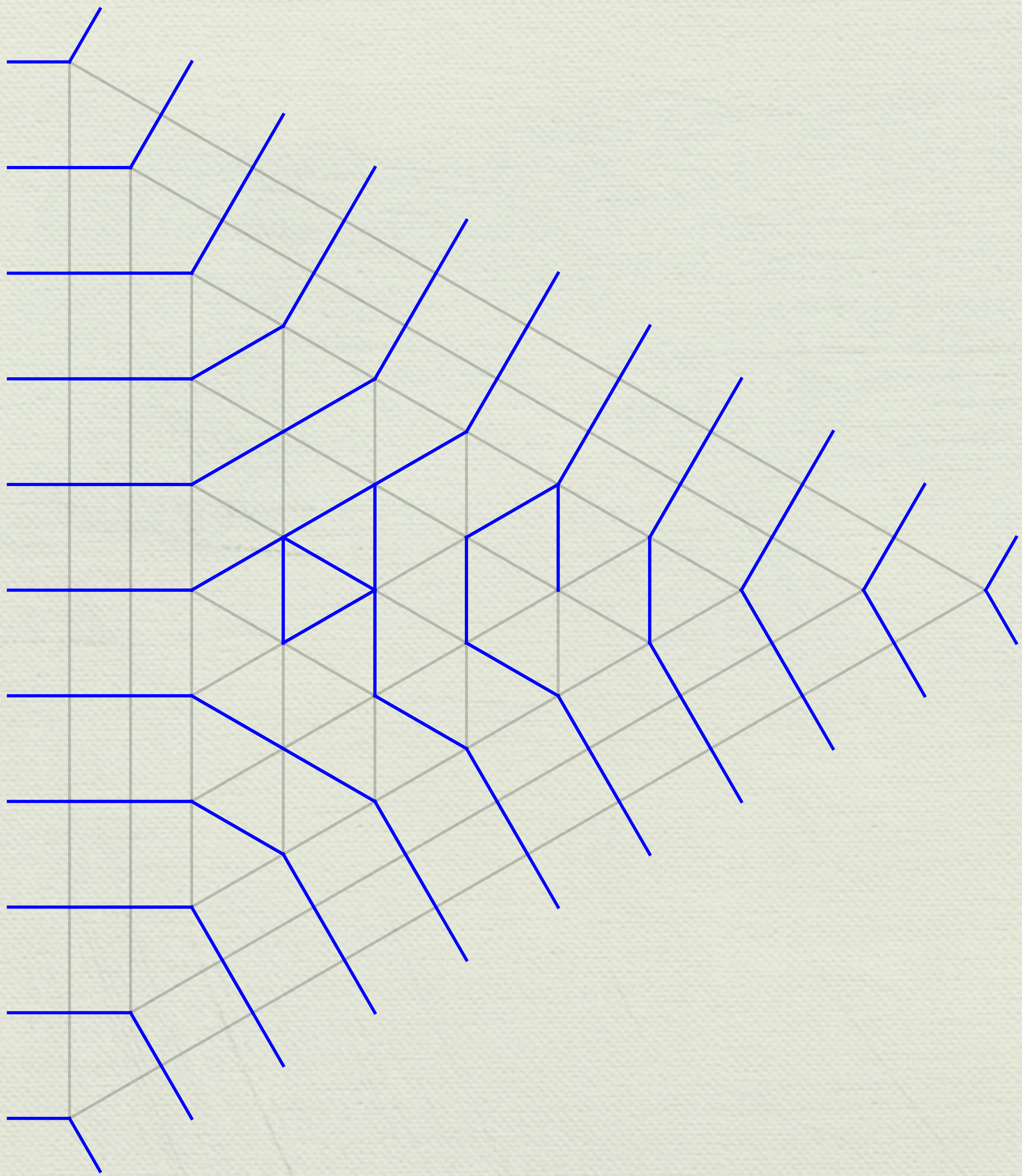
This is the unique configuration with these boundary connections.

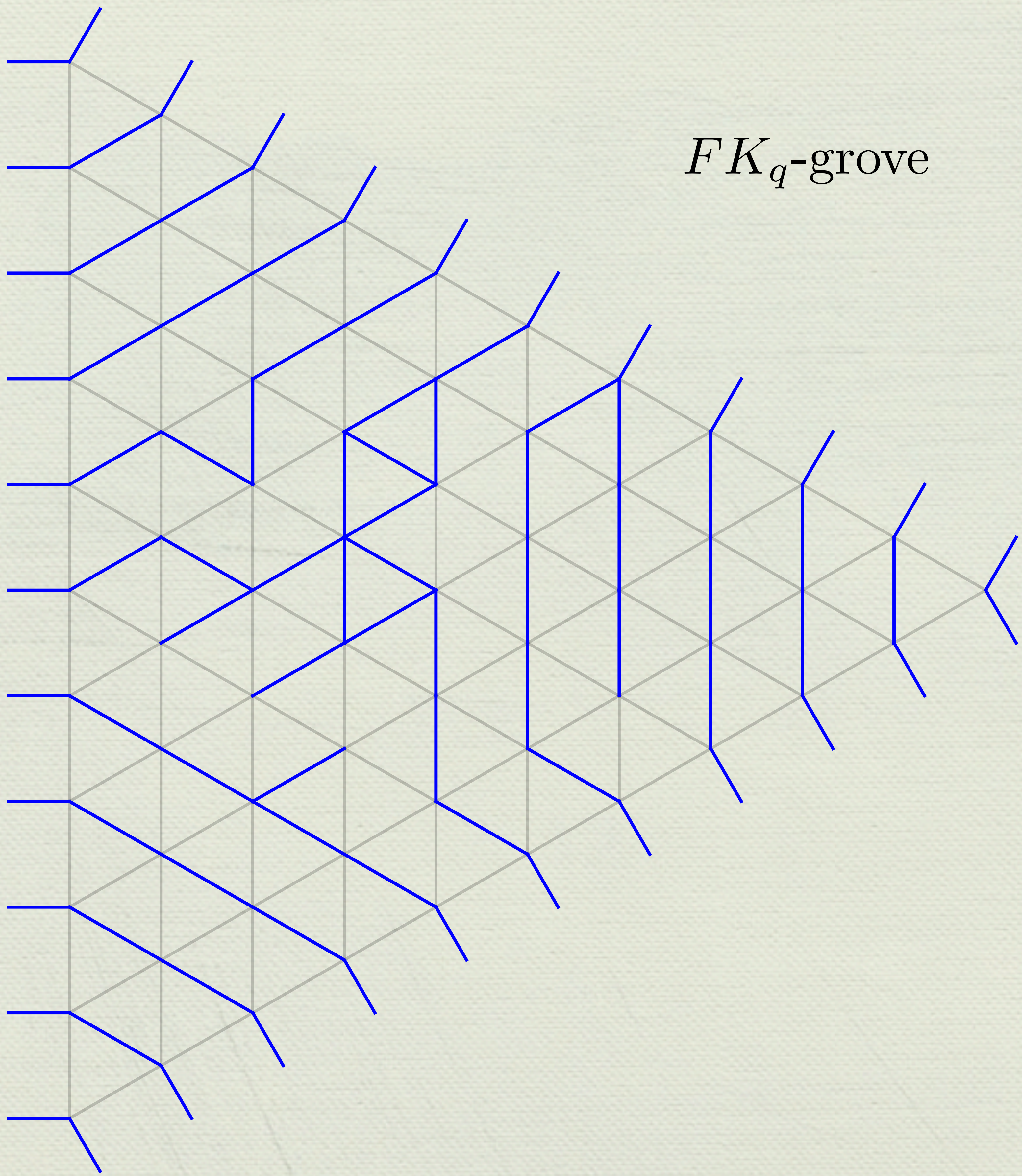




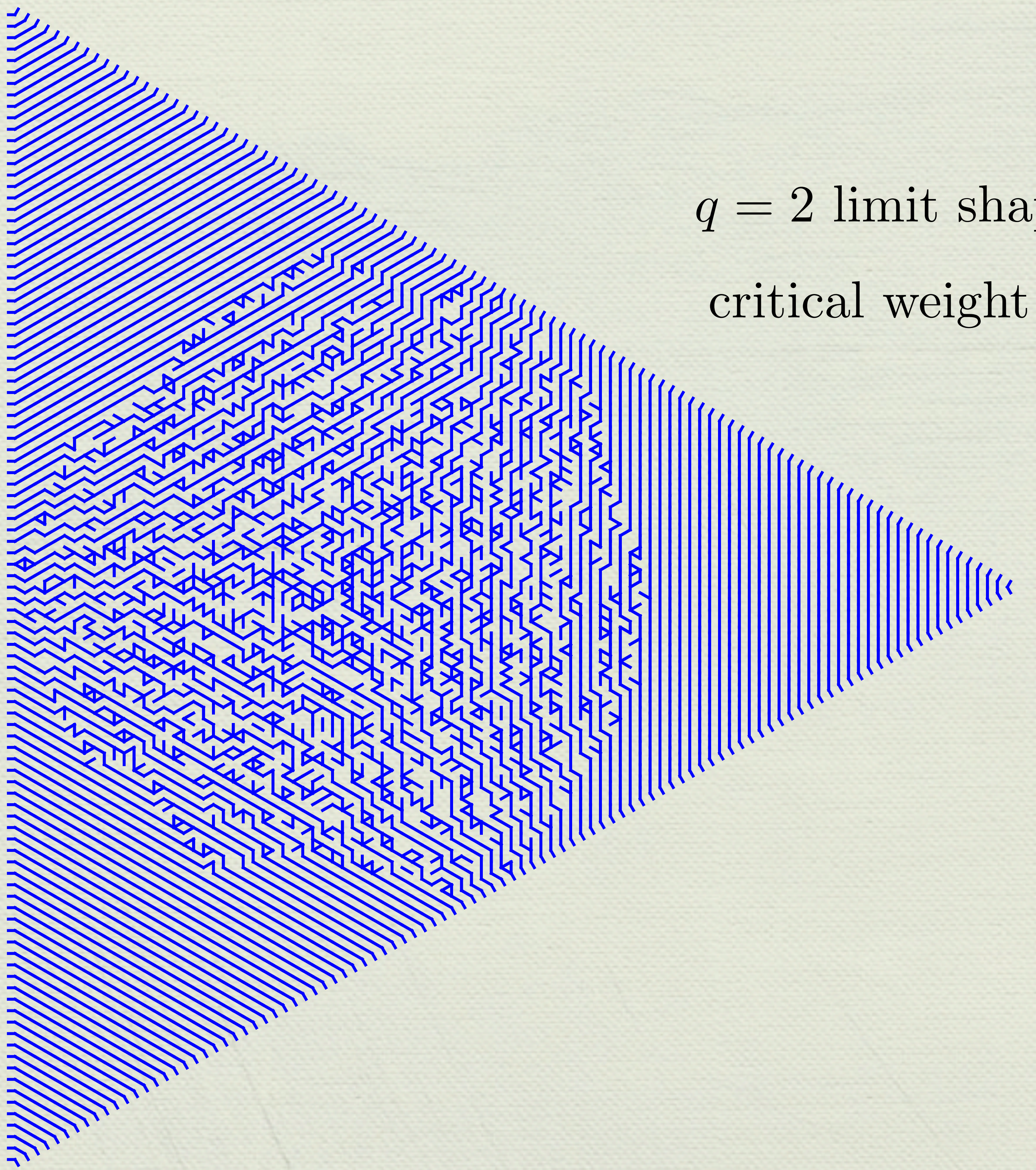






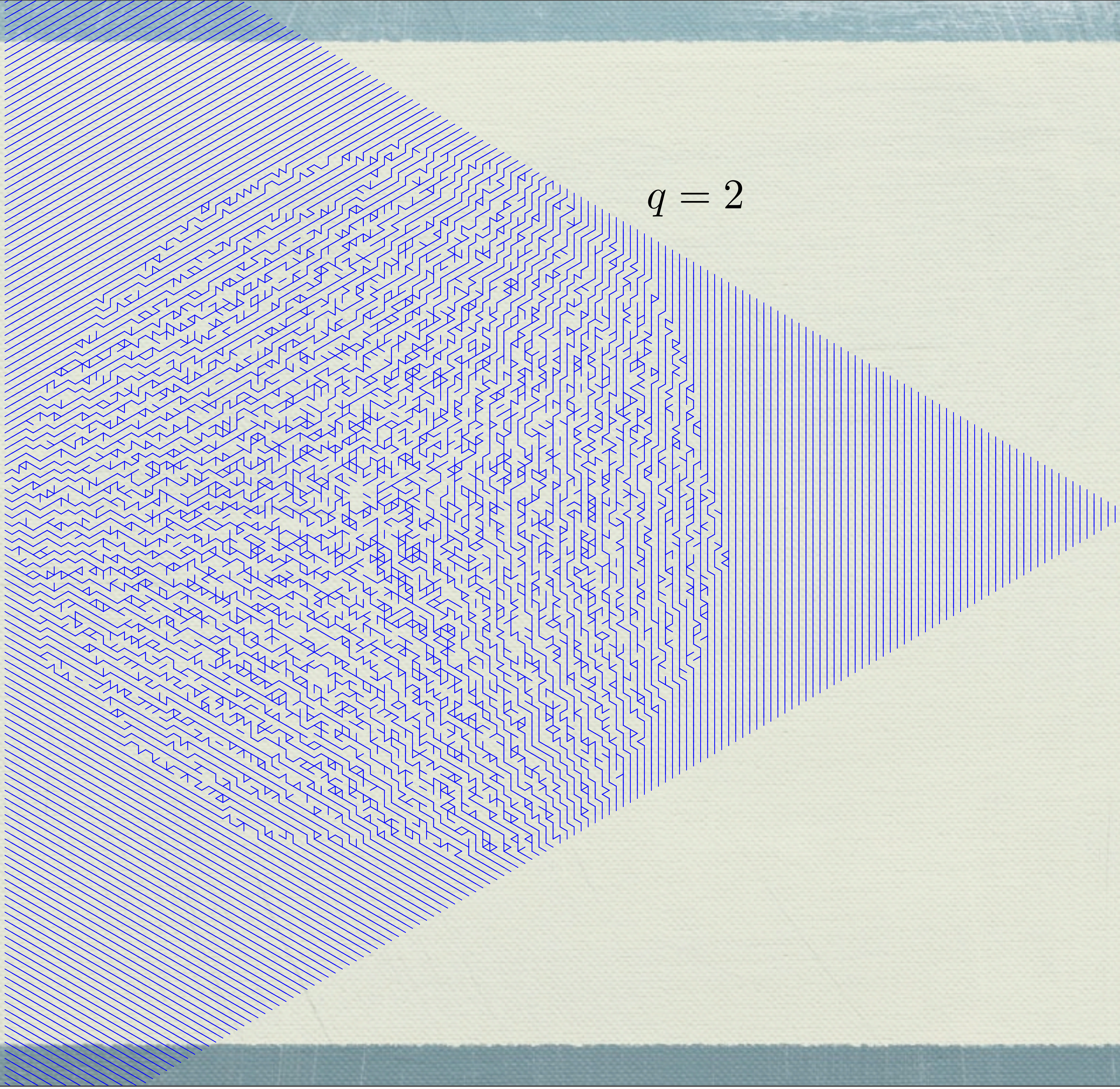


FK_q-grove

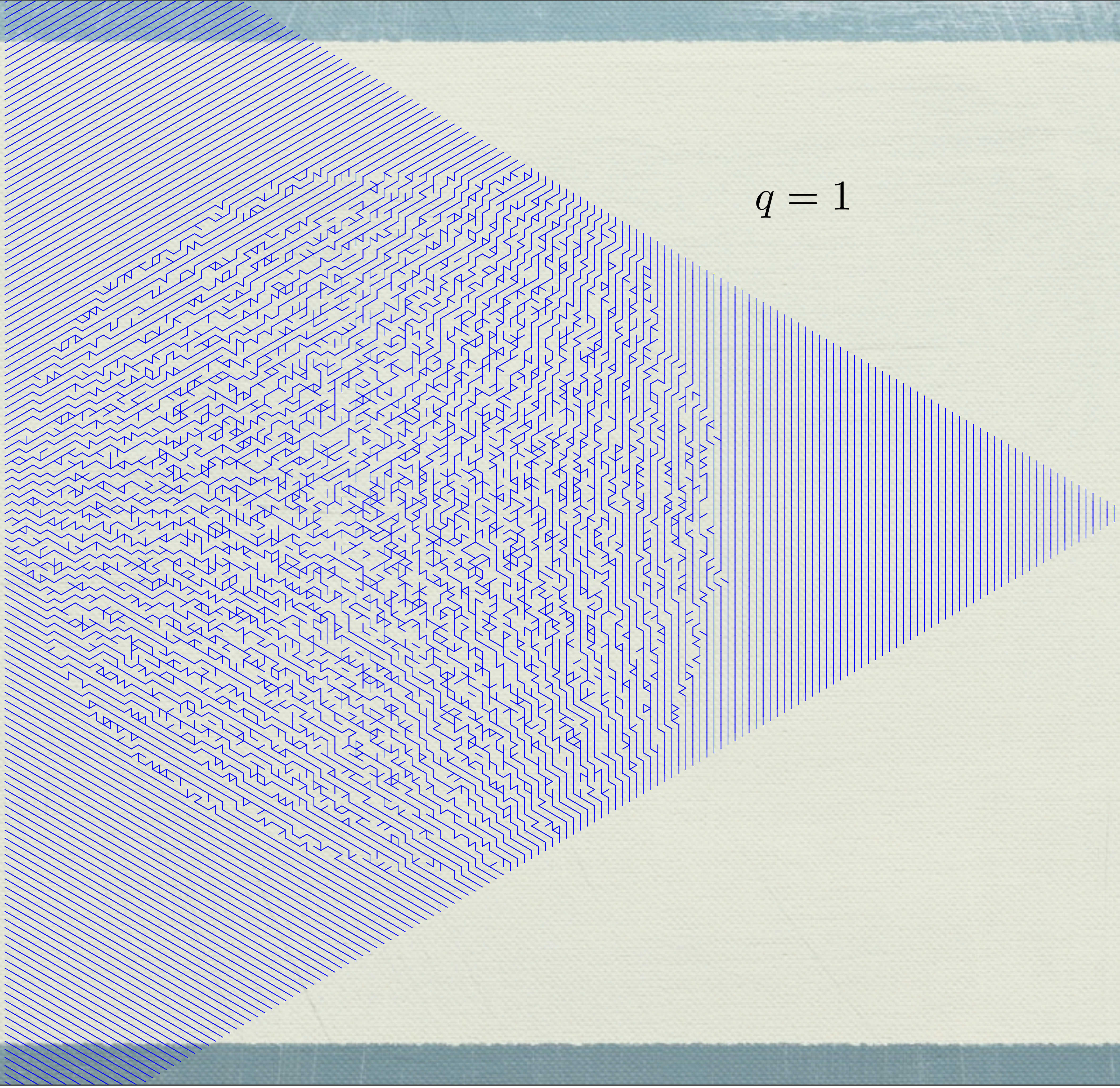


$q = 2$ limit shape.

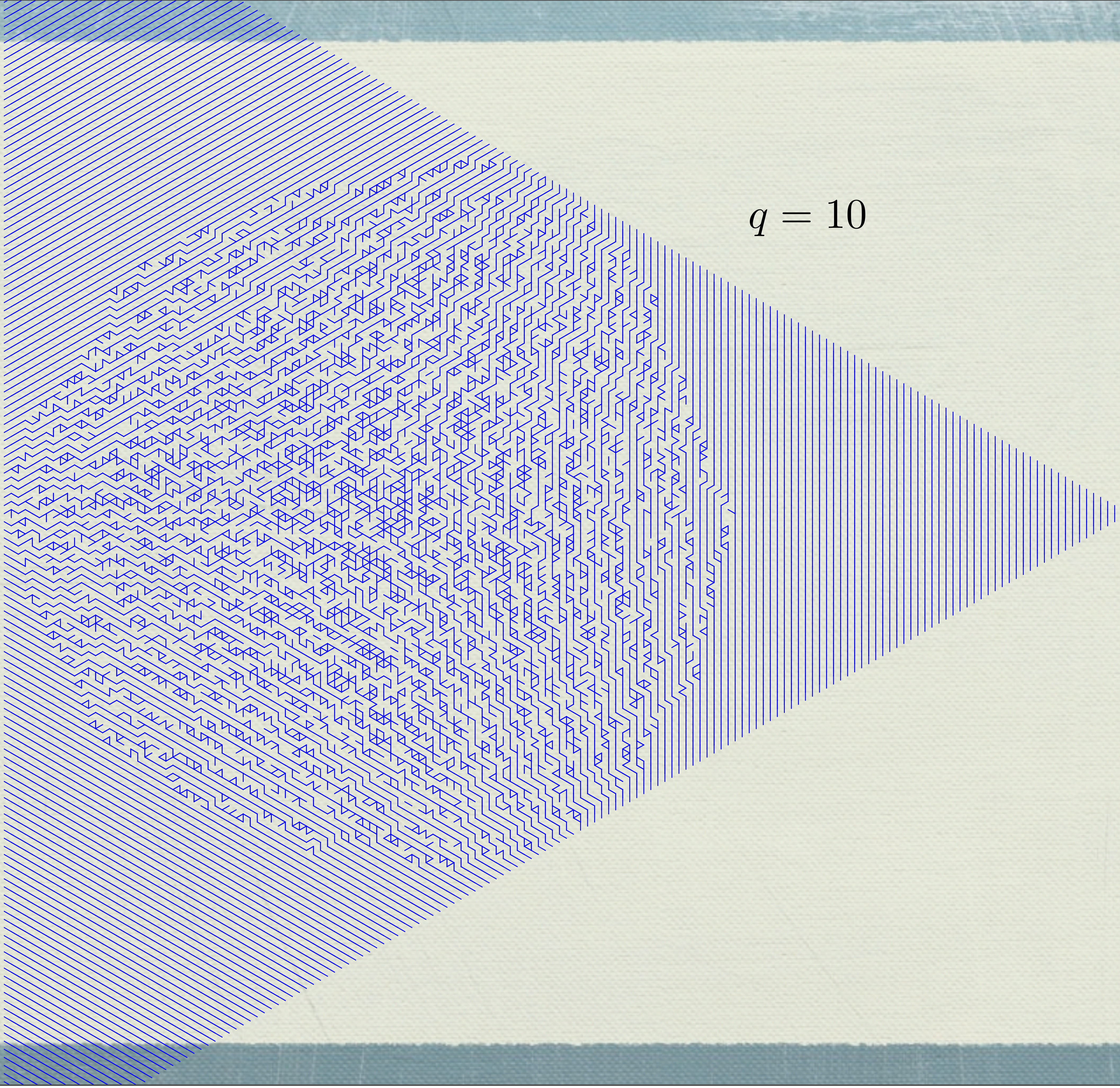
critical weight $w = \sqrt{3} + 1$.



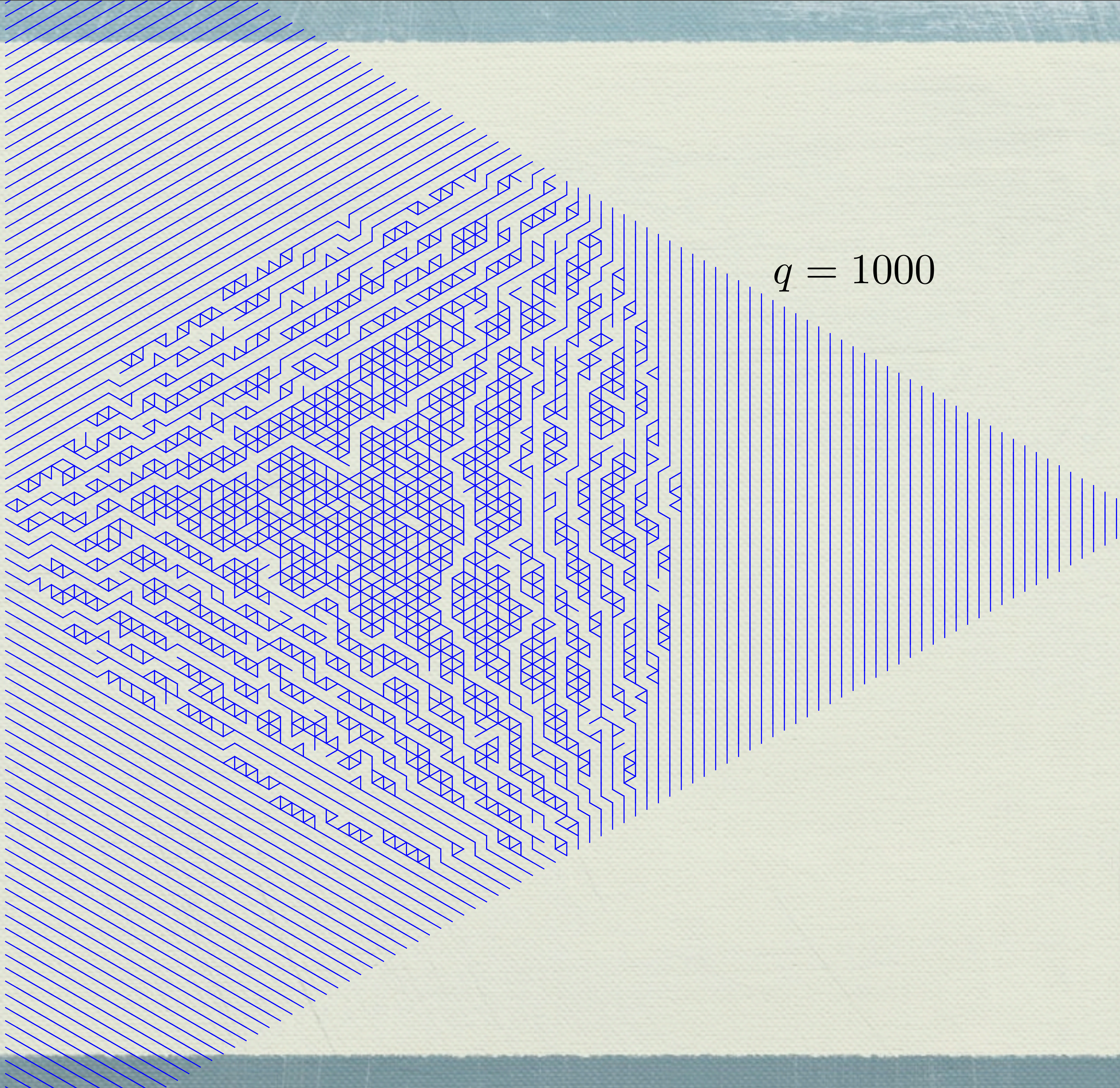
$q = 2$



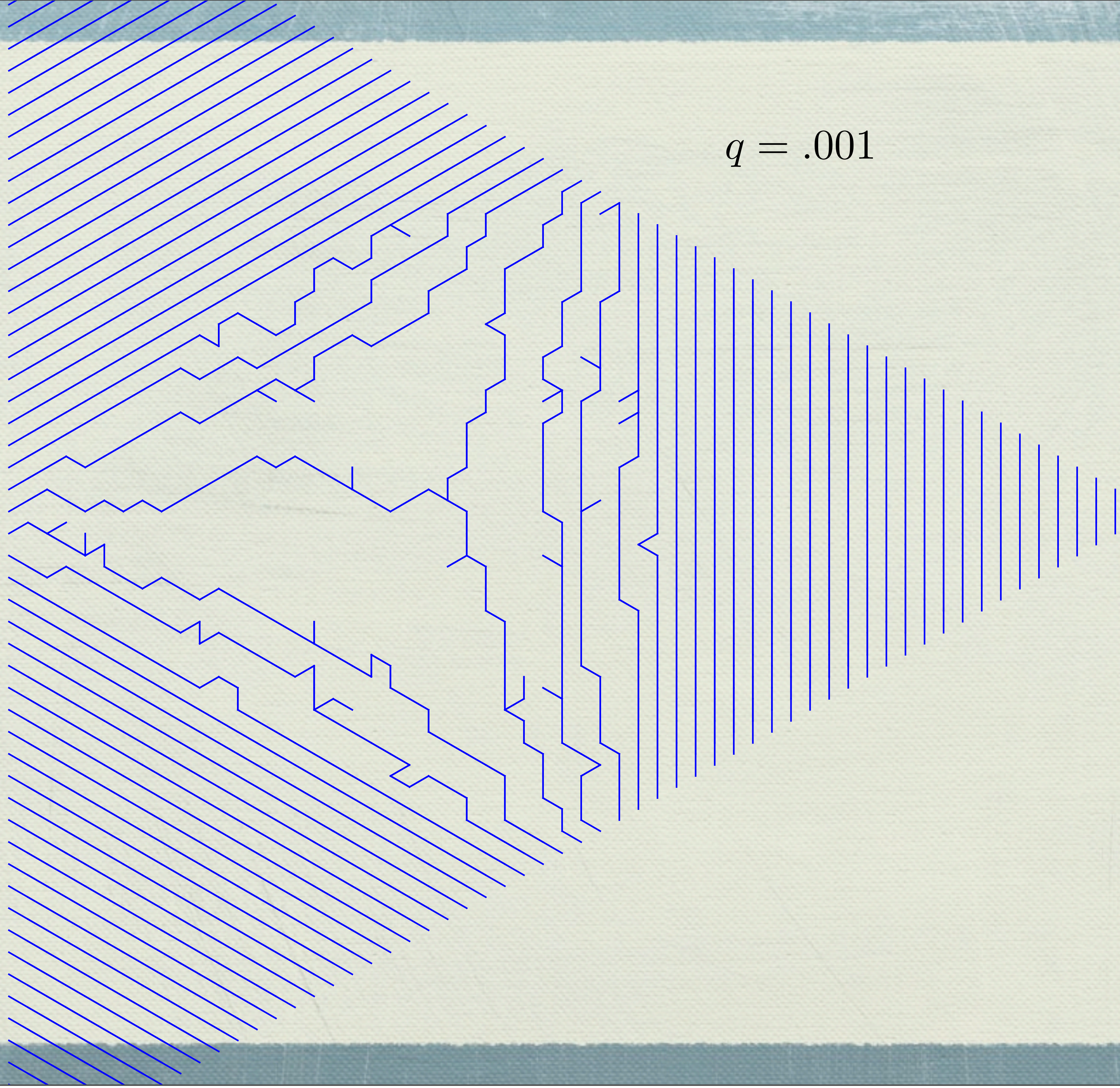
$q = 1$



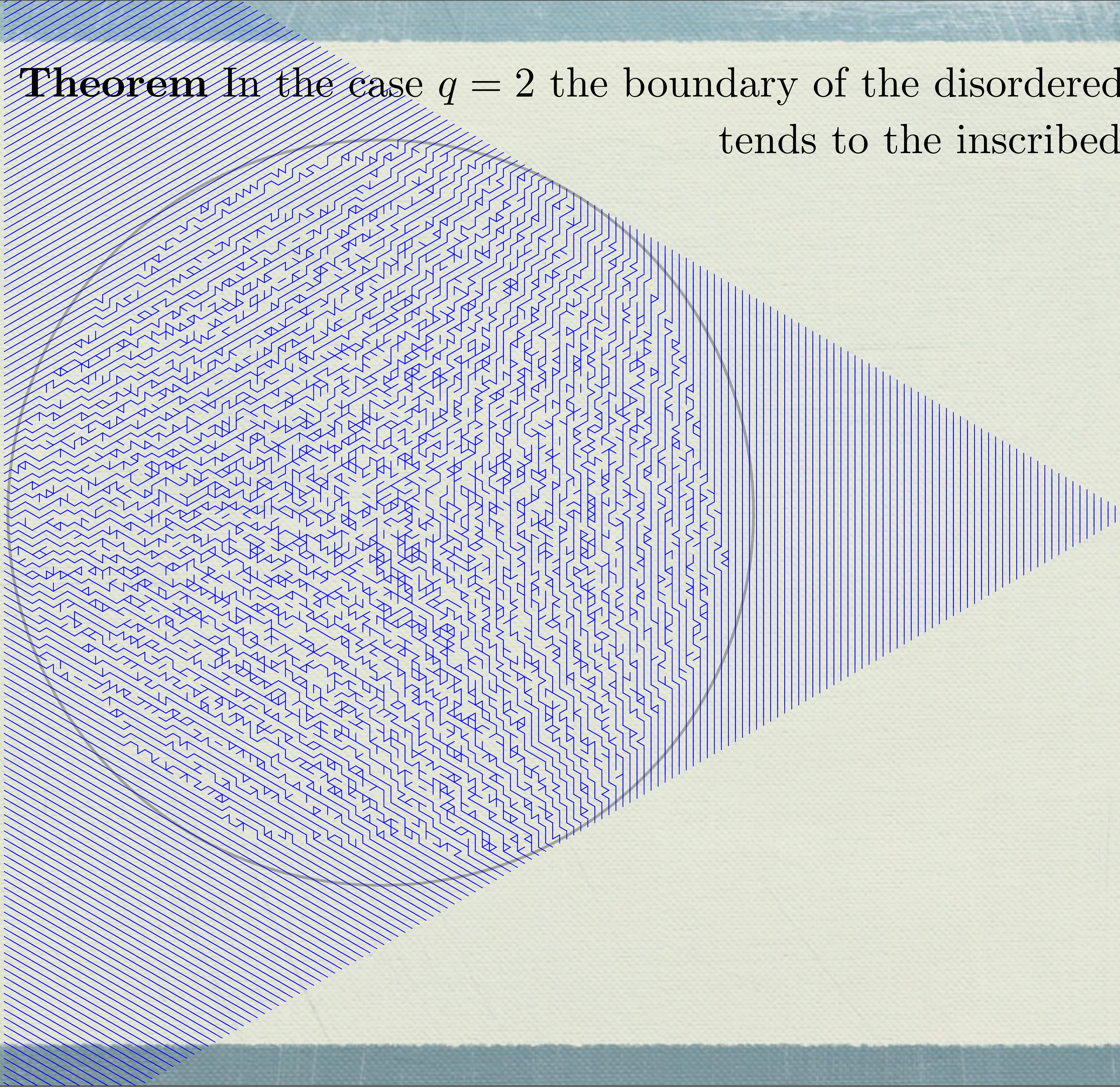
$q = 10$



$q = 1000$

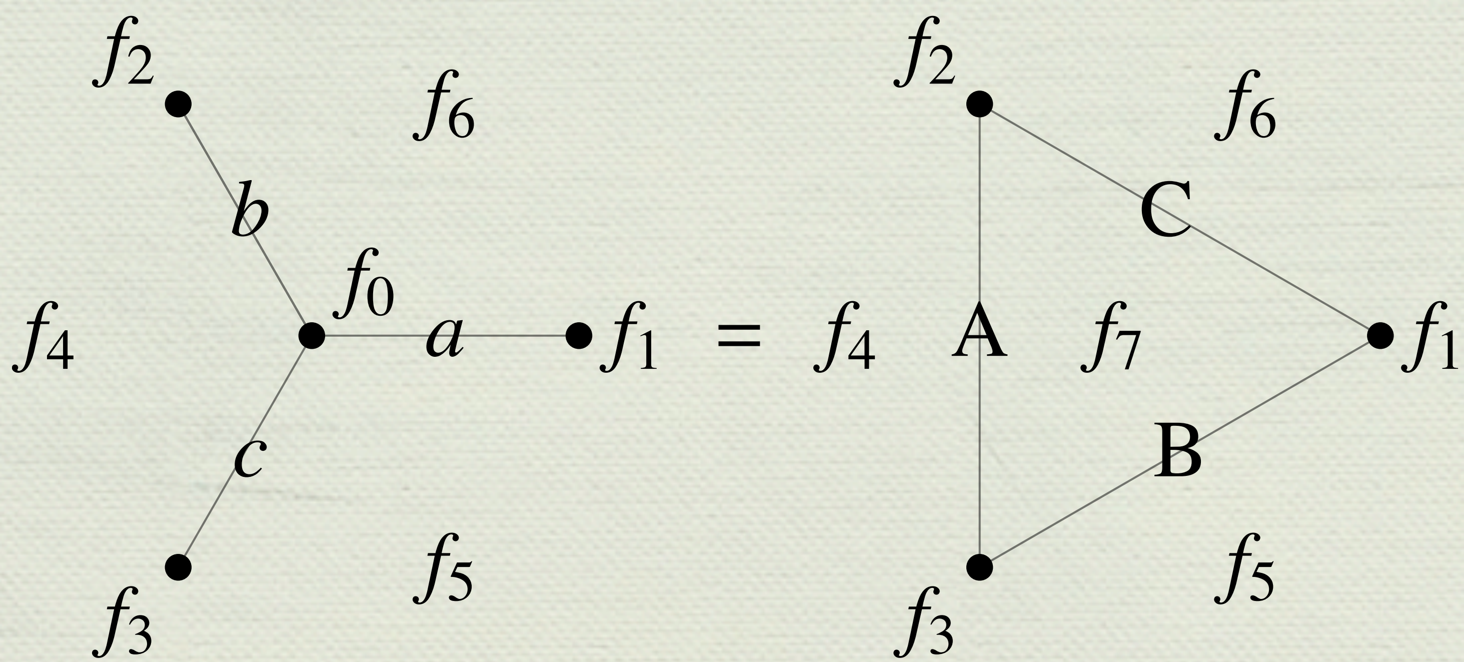

$q = .001$

Theorem In the case $q = 2$ the boundary of the disordered region tends to the inscribed circle.



Remarkable fact about the Ising Y-Delta move (Kashaev):

Define new variables f on vertices and faces:



where ratios of adjacent fs are related to edge weights as:

$$\left(\frac{a-1/a}{2}\right)^2 = \frac{f_0 f_1}{f_5 f_6}, \qquad etc.$$

Theorem [Kashaev] The fs satisfy

$$f_0^2 f_7^2 + f_1^2 f_4^2 + f_2^2 f_5^2 + f_3^2 f_6^2 - 2(f_1 f_2 f_4 f_5 + f_1 f_4 f_3 f_6 + f_2 f_3 f_5 f_6) \\ -2 f_0 f_7 (f_1 f_4 + f_2 f_5 + f_3 f_6) - 4(f_0 f_4 f_5 f_6 + f_7 f_1 f_2 f_3) = 0.$$

We say $f : \mathbb{Z}^3 \to \mathbb{C}$ **satisfies the Kashaev recurrence**

if $P(f_{i,j,k}, f_{i+1,j,k}, \ldots, f_{i+1,j+1,k+1}) = 0$ for all $(i, j, k) \in \mathbb{Z}^3$.

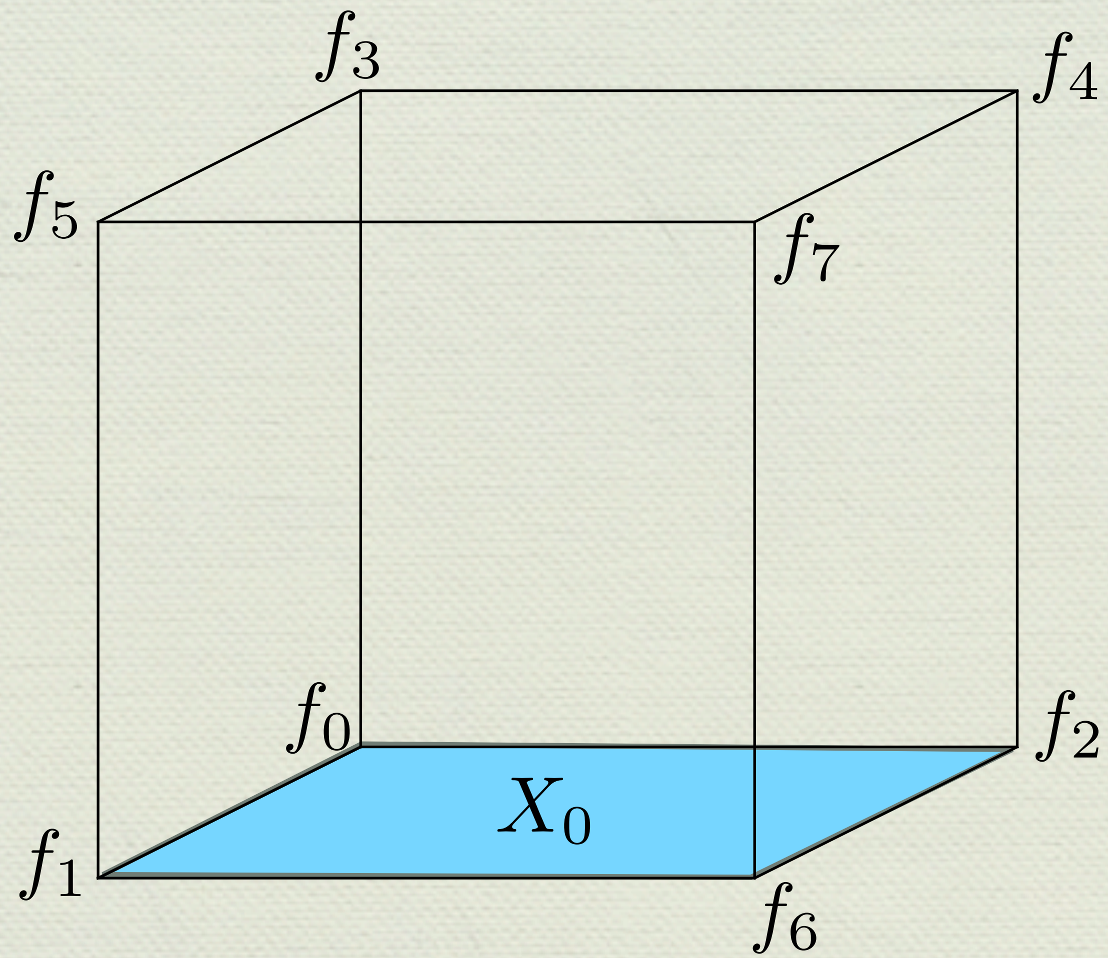
By defining $f_{i,j,k}$ on $0 \leq i + j + k \leq 2$ we can use P to define it everywhere.

Example: Suppose $f_{i,j,k} = f_{i+j+k}$

$f_0 = 1, f_1 = a, f_2 = b$, then

$$\begin{aligned} f_{2n} &= a^{2n} R^{n^2} S^{n^2 - n} \\ f_{2n+1} &= a^{2n+1} R^{n^2+n} S^{n^2} \end{aligned}$$

where $R = b/a^2$ and $S = \frac{2(R+1)^{3/2} + 3R + 2}{R^2}$.


f_3
f_4
f_5
f_7
f_0
f_2
X_0
f_1
f_6

Let $X_{i,j,k} = \sqrt{f_{i,j,k} f_{i,j+1,k+1} + f_{i,j+1,k} f_{i,j,k+1}}$,

and symmetrically for Y, Z.

Then f, X, Y, Z satisfy the recurrence:

$$f_{i,j,k} = \frac{Z^2_{i-1,j-1,k} - f_{i-1,j,k} f_{i,j-1,k}}{f_{i-1,j-1,k}}$$

$$X_{i,j,k} = \frac{f_{i,j,k} X_{i-1,j,k} + Y_{i-1,j,k} Z_{i-1,j,k}}{f_{i-1,j,k}} \qquad \text{\& cyclic}$$

Fact(?): $f_{i',j',k'}$ is a Laurent polynomial in the quantities

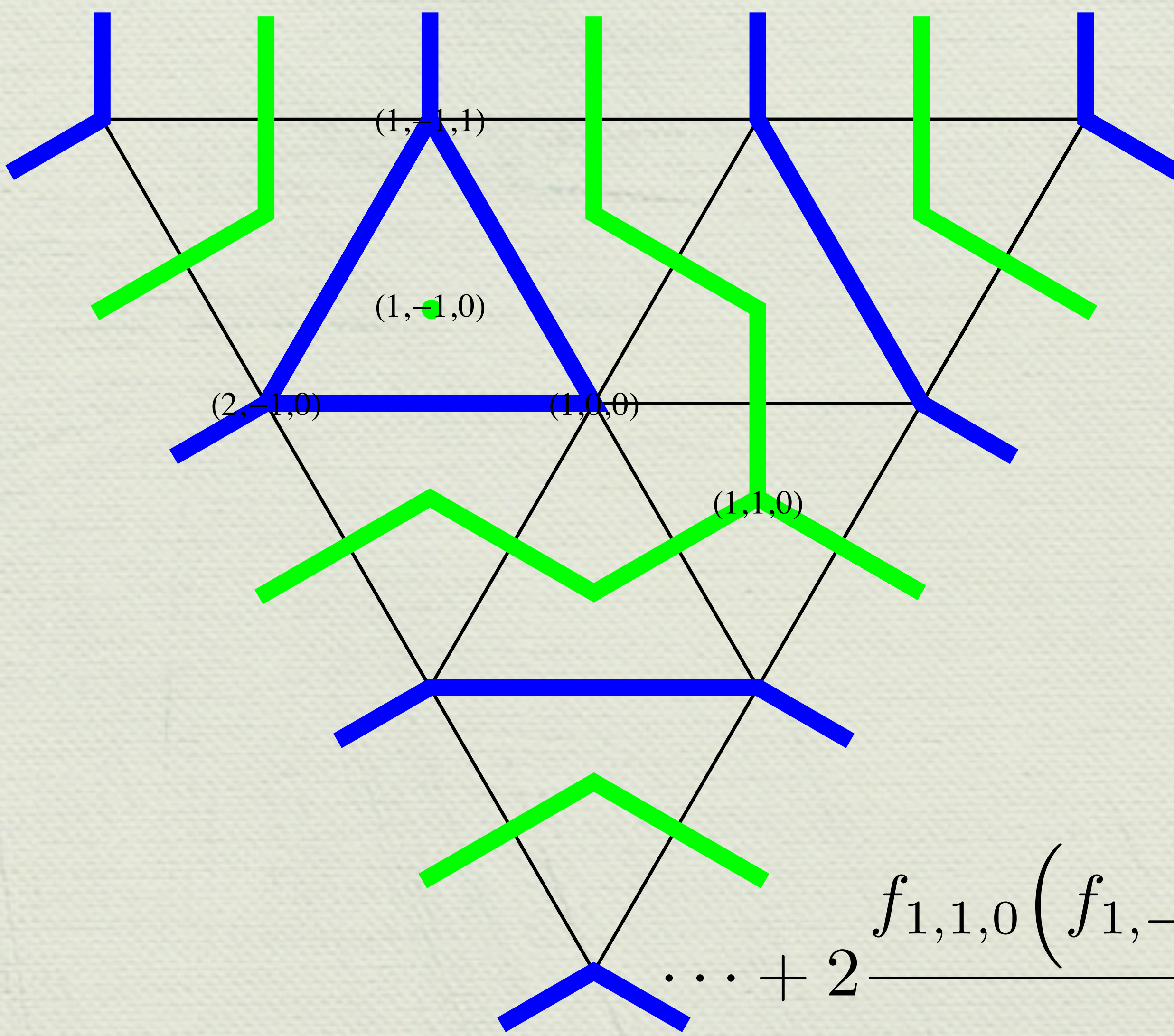
$$f_{i,j,k}, \quad i+j+k = 0,1$$

$$X_{i,j,k}, Y_{i,j,k}, Z_{i,j,k}, \quad i+j+k = 0$$

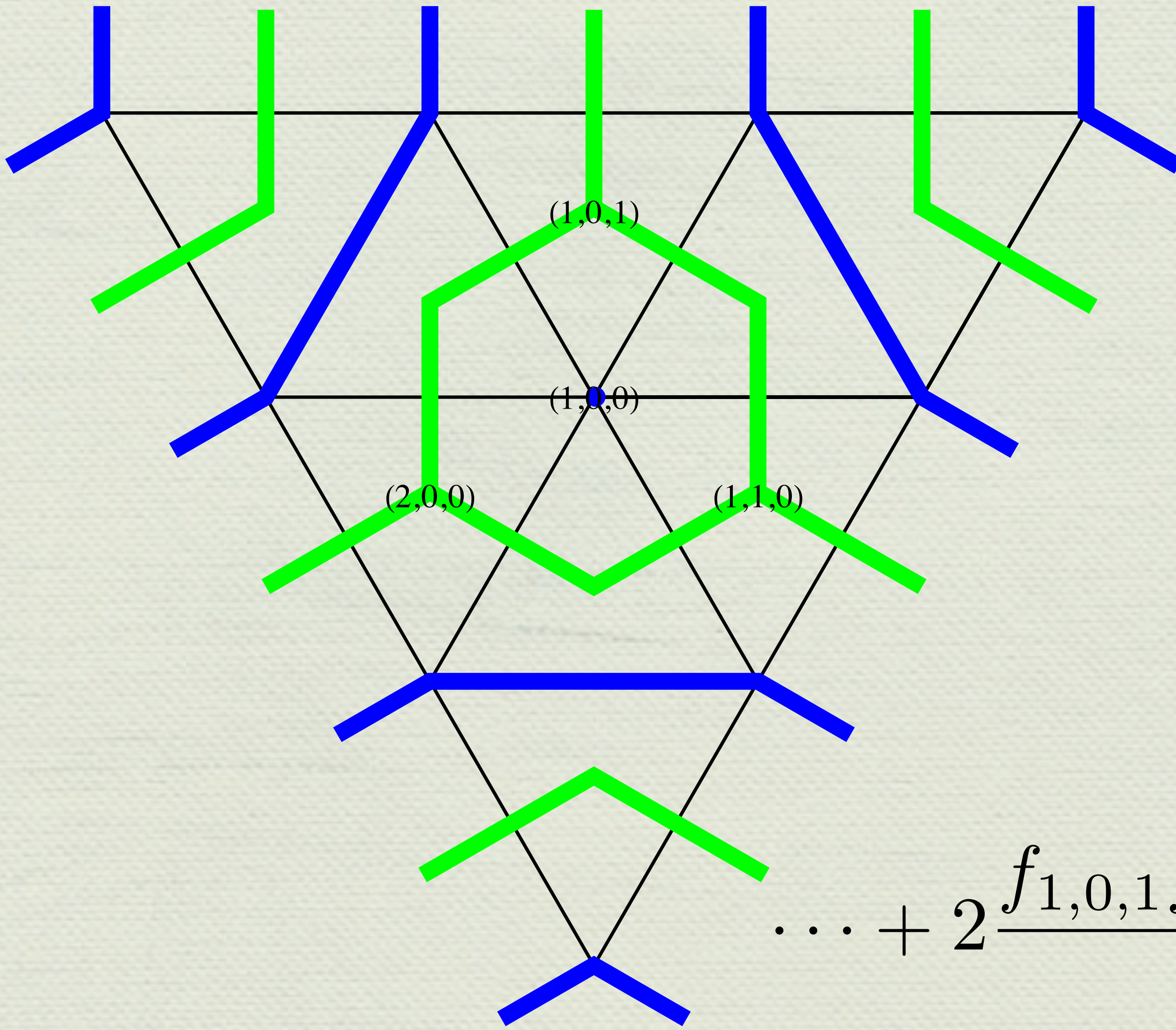
Conjecture: $f_{i,j,k} = \sum_S wt(S)$, where the sum is over edge subsets

having the desired boundary connectivity,

and $wt(S) = \ldots$



$$\cdots + 2\,\frac{f_{1,1,0}\left(f_{1,-1,0}f_{2,-1,0} + \frac{X_{1,-1,0}Y_{1,-1,0}Z_{1,-1,0}}{f_{1,0,0}}\right)}{f_{1,-1,0}^2} + \ldots$$



$$\cdots + 2\frac{f_{1,0,1}f_{2,0,0}f_{1,1,0} + X_{1,0,0}Y_{1,0,0}Z_{1,0,0}}{f_{1,0,0}^2} + \ldots$$

Arctic circle theorem: Use $f_{i,j,k} = 3^{(i+j+k)^2/2}$.

$f'_{n,n,n}$ satisfies a linear recurrence (with coefficients depending on f).

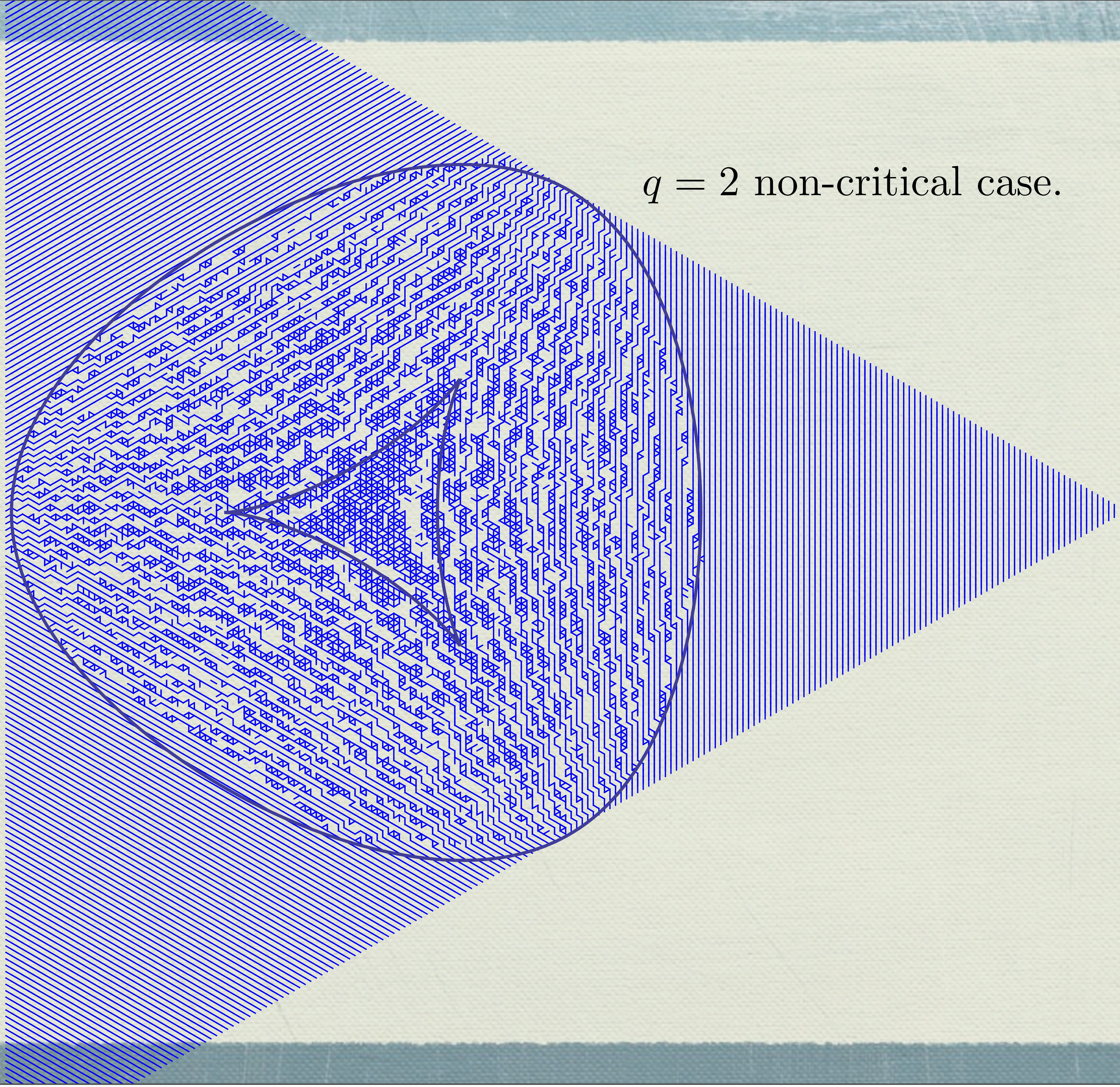
Let $G(x,y,z) = \sum f'_{i,j,k} x^i y^j z^k$.

Then $G(x,y,z)$ satisfies a linear recurrence with characteristic polynomial:

$$P(x,y,z) = xyz + 1 - \tfrac{1}{3}(xy + xz + yz + x + y + z).$$

Analyze growth of coefficients of $1/P$:

- polynomial inside inscribed circle
- exponential decay outside inscribed circle QED.



$q = 2$ non-critical case.

$$Q = 3098115099749559840207375698 41a^6 - 1858374937729039544359650269170a^5b - 1858374937729039544359650269170a^5c + 5883454153820320725807778237007a^4b^2 + 4334397195006546369711336315654a^4bc + 5883454153820320725807778237007a^4c^2 - 8669781452132474330937731075356a^3b^3 - 7427079315358238395356762728212a^3b^2c - 7427079315358238395356762728212a^3bc^2 - 8669781452132474330937731075356a^3c^3 + 5883454153820320725807778237007a^2b^4 - 7427079315358238395356762728212a^2b^3c + 3279754328428189867356873038 7594a^2b^2c^2 - 7427079315358238395356762728212a^2bc^3 + 5883454153820320725807778237007a^2c^4 - 1858374937729039544359650269170ab^5 + 4334397195006546369711336315654ab^4c - 7427079315358238395356762728212ab^3c^2 - 7427079315358238395356762728212ab^2c^3 + 4334397195006546369711336315654abc^4 - 1858374937729039544359650269170ac^5 + 3098115099749559840207375698 41b^6 - 1858374937729039544359650269170b^5c + 5883454153820320725807778237007b^4c^2 - 8669781452132474330937731075356b^3c^3 + 5883454153820320725807778237007b^2c^4 - 1858374937729039544359650269170bc^5 + 3098115099749559840207375698 41c^6$$